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NON-LINEAR VIBRATIONS OF A SLIGHTLY CURVED BEAM RESTING ON A NON-LINEAR ELASTIC FOUNDATION

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In this study, non-linear vibrations of slightly curved beams are investigated. The curvature is taken as an arbitrary function of the spatial variable. The initial displacement is not due to buckling of the beam, but is due to the geometry of the beam itself. The ends of the curved beam are on immovable simple supports and the beam is resting on a non-linear elastic foundation. The immovable end supports result in the extension of the beam during the vibration and hence introduces further non-linear terms to the equations of motion. The integro-differential equations of motion are solved analytically by means of direct application of the method of multiple scales (a perturbation method). The amplitude and phase modulation equations are derived for the case of primary resonances. Both free and forced vibrations with damping are investigated. Effect of non-linear elastic foundation as well as the effect of curvature on the vibrations of the beam are examined. It is found that the effect of curvature is of softening type. For sufficiently high values of the coefficients, the elastic foundation may suppress the softening behaviour resulting in a hardening behaviour of the non-linearity.

1. INTRODUCTION

The vibrations of beams under immovable end conditions have been studied in detail. The immovable ends cause extensions axially in the beam which introduces integral type cubic non-linearities into the equations of motion. For straight beams, the pioneering work is due to Woinowsky-Krieger [1] who investigated the free oscillations of a bar having an initial tensile force. Srinivasan [2] applied the Ritz–Galerkin technique to analyse the large amplitude of free oscillation of beams and plates with stretching. In addition to stretching, Wrenn and Mayers [3] included the effects of transverse shear and rotary inertia. Nayfeh and Mook [4] reviewed the relevant work up to 1979. Pakdemirli and Nayfeh [5] investigated a beam–mass–spring system where the non-linearities arise due to stretching and non-linear spring supporting the mass. Recently Özkaya *et al*. [6] investigated a concentrated mass on a Euler–Bernoulli beam which was supported by immovable end conditions leading to stretching during the vibrations.

The effect of stretching has also been included in the vibrations of slightly curved beams or shallow arcs. Among the many contributions in this area, a few of them are mentioned here: Rehfield [7] derived the equations of motion of a shallow arch with an arbitrