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THE DIRECT AND INDIRECT POWER FLOWS OF THREE NON-CONSERVATIVELY SERIES COUPLED OSCILLATORS

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The formulae of power flows among three non-conservatively series coupled oscillators excited by random forces have been derived and analyzed. The analysis results show that the theory of the power flows of both three conservatively series coupled oscillators and two non-conservatively coupled oscillators are special examples of the theory of the power flows of three non-conservatively series coupled oscillators, and the reciprocity of the indirect power flows among three non-conservatively series coupled oscillators is untenable. The relationships between the power flows and the parameters of coupling have been studied by means of the numerical method. The numerical results show that the indirect power flows correspond to the resonant transmission between the indirectly coupled oscillators, and the indirect power flows should not be ignored in strong coupling conditions. Using the definition of the direct coupling loss factors (coupling loss factor in classical Statistial Energy Analysis) and the power flows of three non-conservatively series coupled oscillators, the influence of the indirectly coupled oscillators on the direct coupling loss factors has been studied by means of the numerical method. It is shown that the existence of the indirectly coupled oscillators has great influence on the direct coupling loss factors. The direct coupling loss factors in multi-coupled oscillator system are different from the coupling loss factors of the two coupled oscillators, and it is necessary to modify the direct coupling loss factors when the energy balance equations of SEA are modified. Experimental verification has been carried out with a floating raft isolation system. The agreement between the results estimated by SEA and those experimentally measured are good.

1. INTRODUCTION

Since Lyon and Maidanik [1] and Lyon [2] proposed the idea of Statistical Energy Analysis (SEA), SEA has not only been widely applied to the prediction of sound and vibration response [3–5], but it has also been developed and improved continuously by many researchers [6–22]. Recently, Heron [9], Finnveden [10] and some other researchers have found the existence of indirect power flows in multistructure systems, and the mechanism of the indirect power flows has been studied. In order to extend the energy balance equations of the classical SEA to multistructure systems, the power flows among three conservatively series coupled oscillators have been investigated by Sun and his co-researchers [17, 18]. The results show that the power flows among coupled oscillators include both direct and indirect power flows: the direct power flows correspond to resonant

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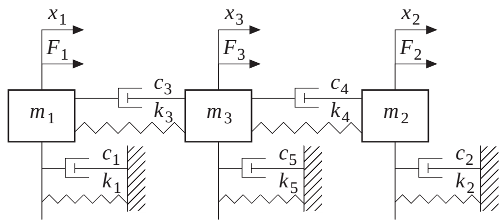

Figure 1. A mechanical model of three non-conservatively series coupled oscillators.

energy transmission between the physically coupled oscillators, and the indirect power flows correspond to non-resonant transmission between the indirectly coupled oscillators. Recently, Beshara and Keane [21] have studied SEA of multiple, non-conservatively coupled systems and some meaningful results have been obtained. However, the influence of the multiple non-conservatively coupled substructures on the power flows has not been studied. By using SEA for non-conservatively coupled systems, it is possible to study the power flows among three non-conservatively series coupled oscillators. Moreover, the energy balance equations of the classical SEA could be modified for the non-conservatively coupled multistructure systems if knowledge of the power flows among three non-conservatively series coupled oscillators are available. In order to apply SEA to predict the responses of non-conservatively and strongly coupled structures, the strong coupling loss factors have been modified by some researchers [11–14], and this modification has improved the precision of prediction to a certain extent. Thus, the energy balance equations of SEA are of universal significance. However, only the coupling loss factors have been modified in this modification, and the energy balance equations of SEA have not been improved essentially. In recent years, indirect power flows have been discovered. The important influence of indirect power flow on the prediction of structural vibration response has been studied [14, 15, 19, 20]. The energy balance equations of SEA have been modified correspondingly. Other wide ranging studies have also been carried out by many researchers [16–18] so as to extend the application of SEA. For example, coupling damping loss factors have been introduced into the main SEA energy balance equations for the limiting case of weak coupling by Beshara and Keane [21]. However, there has been still no research on the influence of indirectly coupled structures on the coupling loss factors

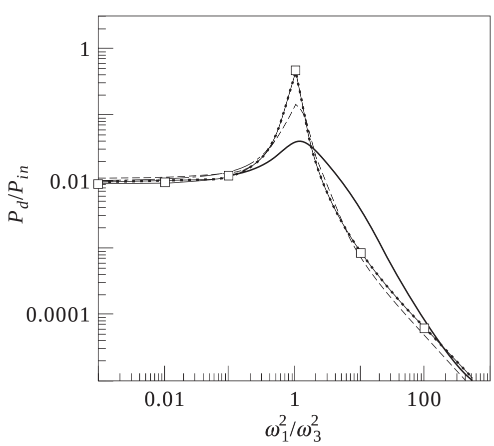

Figure 2. Ratios of direct power flow to input power versus ω_1^2/ω_3^2 for selected m_2. Key: —, $m_2 = 2m_1$; ---, $m_2 = 3m_1$; · · ·, $m_2 = 5m_1$; -●-, $m_2 \gg 10m_1$; -□-, 2 oscillators.

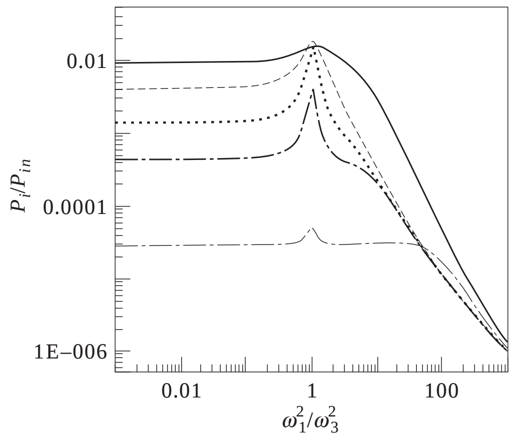

Figure 3. Ratios of indirect power flow to input power versus ω_1^2/ω_3^2 for selected m_2. Key: —, $m_2 = 2m_1$; ---, $m_2 = 3m_1$; · · ·, $m_2 = 5m_1$; ▬ ▪ ▬, $m_2 = 10m_1$; — - —, $m_2 = 100m_1$.

of conservatively or non-conservatively coupled structures, and such influence can be found in Figure 3 and Figure 4 of reference [16].

In this paper, the power flows among three non-conservatively series coupled oscillators are analyzed. The formulae for power flows among three non-conservatively series coupled oscillators are presented. The relationships between power flows and the difference of the stored energies are given. The influence of the parameters of the oscillators and couplings on the power flows and the direct coupling loss factors has been investigated by the numerical method.

Lastly, experimental verifications have also been carried out with a floating raft isolation system. The data obtained corroborate the finding of this paper. The authors hope that this paper would be helpful in the further application of SEA.

2. THE THEORETICAL ANALYSIS

The mechanical model of three non-conservatively series coupled oscillators is shown in Figure 1. The equations of motion of the masses are:

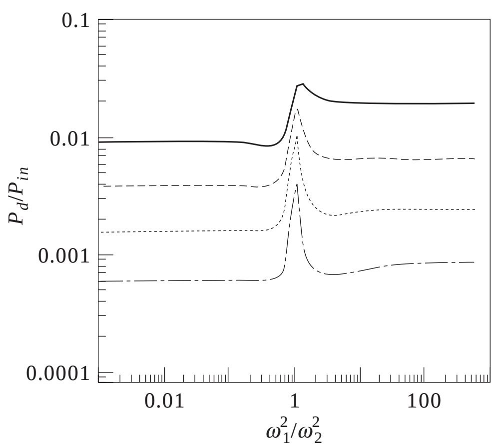

Figure 4. Ratios of direct power flow to input power versus ω_1^2/ω_2^2 for selected m_3. Key: —, $m_3 = 2m_1$; –––, $m_3 = 3m_1$; · · ·, $m_3 = 5m_1$; –·–·–, $m_3 = 10m_1$.

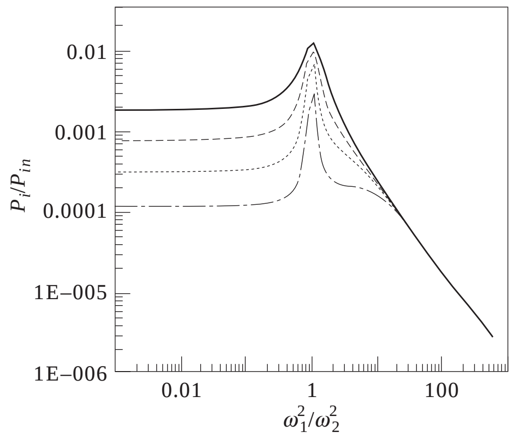

Figure 5. Ratios of indirect power flow to input power versus ω_1^2/ω_2^2 for selected m_3. Key as for Figure 4.

$$\begin{aligned} m_1 \ddot{x}_1 &= -k_1 x_1 - c_1 \dot{x}_1 - k_3 (x_1 - x_3) - c_3 (\dot{x}_1 - \dot{x}_3) + F_1(t), \\ m_2 \ddot{x}_2 &= -k_2 x_2 - c_2 \dot{x}_2 - k_4 (x_2 - x_3) - c_4 (\dot{x}_2 - \dot{x}_3) + F_2(t), \\ m_3 \ddot{x}_3 &= -k_3 (x_3 - x_1) - c_3 (\dot{x}_3 - \dot{x}_1) - k_4 (x_3 - x_2) - c_4 (\dot{x}_3 - \dot{x}_2) - k_5 x_3 \\ &\quad - c_5 \dot{x}_3 + F_3 (t). \end{aligned} \tag{1}$$

Here m_i denotes the mass of oscillator i. k_i, k_4 and k_5 symbolize the stiffness coefficients. c_i, c_4 and c_5 are the damping coefficients, and $F_i(t)$ is the random force supplied to oscillator i ($i = 1, 2, 3$). New parameters are as follows:

$$\omega_1^2 = (k_1 + k_3)/m_1, \qquad \omega_2^2 = (k_2 + k_4)/m_2, \qquad \omega_3^2 = (k_3 + k_4 + k_5)/m_3,$$

$$\Delta_1 = (c_1 + c_3)/m_1, \qquad \Delta_2 = (c_2 + c_4)/m_2, \qquad \Delta_3 = (c_3 + c_4 + c_5)/m_3, \qquad v_1 = k_3/m_1,$$

$$v_2 = k_4/m_2, \qquad v_3 = k_3/m_3, \qquad v_4 = k_4/m_3, \qquad \mu_1 = c_3/m_1, \qquad \mu_2 = c_4/m_2,$$

$$\mu_3 = c_3/m_3, \qquad \mu_4 = c_4/m_3, \qquad f_1 = F_1(t)/m_1, \qquad f_2 = F_2(t)/m_2, \qquad f_3 = F_3(t)/m_3.$$

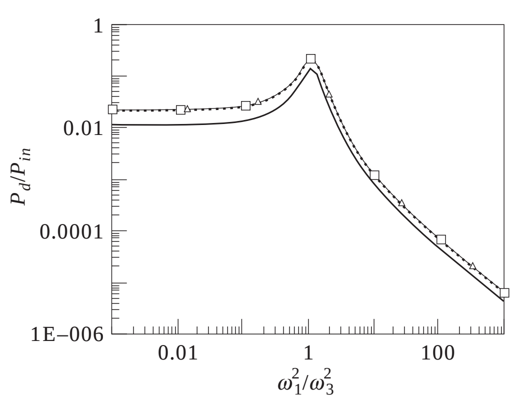

Figure 6. Ratios of direct power flow to input power versus ω_1^2/ω_3^2 for selected c_3. Key: —, $c_3 = c_1$; · · · , $c_3 = 0{\cdot}1c_1$; –□–, $c_3 = 0{\cdot}01c_1$; –△–, $c_3 = 0{\cdot}001c_1$.

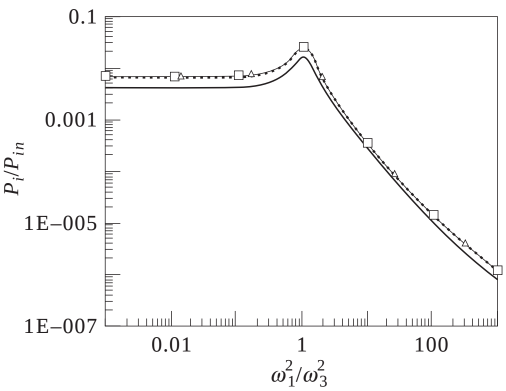

Figure 7. Ratios of indirect power flow to input power versus ω_1^2/ω_3^2 for selected c_3. Key as for Figure 6.

Thus equations (1) may be rewritten as

$$\ddot{x}_1 + \Delta_1 \dot{x}_1 + \omega_1^2 x_1 - \mu_1 \dot{x}_3 - v_1 x_3 = f_1, \qquad \ddot{x}_2 + \Delta_2 \dot{x}_2 + \omega_2^2 x_2 - \mu_2 \dot{x}_3 - v_2 x_3 = f_2,$$

$$\ddot{x}_3 + \Delta_3 \dot{x}_3 + \omega_3^2 x_3 - \mu_3 \dot{x}_1 - v_3 x_1 - \mu_4 \dot{x}_2 - \mu_4 x_2 = f_3. \tag{2}$$

By assuming that the external forces are harmonic with zero averaged value and statistical independence, of which the spectral densities are s_i ($i = 1, 2, 3$) allows the time averaged responses to be obtained by using equations (A1–4) of reference [17] and which are given in Appendix A. Here, as a simplified but general example, suppose that only oscillator 1 is excited by an external force, that is $s_1 \neq 0$, $s_2 = 0$, $s_3 = 0$. Similar to the derivation of reference [17], the total power input to the system can be obtained.

$$P_{in} = (c_1 + c_3)\langle \dot{x}_1^2 \rangle - c_3 \langle \dot{x}_1 \dot{x}_3 \rangle + k_3 \langle x_1 \dot{x}_3 \rangle, \tag{3}$$

and the power flow from oscillator 1 to oscillator 3 (reaching ocillator 3) may be obtained

$$\langle P_{13} \rangle = \beta(\langle E_1 \rangle - \langle E_3 \rangle) + \gamma(\langle E_1 \rangle - \langle E_2 \rangle). \tag{4}$$

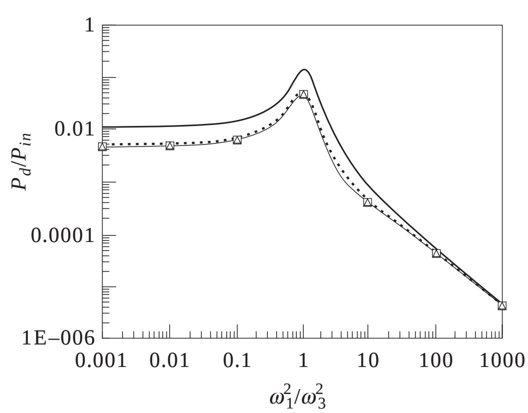

Figure 8. Ratios of direct power flow to input power versus ω_1^2/ω_3^2 for selected c_4. Key: —, $c_4 = c_1$; · · · ·, $c_4 = 0{\cdot}1c_1$; –□–, $c_4 = 0{\cdot}01c_1$; –△–, $c_4 = 0{\cdot}001c_1$.

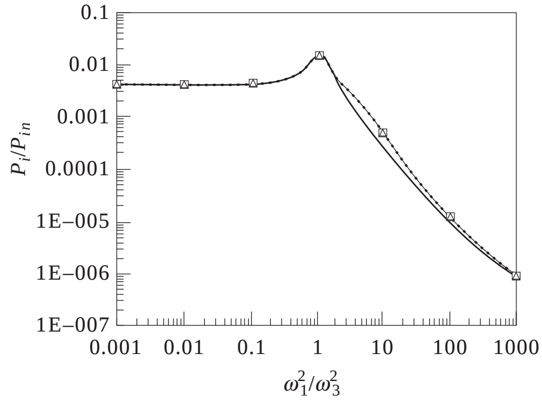

Figure 9. Ratios of indirect power flow to input power versus ω_1^2/ω_3^2 for selected c_4. Key as for Figure 8.

Here

$$\beta = [A - \gamma(A_1 - A_2)]/(A_1 - A_3),$$

$$\gamma = \Delta_1 \{-k_4 [\mu_2 \mu_3^2 \lambda_2 + (\omega_2^2 \mu_3^2 - v_3^2 - \Delta_2)\mu_2 \lambda_3 + (\Delta_2 v_2 v_3^2 - \omega_2^2 v_3^2 \mu_2)\lambda_4] + c_4 [(\mu_3^2 v_2 - \Delta_2 \mu_3^4 \mu_2 - \mu_2^2 \mu_3^2)\lambda_2 + (\Delta_2 \mu_2 v_3^2 + \omega_2^2 \mu_3^2 v_2 - v_2 v_3^2 - \mu_2^2 v_3^2 + \mu_3^2 v_2^2)\lambda_3 + (v_2 \omega_2^2 v_3^2 + v_2^2 v_3^2)\lambda_4]\}/[m_1 \Delta_6 (A_1 - A_2)],$$

$$A = k_3 \sum_{n=1}^{4} b_{11n} \lambda_n + c_3 \sum_{n=1}^{4} b_{31n} \lambda_n + c_3 \sum_{n=1}^{4} b_{71n} \lambda_n, \qquad \langle E_i \rangle = A_i \frac{\pi s_1}{m_1 \Delta_1}, \quad (i = 1, 2, 3),$$

$$A_1 = m_1 \sum_{n=1}^{4} b_{51n} \lambda_n, \qquad A_2 = m_2 \sum_{n=1}^{4} b_{61n} \lambda_n, \qquad A_3 = m_3 \sum_{n=1}^{4} b_{71n} \lambda_n.$$

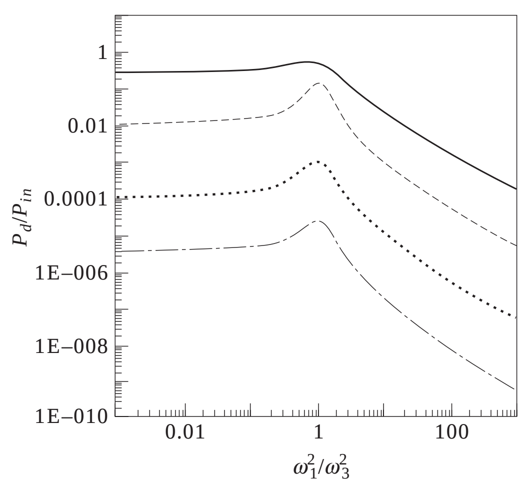

Figure 10. Ratios of direct power flow to input power versus ω_1^2/ω_3^2 for selected k_3. Key: —, $k_3 = k_1$; - - -, $k_3 = 0{\cdot}1k_1$; · · ·, $k_3 = 0{\cdot}01k_1$; –·–·, $k_3 = 0{\cdot}001k_1$.

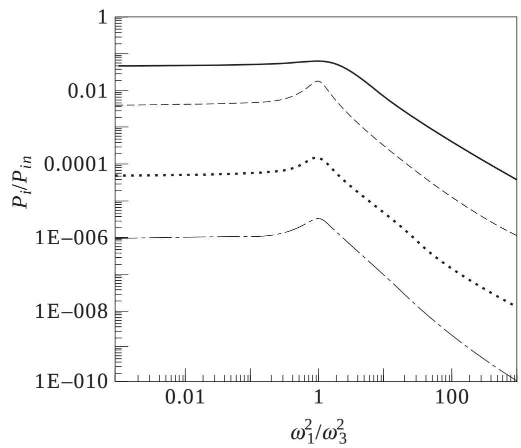

Figure 11. Ratios of indirect power flow to input power versus ω_1^2/ω_3^2 for selected k_3. Key as for Figure 10.

The power flow from oscillator 2 to oscillator 3 (reaching oscillator 3) can be obtained from

$$\langle P_{23} \rangle = \beta'(\langle E_2 - E_3 \rangle) + \gamma'(\langle E_2 \rangle - \langle E_1 \rangle) \tag{5}$$

$$\beta' = [B - \gamma'(A_2 - A_1)]/(A_2 - A_3),$$

$$\gamma' = \Delta_1 \{-k_4 [\mu_2 \mu_3^2 \lambda_2 + ((\omega_2^2 \mu_3^2 - v_3^2)\mu_2 - \Delta_2 \mu_2)\lambda_3 + (\Delta_2 v_2 v_3^2 - \omega_2^2 v_3^2 \mu_2)\lambda_4] + c_4 [(\mu_3^2 v_2 - \Delta_2 \mu_3^4 \mu_2)\lambda_2 + (\Delta_2 \mu_2 v_3^2 + v_2 (\omega_2^2 \mu_3^2 - v_3^2))\lambda_3 + v_2 \omega_2^2 v_3^2 \lambda_4] - (\mu_2^2 \mu_3^2 \lambda_2 + \mu_2^2 v_3^2 - \mu_3^2 v_2^2)\lambda_3 - v_2^2 v_3^2 \lambda_4\}/[m_1 \Delta_6 (A_2 - A_1)],$$

$$B = k_4 \sum_{n=1}^{4} b_{21n} \lambda_n + c_3 \sum_{n=1}^{4} b_{41n} \lambda_n + c_3 \sum_{n=1}^{4} b_{71n} \lambda_n .$$

Here $\pi s_1 / m_1 \Delta_1$ is called the time averaged total energy of the uncoupled oscillator 1 when subjected to external force. The coefficients b_{ijk}, λ_n and the time averaged response are given in Appendix A. The first items in equation (4) and (5) are the direct power flows, and the

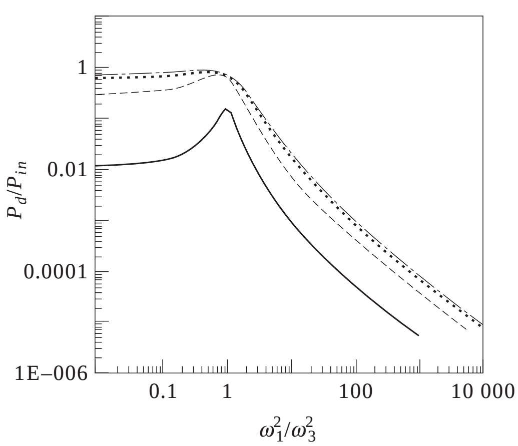

Figure 12. Ratios of direct power flow to input power versus ω_1^2/ω_3^2 for selected k_4. Key: —, $k_4 = k_1$; --, $k_4 = 0{\cdot}1k_1$; · · ·, $k_4 = 0{\cdot}01k_1$; –·–·, $k_4 = 0{\cdot}001k_1$.

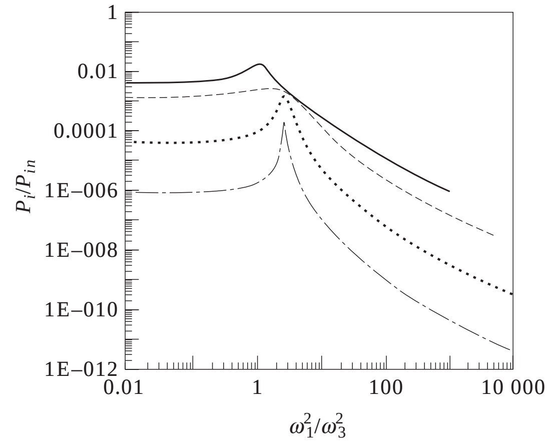

Figure 13. Ratios of indirect power flow to input power versus ω_1^2/ω_3^2 for selected k_4. Key as for Figure 12.

second items are the indirect power flows. For an arbitrary system of three non-conservatively series coupled oscillators, the equation $\gamma = \gamma'$ is no longer available. Therefore, the amplitudes of the indirect power flows in different directions are not the same.

Equations (4) and (5) are very revealing. They indicate that:

(1) The power flows among three non-conservatively series coupled oscillators include the direct and indirect power flows. The direct and indirect power flows are proportional to the differences of the stored energies in the directly and indirectly coupled oscillators respectively. The proportional coefficients are not only related to the parameters of the directly coupled oscillators and couplings, but also related to the parameters of the indirectly coupled oscillators and couplings.

(2) The reciprocity, which exists in three conservatively series coupled oscillators, is untenable.

(3) When the coupling damping coefficients $c_3 = c_4 = c_5 = 0$, it can be verified that equation (4) is the same as equation (12a) of reference [17]. Additionally, when m_2 tends to reach infinity, equation (4) reduces to the expression of the power flows with coservative coupling (see equation (17) of reference [11] and equation (15) of reference [12]). If the coupling damping coefficients $c_3 = c_4 = 0$ and m_2 tends to reach infinity, equation (4) is the

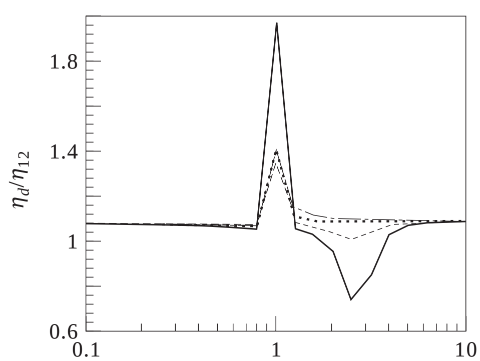

Figure 14. The influence of c_2 on η_d/η_{12} for conservative coupling ($c_3 = c_4 = 0$). Key: —, $c_2 = c_1$; –––, $c_2 = 0{\cdot}1c_1$; · · · , $c_2 = 0{\cdot}01c_1$; –·–·, $c_2 = 0{\cdot}001c_1$.

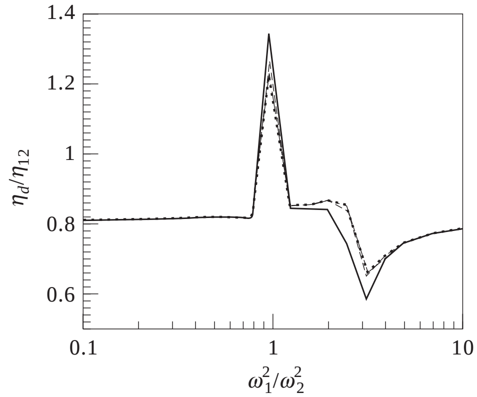

Figure 15. The influence of c_2 on η_d/η_{12} for non-conservative coupling ($c_3 = c_4 = c_1$). Key as for Figure 14.

same as the expression of the power flows between two conservatively coupled oscillators (see equation (3.3) of reference [1]).

3. THE NUMERICAL RESULTS

The power flows of three non-conservatively series coupled oscillators are complex functions of the parameters of the oscillators and couplings. Therefore, it is difficult to obtain the relationships between the power flows and the parameters of the oscillators and couplings theoretically. Meanwhile, in order to understand the general characteristics of power flows penetratively, it is necessary to carry out some further analysis on the power flows of three non-conservatively series coupled oscillators. The main purpose of the numerical analysis is to research in greater depth the power flows of three non-conservatively series coupled oscillators, especially the relationships between the indirect power flows and the parameters of the oscillators. Thus, knowledge of the power flows, especially the indirect power flows, can be available.

In reference [17], the relationship between the power flows of the three conservatively series coupled oscillators and the parameters of the oscillators and couplings have been investigated in detail. Here the emphasis has been laid on the study of the influence of the coupling stiffness and the coupling damping on the direct and indirect power flows of three non-conservatively series coupled oscillators. As a general example, suppose that only

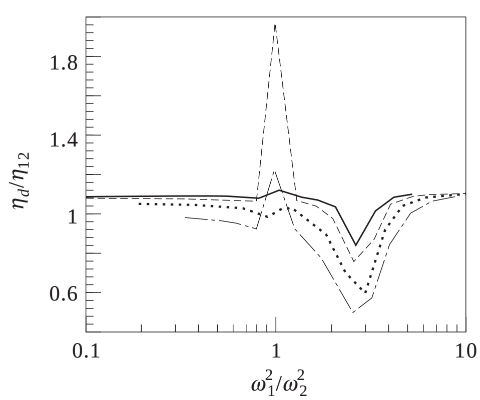

Figure 16. The influence of k_2 on η_d/η_{12} for conservative coupling ($c_3 = c_4 = 0$). Key: —, $k_2=2k_1$; ---, $k_2 = k_1$; $\cdots$, $k_2 = 0{\cdot}5k_1$; –·– $k_2 = 0{\cdot}25k_1$.

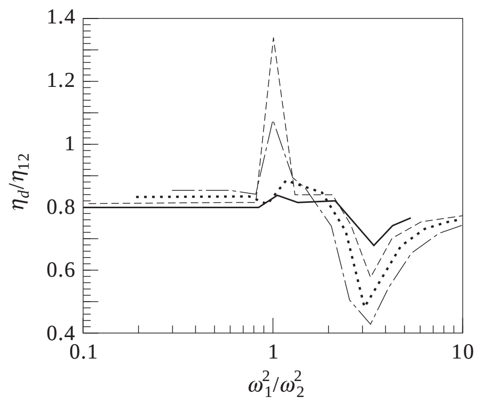

Figure 17. The influence of k_2 on η_d/η_{12} for non-conservative coupling ($c_3 = c_4 = c_1$). Key as for Figure 16.

oscillator 1 is excited by the external force and the basic parameters used here are, $m_1 = 1{\cdot}5$ kg, $m_2 = m_3 = 3m_1$, $c_1 = 0{\cdot}15\ \mathrm{Nsm}^{-1}$, $c_2 = c_3 = c_4 = c_5 = c_1$, $k_1 = 10^4$ N/m, $k_2 = k_5 = k_1$, $k_3 = k_4 = 0{\cdot}1\ k_1$, $s_1 = 1{\cdot}0$. There are supplements in each figure if the parameters have been changed. The ratios of the direct power flow P_d to the input power flow P_{in} versus the ratios of ω_1^2 to ω_3^2 (that is m_3/m_1) for selected m_2 are shown in Figure 2. The power flow between two non-conservatively coupled oscillators is shown in Figure 2 as well [11, 12]. It indicates that the direct power flows of three non-conservatively series coupled oscillators tends to reach the power flows of two non-conservatively coupled oscillators when $m_2 > 5m_1$. From the numerical analysis, conclusions are as follows.

(1) Figures 2, 3 show that, for fixed m_2, the direct and indirect power flows vary slowly when $\omega_1^2/\omega_3^2 < 1$. The direct and indirect power flows decrease obviously with the increase in m_2 when $\omega_1^2/\omega_3^2 > 1$, and reach their extreme values when $\omega_1^2/\omega_3^2 = 1$. These show that the direct and indirect power flows are related to resonant energy transmission between the directly coupled oscillators, and there are similar results in Figures 4–13. The bigger m_2 is, the less the ratio of the indirect power flow to the direct power flow, and the same results are shown in Figure 3 of reference [17]. If ω_1^2/ω_3^2 is constant, the influence of m_2 on the power flows is small for non-resonant frequency and great for resonant frequency, and the bigger m_2 is, (which means the natural frequency of oscillator 2 is far away from the natural frequencies of oscillator 1 and 3), the higher the peak of the direct power flow

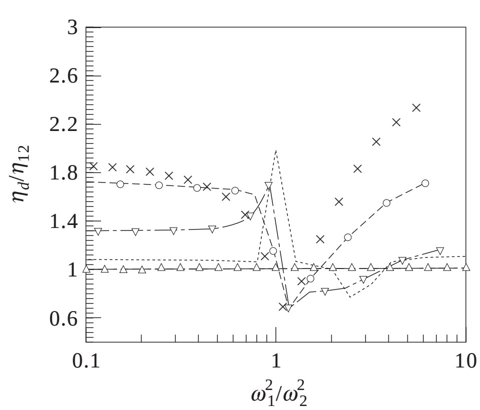

Figure 18. The influence of k_4 on η_d/η_{12} for conservative coupling ($c_3 = c_4 = 0$). Key: $\times$, $k_4 = k_1$; ‒‒○‒‒, $k_4 = 0{\cdot}8k_1$; ▽‒‒‒▽, $k_4 = 0{\cdot}5k_1$; · · ·, $k_4 = 0{\cdot}1k_1$; ‒·‒·, $k_4 = 0{\cdot}01k_1$; ———, $k_4 = 0{\cdot}001k_1$; △‒‒△, $k_4 = 0{\cdot}0001k_1$.

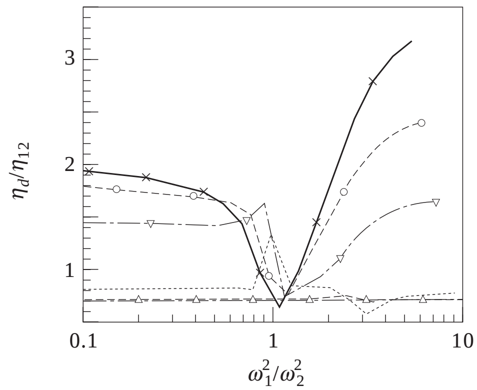

Figure 19. The influence of k_4 on η_d/η_{12} for non-conservative coupling ($c_3 = c_4 = c_1$). Key as for Figure 18.

becomes. The curve tends to reach the curve of the power flow of two non-conservatively coupled oscillators when m_2 tends to reach infinity. However, the indirect power flow decreases rapidly as m_2 increases (which could be seen from Figures 4, 5 as well).

(2) When m_3 is constant (see Figures 4, 5), the direct and indirect power flows vary slowly as the change in ω_1^2/ω_2^2 when $\omega_1^2/\omega_2^2 < 1$, and reach their maxima at the resonant frequency. P_d/P_{in} decreases rapidly with the increase in m_2, and then reaches a new steady value. However, P_i/P_{in} decreases linearly and rapidly when $\omega_1^2/\omega_2^2 > 1$. Similar results obtained when m_2 is fixed show that the direct and indirect power flows reach their extreme values when $\omega_1^2/\omega_2^2 = 1$, and the direct and indirect power flows are also related to resonant energy transmission between the indirectly coupled oscillators. Additionally, the indirect power flow is less than the direct power flow when m_3 is big, and the same result are shown in Figure 9 of reference [17]. For fixed ω_1^2/ω_2^2, the direct power is approximately inversely proportional to m_3. When $\omega_1^2/\omega_2^2 > 1$, the indirect power flow is influenced slightly by m_3. When $\omega_1^2/\omega_2^2 < 1$, the indirect power flow is approximately inversely proportional to m_3.

(3) The influence of the coupling dampings on the direct and indirect power flows is relatively small (see Figures 6–9). The direct and indirect power flows increase a little with a decrease in c_3. The influence of c_4 on the direct and indirect power flows are small.

(4) The direct and indirect power flows are approximately proportional to k_3 (see Figures 10, 11). It is shown that the coupling between the oscillators decreases as k_3

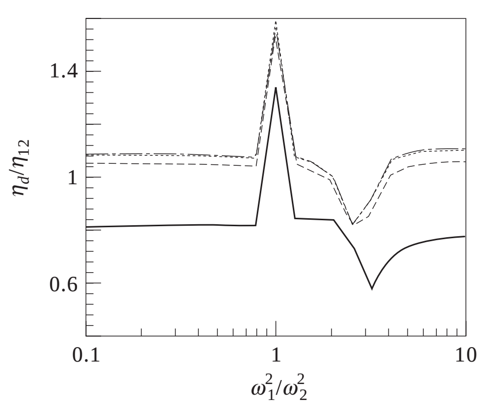

Figure 20. The influence of c_4 on η_d/η_{12} for non-conservative coupling ($c_3 = c_1$). Key: —, $c_4 = c_1$; – – –, $c_4 = 0{\cdot}1c_1$; · · ·, $c_4 = 0{\cdot}01c_1$; — · —, $c_4 = 0{\cdot}001c_1$.

decreases. Therefore, the indirect power flow cannot be ignored in the condition of strong coupling. The indirect power flows are of the same level as the direct power flows even if k_3 is very small.

(5) The indirect power flows decrease as k_4 decreases, while the direct power flows increase as k_4 decreases. The indirect power flows are less than the direct power flows when k_4 is very small and thus may be ignored.

4. THE INFLUENCES OF THE INDIRECTLY COUPLED OSCILLATORS ON THE DIRECT COUPLING LOSS FACTORS

According to the definition of coupling loss factor [2], the direct coupling loss factor from oscillator 1 to oscillator 3 is given

$$\eta_d = \beta/\omega_1, \tag{6}$$

where β is expressed in equation (4).

Reference [7] gives the coupling loss factor of two coupled oscillators

$$\eta_{12} = \gamma_{12} \tag{7}$$

where γ_{12} is expressed in reference [7]. η_d and η_{12} are complex functions of the parameters of the oscillators and couplings. Therefore, it is difficult to analyze the difference between η_d and η_{12} theoretically. In order to study the influence of the parameters of the oscillators and couplings on the direct coupling loss factor, numerical calculations are carried out. The basic parameters used here are $m_1 = 1{\cdot}5$ kg, $m_2 = m_1$, $m_3 = 3m_1$, $c_1 = 0{\cdot}15$ Nsm^{-1}, $c_2 = c_3 = c_4 = c_5 = c_1$, $k_1 = 10^4$ N/m, $k_2 = k_5 = k_1$, $k_3 = k_4 = 0{\cdot}1\ k_1$, $s_1 = 1{\cdot}0$. There are supplements in each figure if the parameters have been changed.

The ratios of η_d to η_{12} are shown in Figures 14–21 for selected values of the parameters c_2, k_2, c_4 and k_4 respectively. For graphs with abscissa ω_1^2/ω_2^2, m_2 is varied. The influence of m_2 on the direct coupling loss factor is negligible when ω_1^2/ω_2^2 is small. As ω_1^2/ω_2^2 tends to reach 1, η_d/η_{12} increases obviously. When $\omega_1^2/\omega_2^2 = 1$, η_d/η_{12} shows a peak, which illustrates that η_d/η_{12} reaches an extreme value when the indirectly coupled oscillators are in resonance. As ω_1^2/ω_2^2 increases, η_d/η_{12} decreases. The difference between η_d and η_{12} is maximum when the directly or indirectly coupled oscillators are in resonance.

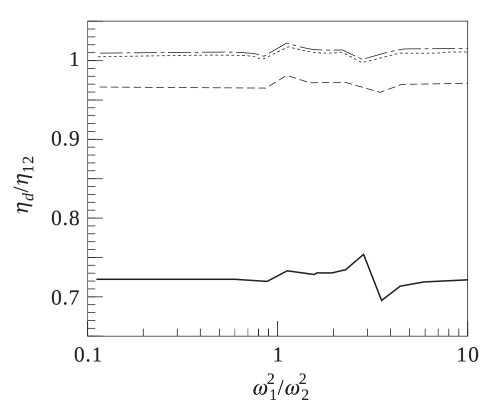

Figure 21. The influence of c_4 on η_d/η_{12} for non-conservative coupling ($c_3 = c_1, k_4 = 0{\cdot}01k_1$). Key as for Figure 20.

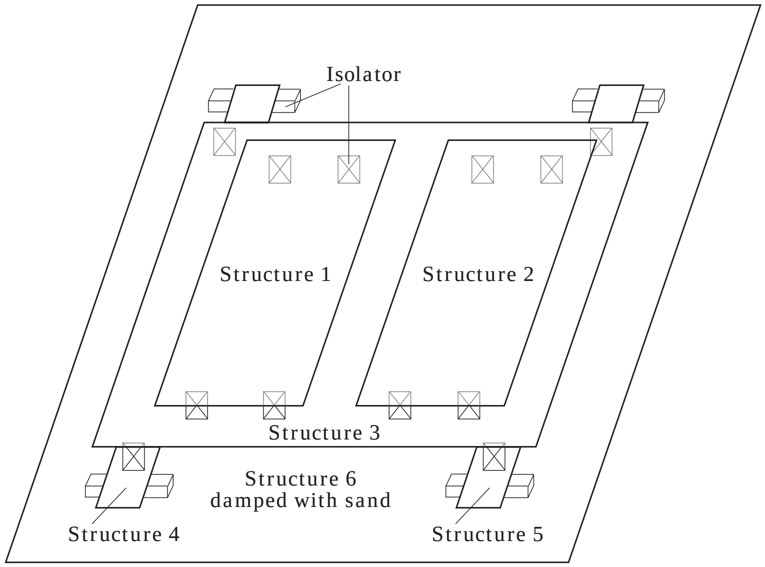

Figure 22. Schematic diagram of the floating raft isolation system.

The influences of the idividual parameters are:

(1) it is shown from Figure 14 that the damping of oscillator 2 affects η_d/η_{12} greatly when the oscillators are conservatively coupled. However, this influence is very little when the oscillators are non-conservatively coupled (see Figure 15).

(2) Figures 16 and 17 show the influence of k_2 on the direct coupling loss factor. The relationship between η_d/η_{12} and k_2 is complicated. It is shown that η_d/η_{12} increases with the decrease in k_2 when $\omega_1^2/\omega_2^2 < 1$, and η_d/η_{12} is maximum when the parameters of oscillator 1 and 2 are the same. Since the stiffness of indirectly coupled oscillator affects the direct coupling loss factor greatly, it is necessary to make a further investigation.

(3) Figures 18 and 19 show the influence of k_4 on the direct coupling loss factor. When k_4 is small, η_d/η_{12} changes slightly with ω_1^2/ω_2^2. η_d/η_{12} tends to reach 1 for conservative couplings and tends to reach 0·7 for non-conservative couplings. When the directly or indirectly coupled oscillators are in resonance, the difference between η_d and η_{12} becomes more and more obvious as k_4 increases. Furthermore, when k_4 is large (for example $k_4 > 0{\cdot}1k_1$), η_d/η_{12} changes greatly with ω_1^2/ω_2^2, and tends to become greater than 1 for both conservative and non-conservative couplings.

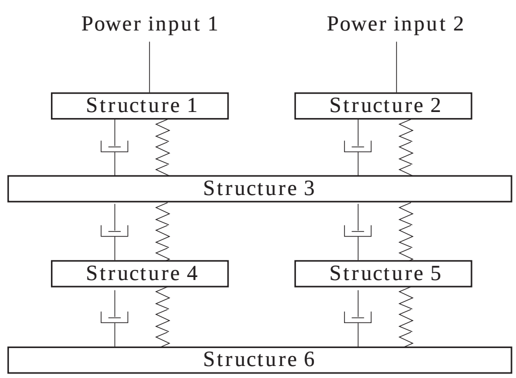

Figure 23. The mechanical model of the floating raft isolation system used for SEA.

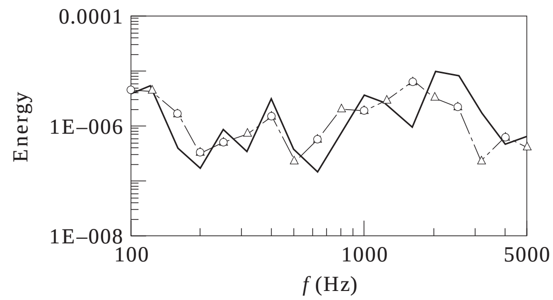

Figure 24. The energy of the structure 1. Key: △, power injection method; —, experiment; ○, CSEA; · – · –, SEA.

(4) Figure 20 shows the influence of c_4 on η_d/η_{12}, which increases with the decrease of c_4. η_d/η_{12} is maximum when the directly or indirectly coupled oscillators are in resonance.

(5) Figure 21 shows the influence of c_4 on η_d/η_{12} when the coupling stiffness $k_4 = 0{\cdot}01k_1$. When k_4 is small, η_d/η_{12} increases with the decrease of c_4 and tends to reach 1. However, η_d/η_{12} varies slightly with the change of ω_1^2/ω_2^2 even though the directly or indirectly coupled oscillators are in resonance.

5. ENERGY BALANCE EQUATIONS OF SEA

The energy balance equations for a N substructures system in the classical Statistical Energy Analysis (CSEA) have been modified by using the equations of the power flows among three series coupled oscillators [18], and are given

$$\begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_N \end{bmatrix} = \omega \begin{bmatrix} \varphi_{11} & -\varphi_{21} & \cdots & -\varphi_{N1} & \bar{E}_1 \\ -\varphi_{12} & \varphi_{22} & \cdots & -\varphi_{N2} & \bar{E}_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\varphi_{1N} & -\varphi_{2N} & \cdots & \varphi_{NN} & \bar{E}_N \end{bmatrix}$$

or written as

$$[\mathbf{P}] = \omega[\mathbf{\Phi}]\,[\mathbf{E}] \tag{8}$$

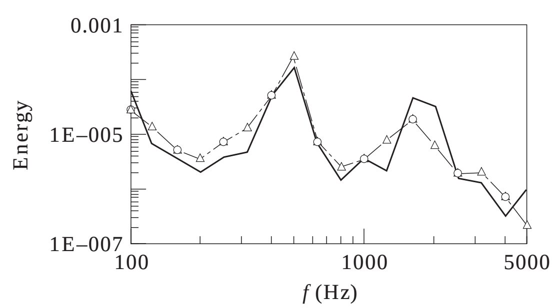

Figure 25. The energy of the structure 2. Key as for Figure 24.

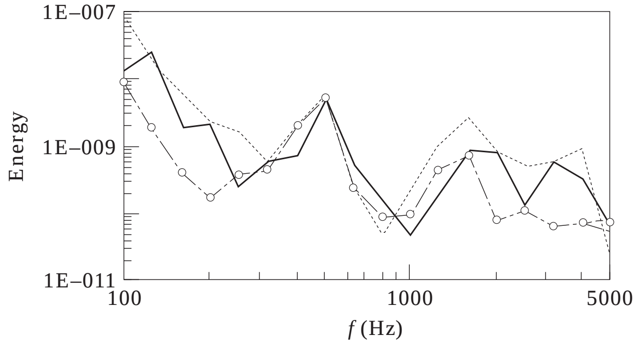

Figure 26. The energy of the structure 3. Key: · · · , power injection method; —, experiment; ○, CSEA; – · – · –, SEA.

Here

$$\left\{\begin{aligned} \varphi_{ii} &= \eta_i + \sum_{\substack{k=1 \\ k \neq i}}^{k+N} (\eta_{ik} + \eta'_{ik}), && (i = 1, 2, 3, \ldots, N), \\ \varphi_{ij} &= \eta_{ij} + \eta'_{ij}, && (j = 1, 2, 3, \ldots, N, j \neq i), \end{aligned}\right\} \tag{9}$$

Here η_i is called the effective internal loss factors [15]. η_{ik} and η'_{ik} are called the direct and indirect coupling loss factors respectively. The indirect coupling loss factors could also be expressed by using the direct coupling loss factors and the internal loss factors. Equation (8) is also called the modified energy balance equations of the SEA and suitable for the multiple, strongly coupled systems, because the influence of the strong and non-conservative couplings on the power flows have been taken into account here. Moreover, the effective internal loss factors derived by Sheng [22] are used in these equations. When the influence of the indirectly coupled substructures on the direct coupling loss factors are considered, the power injection method could be used to determine the effective loss factors and coupling loss factors [15].

Obviously, with the coupling damping loss factors being taken into account, the energy balance equations of SEA could also be modified as the form given by Beshara and Keane [21].

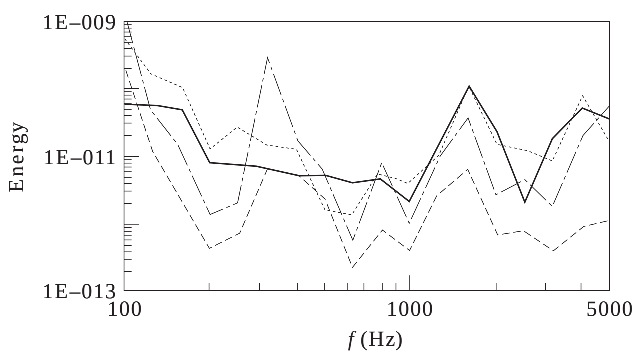

Figure 27. The energy of the structure 4. Key as for Figure 26 except –––, CSEA.

6. EXPERIMENTAL VERIFICATION

A floating raft isolation system mounted on a damped plate is used (as shown in Figure 22) to verify the influence of the non-conservative coupling and the indirectly coupled oscillators on the power flows of the multiple, non-conservatively coupled system. The floating raft isolation system consists of three plates and two beams. Two of the plates are used to simulate two machines, and the other plate is used to simulate the raft. The beams act as the foundations and the damped plate serves as the shell of a ship. Because of the use of isolators, the connections between the components become non-conservative. Thus the whole system is a multiple, non-conservatively coupled system.

A SEA model of the floating raft isolation system is shown in Figure 23. The third octave band data are obtained when substructures 1 and 2 are excited. The power injection method [14] has also been used here.

Figures 24–29 show the vibrational energies of substructures 1–6 respectively. The test frequency range is from 100–5 kHz. The data of the third octave band are given. The dotted line denotes the estimated vibrational energy by means of the power injection method. The solid line denotes the measured vibrational energy averaged by time and space. The dashed and the dash dot line denote the theoretically predicted results with or without considering the indirect power flows and the influence of the indirectly coupled structures of multiple, non-conservatively coupled system respectively.

Figures 24 and 25 show the estimated results of SEA, CSEA [15] and the power injection method (see Figures 24 and 25), which indicates that the results are identical in the directly excited substructures. In Figure 26, the results estimated by SEA and CSEA are similar except for a few points. For the indirectly excited substructures, because the indirect power flows have not been taken into account in the CSEA method, the results obtained by the SEA method are almost always greater than the results by CSEA (see Figures 27–29). These figures show that when the influence of the indirectly coupling structures on the loss factors is considered, the estimated data agree well with the test data. However, for the indirectly excited structures, there is great difference between the experimental results and the theoretical results when the influence of the indirectly coupled structures are not considered, and the more the substructures between the directly and indirectly excited structures are, the greater the difference is (see Figures 27–29).

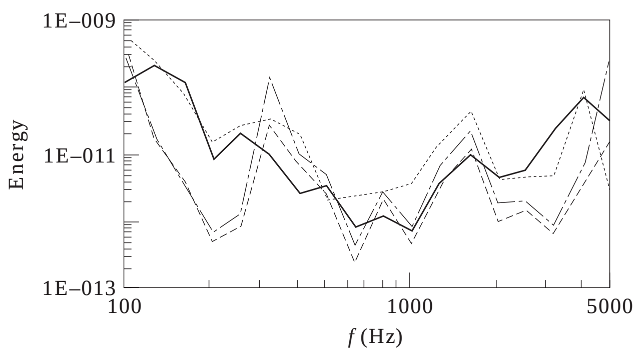

Figure 28. The energy of the structure 5. Key as for Figure 27.

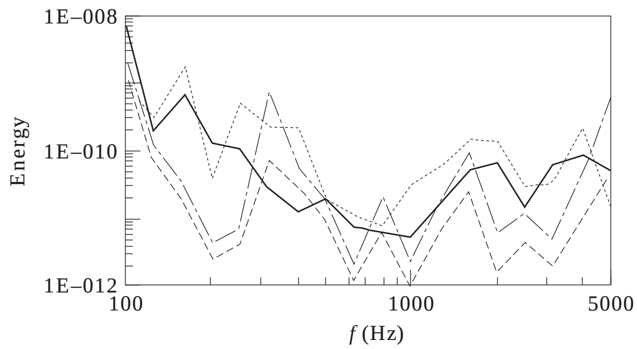

Figure 29. The energy of the structure 6. Key as for Figure 27.

Because of the errors caused by the measured input power and vibrational energies, there is some difference between the estimated results and the experiments results. However, such a difference is acceptable in SEA.

7. CONCLUSIONS

According to the analysis of the characteristics of three non-conservatively series coupled oscillators and the influence of the indirectly coupled oscillator on the direct coupling loss factor it is possible to draw the following conclusions.

The theories of the power flows of both three conservatively series coupled oscillators [17] and two non-conservatively coupled oscillators [1] are special examples of the theory of the power flows of three non-conservatively series coupled oscillators [11, 12]. The reciprocity of the indirect power flows among three non-conservatively series coupled oscillators is untenable.

The theory of the power flows of three non-conservatively series coupled oscillators provides the theoretical basis for the modification of the energy balance equations of non-conservatively coupled systems, and it is the basis of Statistical Energy Analysis under general conditions.

The indirect power flow reaches its maximum when the indirectly coupled oscillators are in resonance, which illustrates that the indirect power flow corresponds to resonant transmission between indirectly coupled oscillators.

If the parameters of oscillators are fixed, the indirect power flows are mainly dependent on the coupling stiffness coefficients k_3 and k_4, and slightly affected by the coupling damping coefficients c_3 and c_4. The indirect power flows cannot be ignored in the condition of strong coupling.

The difference between η_d and η_{12} is maximum when the directly or indirectly coupled oscillators are in resonance. For conservative coupling, the influence of the indirectly coupled oscillator on the direct coupling loss factor could be neglected if the indirect coupling stiffness is small. The influence must be taken into account when the indirect coupling stiffness is large. For non-conservative coupling, the influence of the indirectly coupled oscillator on the direct coupling loss factor can be ignored provided that both the indirect coupling stiffness and damping are small.

The stiffness of the indirectly coupled oscillator has a great influence on the direct coupling loss factor. When the indirect power flows and the influence of the indirectly coupled structures of multiple, non-conservatively coupled systems are taken into account in the SEA model, better agreements between the predicted result and the experimental result are obtained. It is necessary to modify the direct coupling loss factors when energy balance equations of SEA are modified.

ACKNOWLEDGMENT

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REFERENCES

1. R. H. Lyon and G. Maidanik 1962 *Journal of the Acoustic Society of America* **34**, 1265–1269. Power flow between linearly coupled oscillators.
2. R. H. Lyon 1975 *Statistical Energy Analysis of Dynamical Systems*: *Theory and Applications*. MIT Press.
3. A. J. Price and M. J. Crocker 1970 *Journal of the Acoustic Society of America* **47**, 633–693. Sound transmission through double panels using statistical energy analysis.
4. V. R. Miller and L. L. Faulkner 1983 *American Society of Mechanical Engineers*, *Journal of Vibration*, *Acoustics*, **105**, 512–518. Prediction of aircraft internal noise using statistial energy analysis.
5. J. M. Cuschieri and J. C. Sun 1994 *Journal of Sound and Vibration* **170**, 203–214. Use of statistical energy analysis for rotating machinery, part III: experimental verification with machine on foundation.
6. J. Zhang and C.-X. Gu 1993 *Acta Acustica* **18**, 186–195. Power flow and energy distribution in conservatively and non-conservatively coupled oscillators under correlative excitations.
7. J. C. Sun, C. Wang and Z. H. Sun 1995 *Acta Acustica* **20**, 33–41. Coupling loss factors between indirectly coupled machinery structures.
8. A. J. Keane 1992 *Proceedings of the Royal Society of London* **A436**, 537–568. Energy flow between arbitrary configurations of conservatively coupled multi-modal elastic subsystems.
9. K. Heron 1994 *Philosophical Trnasactions of the Royal Society of London* **346(A)**, 501–510. Advanced statistical energy analysis.
10. S. Finnveden 1994 *Royal Institute of Technology*, *Stockholm*, *ISRN KTH/FKT/TR*-94/22-*SE*. Ensemble averaged vibration energy flows in a three element structure.
11. J. C. Sun, N. Lalor and E. J. Richards 1987 *Journal of Sound and Vibration* **112**, 321–343. Power flow and energy balance of non-conservatively coupled oscillators.
12. F. J. Fahy and D. Y. Yao 1987 *Journal of Sound and Vibration* **114**, 1–11. Power flow between non-conservatively coupled oscillators.
13. A. J. Keane and W. G. Price 1987 *Journal of Sound and Vibration* **117**, 363–386. Statistical energy analysis for strongly coupled systems.
14. F. J. Fahy 1987 *Statistical Energy Analysis in Noise and Vibration*. Chichester: John Wiley.
15. J. M. Cuschieri and J. C. Sun 1994 *Journal of Sound and Vibration* **170**, 181–190. Use of statistical energy analysis for rotating machinery, part I: determination of dissipation and coupling loss factors using energy ratios.
16. J. M. Cuschieri and J. C. Sun 1994 *Journal of Sound and Vibration* **170**, 190–201. Use of statistical energy analysis for rotating machinery, part II: coupling loss factors between indirectly coupled substructures.
17. J. C. Sun, C. Wang and Z. H. Sun 1996 *Journal of Sound and Vibration* **189**, 215–229. Power flow between three series coupled oscillators.
18. J. C. Sun, C. Wang and Z. H. Sun 1996 *Chinese Journal of Acoustics* **15**, 1–7. Modification of energy balance equation in statistical energy analysis.
19. M. Z. Liu 1987 *Ph.D. Thesis*, *Nanjing Aeronautical Engineering Institute*. The study of statistical energy analysis for non-conservatively coupled systems.
20. J. Zhang and C. X. Gu 1991 *Journal of Applied Mechanics* **8**, 11–19. The influence of the coupling damping on the energy distribution and the power flow in non-conservatively coupled oscillators.
21. M. Beshara and A. J. Keane 1996 *Journal of Sound and Vibration* **198**, 95–122. Statistical energy analysis of multiple, non-conservatively coupled system.
22. M. P. Sheng, M. Q. Wang and J. C. Sun 1997 *Acta Acustica* **22**, 555–561. The effective internal loss factors of the non-conservatively coupled system.

APPENDIX A: THE TIME-AVERAGED RESPONSES

When the three non-conservatively series coupled oscillators are excited by external forces with zero-average value, which are statistically independent and of spectral densities s_i, $i = 1, 2, 3$, the time-averaged responses can be obtained by using equations (A1–4) in reference [17].

$$\langle x_3 \dot{x}_1 \rangle = \sum_{i=1}^{3} \left(\frac{\pi s_i}{m_i^2 \Delta_6} \right) \sum_{n=1}^{4} b_{1in} \lambda_n, \tag{A1}$$

where

$$\Delta_6 = a_6^2 a_1^3 + 3a_6 a_5 a_4 a_1 a_0 - 2a_6 a_5 a_2 a_1^2 - a_6 a_4 a_3 a_1^2 - a_6 a_3^3 a_0 + a_6 a_3^2 a_2 a_1 + a_5^3 a_0^2 - 2a_5^2 a_4 a_1 a_0 - a_5^2 a_3 a_2 a_0 + a_5^2 a_2^2 a_1 + a_5 a_4 a_3^2 a_0 - a_5 a_4 a_3 a_2 a_1$$

$$\lambda_0 = -a_6 a_3 a_1 a_0 + a_6 a_2 a_1^2 - a_5^2 a_0^2 + 2a_5 a_4 a_1 a_0 + a_5 a_3 a_2 a_0 - a_5 a_2^2 a_1 - a_4^2 a_1^2 - a_4 a_3^2 a_0 + a_4 a_3 a_2 a_1,$$

$$\lambda_1 = -a_5 a_1 a_0 + a_4 a_1^2 + a_3^2 a_0 - a_3 a_2 a_1, \qquad \lambda_2 = -a_6 a_1^2 - a_5 a_3 a_0 + a_5 a_2 a_1,$$

$$\lambda_3 = a_6 a_3 a_1 + a_5^2 a_0 - a_5 a_4 a_1, \qquad \lambda_4 = a_6 a_5 a_1 - a_6 a_3^2 - a_5^2 a_2 + a_5 a_4 a_3,$$

$$\lambda_5 = a_6^2 a_1^2 + a_6 a_5 a_3 a_0 - 2a_6 a_5 a_2 a_1 - 2a_6 a_4 a_3 a_1 + a_6 a_3^2 a_2 - a_5^2 a_4 a_0 + a_5^2 a_2^2 + a_5 a_4^2 a_1 - a_5 a_4 a_3 a_2,$$

$$a_0 = 1, \qquad a_1 = \Delta_1 + \Delta_2 + \Delta_3$$

$$a_2 = \omega_1^2 + \omega_2^2 + \omega_3^2 + \Delta_1 \Delta_2 + \Delta_2 \Delta_3 + \Delta_1 \Delta_3 - \mu_1 \mu_3 - \mu_2 \mu_4,$$

$$a_3 = \Delta_1 (\omega_2^2 + \omega_3^2) + \Delta_2 (\omega_1^2 + \omega_3^2) + \Delta_3 (\omega_1^2 + \omega_2^2) + \Delta_1 \Delta_2 \Delta_3 - \mu_1 v_3 - \mu_3 v_1 - \mu_2 v_4 - \mu_4 v_2 - \Delta_2 \mu_1 \mu_3 - \Delta_1 \mu_2 \mu_4,$$

$$a_4 = \omega_1^2 (\omega_2^2 + \Delta_2 \Delta_3) + \omega_2^2 (\omega_3^2 + \Delta_1 \Delta_3) + \omega_3^2 (\omega_1^2 + \Delta_1 \Delta_2) - v_1 v_3 - v_2 v_4 - \Delta_2 (\mu_1 v_3 + \mu_3 v_1) - \Delta_1 (\mu_2 v_4 + \mu_4 v_2) - \omega_1^2 \mu_2 \mu_4 - \omega_2^2 \mu_1 \mu_3,$$

$$a_5 = \Delta_1 (\omega_2^2 + \omega_3^2 - v_2 v_4) + \Delta_2 (\omega_1^2 + \omega_3^2 - v_1 v_3) + \Delta_3 \omega_1^2 \omega_2^2 - \omega_2^2 (\mu_1 v_3 + \mu_3 v_1) - \omega_1^2 (\mu_2 v_4 - \mu_4 v_2)$$

$$a_6 = \omega_1^2 \omega_2^2 \omega_3^2 - \omega_2^2 v_1 v_3 - \omega_1^2 v_2 v_4, \qquad b_{111} = -\mu_3,$$

$$b_{112} = \mu_3 (\Delta_2^2 - 2\omega_2^2 - \omega_3^2 + \mu_2 \mu_4) + \Delta_3 v_3,$$

$$b_{113} = \mu_3 [\omega_3^2 (\Delta_2^2 - 2\omega_2^2) - \omega_2^4] - \Delta_3 v_3 (\Delta_2^2 - 2\omega_2^2) - \mu_2 v_4 v_3 + v_2 (\mu_4 v_3 - \mu_3 v_4) + \mu_2 \mu_3 \mu_4 \omega_2^2 + \Delta_2 \mu_2 (\mu_4 v_3 - \mu_3 v_4) - \Delta_2 \mu_3 \mu_4 v_2,$$

$$b_{114} = (v_2 v_4 - \omega_2^2 \omega_3^2)(\Delta_2 v_3 + \mu_3 \omega_2^2) + v_3 \omega_2^2 (\Delta_2 \omega_3^2 + \Delta_3 \omega_2^2 - \mu_2 v_4 - \mu_4 v_2),$$

$$b_{121} = 0, \qquad b_{122} = -\mu_1 \mu_4^2, \qquad b_{123} = \Delta_1 \mu_4^2 v_1 - \mu_1 (\mu_4^2 \mu_1^2 - v_4^2),$$

$$b_{124} = v_1^2 (\mu_1 \omega_1^2 - \Delta_1 v_1), \qquad b_{131} = \mu_1, \qquad b_{132} = \mu_1 (2\omega_2^2 - \Delta_2^2) + \mu_1 \omega_1^2 - \Delta_1 v_1,$$

$$b_{133} = \mu_1 \omega_2^4 + (\Delta_1 v_1 - \mu_1 \omega_1^2)(\Delta_2^2 - 2\omega_2^2), \qquad b_{134} = \omega_2^4 (\mu_1 \omega_1^2 - \Delta_1 v_1).$$

$$\langle x_2 \dot{x}_3 \rangle = \sum_{i=1}^{3} \left(\frac{\pi s_i}{m_i^2 \Delta_6} \right) \sum_{n=1}^{4} b_{2in} \lambda_n, \tag{A2}$$

where

$$b_{211} = 0, \qquad b_{212} = \mu_2\,\mu_3^2, \qquad b_{213} = \mu_2\,(\mu_3^2\,\omega_2^2 - v_3^2) - \Delta_2\,\mu_3^2\,v_2, \qquad b_{214} = v_3^2\,(\Delta_2\,v_2 - \mu_2\,\omega_2^2),$$

$$b_{221} = \mu_4, \qquad b_{222} = \mu_4\,(\Delta_1^2 - 2\omega_1^2 - \omega_3^2 - \mu_1\,\mu_3) - \Delta_3\,v_4,$$

$$b_{223} = -\mu_4\,[\omega_3^2\,(\Delta_1^2 - 2\omega_1^2) - \omega_1^4] + \Delta_3\,v_4\,(\Delta_1^2 - 2\omega_1^2) + \mu_1\,v_3\,v_4 + v_1\,(\mu_3\,v_4 - \mu_4\,v_3)$$
$$+ \mu_1\,\mu_3\,\mu_4\,\omega_1^2 + \Delta_1\,\mu_1\,(\mu_3\,v_4 - \mu_4\,v_3),$$

$$b_{224} = \mu_4\,\omega_1^4\,\omega_3^2 - v_4\,\Delta_3\,\omega_1^4 + \omega_1^2\,[\mu_1\,v_3\,v_4 + v_1\,(\mu_3\,v_4 - \mu_4\,v_3)] - \Delta_1\,v_1\,v_3\,v_4,$$

$$b_{231} = -\mu_2, \qquad b_{232} = -\mu_2\,(2\omega_1^2 - \Delta_1^2 + \omega_2^2) - \Delta_2\,v_2,$$

$$b_{233} = \mu_2\,[\omega_2^2\,(\Delta_1^2 - 2\omega_1^2) - \omega_1^4] - \Delta_2\,v_2\,(\Delta_1^2 - 2\omega_1^2), \qquad b_{234} = -\omega_1^4\,(\mu_2\,\omega_2^2 - \Delta_2\,v_2).$$

$$\langle \dot{x}_1\,\dot{x}_3 \rangle = \sum_{i=1}^{3} \left(\frac{\pi S_i}{m_i^2\,\Delta_6}\right) \sum_{n=1}^{4} b_{3in}\,\lambda_n, \tag{A3}$$

where

$$b_{311} = \mu_3\,\Delta_3 - v_3, \qquad b_{314} = \omega_2^2\,(v_2\,v_3\,v_4 - v_3\,\omega_2^2\,\omega_3^2),$$

$$b_{312} = (v_3 - \mu_3\,\Delta_3)\,(\Delta_2^2 - 2\omega_2^2) - v_3\,\omega_3^2 - (\mu_2\,\mu_3\,v_4 + \mu_3\,\mu_4\,v_2 - \mu_2\,\mu_4\,v_3) + \Delta_2\,\mu_2\,\mu_3\,\mu_4,$$

$$b_{313} = v_3\,\mu_3\,(\Delta_2^2 - 2\omega_2^2) + \omega_2^4\,(\mu_3\,\Delta_3 - v_3) - \omega_2^2\,(\mu_2\,\mu_3\,v_4 - \mu_3\,\mu_4\,v_2 + \mu_2\,\mu_4\,v_3)$$
$$+ \Delta_2\,(\mu_3\,v_2\,v_4 - \mu_4\,v_2\,v_3 - \mu_2\,v_3\,v_4) + v_2\,v_3\,v_4,$$

$$b_{321} = 0, \qquad b_{322} = \mu_4^2\,(v_1 - \Delta\mu_1\,\mu_4^2),$$

$$b_{323} = v_1\,(\omega_1^2\,\mu_4^2 - v_4^2) + \Delta_1\,\mu_1\,v_4^2,$$

$$b_{324} = -v_1\,v_4^2\,\omega_1^2, \qquad b_{331} = -v_1 + \Delta_1\,\mu_1,$$

$$b_{332} = -v_1\,(\omega_1^2 - \Delta_2^2 + 2\omega_2^2) - \Delta_1\,\mu_1\,(\Delta_2^2 - 2\omega_2^2),$$

$$b_{333} = v_1\,\omega_1^2\,(\Delta_2^2 - 2\omega_2^2) + \omega_2^4\,(\Delta_1\,\mu_1 - v_1), \qquad b_{334} = -v_1\,\omega_1^2\,\omega_2^4.$$

$$\langle \dot{x}_2\,\dot{x}_3 \rangle = \sum_{i=1}^{3} \left(\frac{\pi S_i}{m_i^2\,\Delta_6}\right) \sum_{n=1}^{4} b_{4in}\,\lambda_n, \tag{A4}$$

where

$$b_{411} = 0, \qquad b_{412} = \mu_3^2\,(v_2 - \Delta_2\,\mu_2), \qquad b_{413} = v_2\,(\omega_2^2\,\mu_3^2 - v_3^2) + \Delta_2\,\mu_2\,v_3^2$$

$$b_{414} = -v_2\,v_3^2\,\omega_2^2, \qquad b_{421} = \mu_2\,\Delta_2 - v_2,$$

$$b_{422} = (v_4 - \mu_4\,\Delta_3)\,(\Delta_1^2 - 2\omega_1^2) - v_4\,\omega_3^2 - (\mu_2\,\mu_3\,v_4 + \mu_3\,\mu_4\,v_2 - \mu_2\,\mu_4\,v_3) + \Delta_1\,\mu_1\,\mu_3\,\mu_4,$$

$$b_{423} = v_4\,\mu_4\,(\Delta_1^2 - 2\omega_1^2) + \omega_1^4\,(\mu_4\,\Delta_3 - v_4) - \omega_1^2\,(\mu_1\,\mu_4\,v_3 - \mu_3\,\mu_4\,v_1$$
$$+ \mu_1\,\mu_3\,v_4) + \Delta_1\,(\mu_4\,v_1\,v_3 - \mu_3\,v_1\,v_4 - \mu_1\,v_3\,v_4) + v_1\,v_3\,v_4,$$

$$b_{424} = -\omega_1^2\,(v_1\,v_3\,v_4 - v_4\,\omega_1^2\,\omega_4^2), \qquad b_{431} = -v_1 + \Delta_1\,\mu_1,$$

$$b_{432} = -v_1\,(\omega_1^2 - \Delta_2^2 + 2\omega_2^2) - \Delta_1\,\mu_1\,(\Delta_2^2 - 2\omega_2^2),$$

$$b_{433} = v_1\,\omega_1^2\,(\Delta_2^2 - 2\omega_2^2) + \omega_2^4\,(\Delta_1\,\mu_1 - v_1), \qquad b_{434} = -v_1\,\omega_1^2\,\omega_2^4.$$

$$\langle \dot{x}_1^2 \rangle = \sum_{i=1}^{3} \left(\frac{\pi S_i}{m_i^2\,\Delta_6}\right) \sum_{n=0}^{4} b_{5in}\,\lambda_n, \tag{A5}$$

where

$$b_{510} = -1, \qquad b_{511} = 2[\mu_2 \mu_3 - (\omega_2^2 + \omega_3^2 + \Delta_2 \Delta_3) + (\Delta_2 + \Delta_3)^2],$$

$$b_{512} = 2(\Delta_2 + \Delta_3)(\omega_2^2 \Delta_3 + \omega_3^2 \Delta_2 - \mu_2 v_3 - \mu_3 v_2)$$
$$-[\mu_2 \mu_3 - (\omega_2^2 + \omega_3^2 + \Delta_2 \Delta_3)]^2 - 2(\omega_2^2 \omega_3^2 - v_2 v_3),$$

$$b_{513} = 2(\omega_2^2 \omega_3^2 - v_2 v_3)(\mu_2 \mu_3 - \omega_2^2 - \omega_3^2 - \Delta_2 \Delta_3) + (\omega_2^2 \Delta_3 + \omega_3^2 \Delta_2 - \mu_2 v_3 - \mu_3 v_2)^2$$

$$b_{514} = -(\omega_2^2 \omega_3^2 - v_2 v_3)^2, \qquad b_{520} = b_{521} = 0, \qquad b_{522} = -\mu_1^2 \mu_4^2,$$

$$b_{523} = \mu_1^2 v_4^2 + \mu_4^2 v_1^2, \qquad b_{524} = -v_1^2 v_4^2, \qquad b_{530} = 0, \qquad b_{531} = \mu_1^2,$$

$$b_{532} = -[v_1^2 + \mu_1^2 (\Delta_2^2 - 2\omega_2^2)], \qquad b_{533} = v_1^2 (\Delta_2^2 - \omega_2^2), \qquad b_{534} = -\omega_2^4 v_1^2.$$

$$\langle \dot{x}_2^2 \rangle = \sum_{i=1}^{3} \left(\frac{\pi s_i}{m_i^2 \Delta_6} \right) \sum_{n=0}^{4} b_{6in} \lambda_n, \tag{A6}$$

where

$$b_{610} = b_{611} = 0, \qquad b_{612} = -\mu_2^2 \mu_3^2, \qquad b_{613} = \mu_2^2 v_3^2 + \mu_3^2 v_2^2, \qquad b_{614} = -v_2^2 v_3^2,$$

$$b_{620} = -1, \qquad b_{621} = 2[\mu_1 \mu_3 - (\omega_1^2 + \omega_3^2 + \Delta_1 \Delta_3) + (\Delta_1 + \Delta_3)^2,$$

$$b_{622} = 2(\Delta_1 + \Delta_3)(\omega_1^2 + \omega_3^2 + \Delta_1 \Delta_3 - \mu_1 v_3 - \mu_3 v_1)$$
$$-[\mu_1 \mu_3 - (\omega_1^2 + \omega_3^2 + \Delta_1 \Delta_3)]^2 - 2(\omega_1^2 \omega_3^2 - v_1 v_3),$$

$$b_{623} = 2(\omega_1^2 \omega_3^2 - v_1 v_3)(\mu_1 \mu_3 - \omega_1^2 - \omega_3^2 - \Delta_1 \Delta_3) + (\omega_1^2 \Delta_3 - \omega_3^2 \Delta_2 - \mu_1 v_3 - \mu_3 v_1)^2,$$

$$b_{624} = -(\omega_1^2 \omega_3^2 - v_1 v_3)^2, \qquad b_{630} = 0, \qquad b_{631} = \mu_2^2,$$

$$b_{632} = -[v_2^2 + \mu_2^2 (\Delta_1^2 - 2\omega_1^2)], \qquad b_{633} = v_2^2 (\Delta_1^2 - \omega_1^2), \qquad b_{634} = -\omega_1^4 v_2^2.$$

$$\langle \dot{x}_3^2 \rangle = \sum_{i=1}^{3} \left(\frac{\pi s_i}{m_i^2 \Delta_6} \right) \sum_{n=0}^{4} b_{7in} \lambda_n, \tag{A7}$$

where

$$b_{710} = 0, \qquad b_{711} = \mu_3^2, \qquad b_{712} = -[v_3^2 + \mu_3^2 (\Delta_2^2 - 2\omega_2^2)],$$

$$b_{713} = v_3^2 (\Delta_2^2 - \omega_2^2) + \mu_3^2 \omega_2^4, \qquad b_{714} = -\omega_2^4 v_3^2, \qquad b_{720} = 0, \qquad b_{721} = \mu_4^2,$$

$$b_{722} = -[v_4^2 + \mu_4^2 (\Delta_1^2 - 2\omega_1^2)], \qquad b_{723} = v_4^2 (\Delta_1^2 - \omega_1^2), \qquad b_{724} = -\omega_1^4 v_4^2, \qquad b_{730} = -1,$$

$$b_{731} = (\Delta_1 + \Delta_2)^2 - 2(\omega_1^2 + \omega_2^2 + \Delta_1 \Delta_2),$$

$$b_{732} = 2(\Delta_1 + \Delta_2)(\omega_1^2 \Delta_2 + \omega_2^2 \Delta_1) - (\omega_1^2 + \omega_2^2 + \Delta_1 \Delta_2)^2 - 2\omega_2^2 \omega_3^2,$$

$$b_{733} = -2\omega_1^2 \omega_2^2 (\omega_1^2 + \omega_2^2 + \Delta_1 \Delta_2) + (\omega_1^2 \Delta_2 + \omega_1^2 \Delta_1)^2, \qquad b_{734} = -\omega_2^4 \omega_3^4.$$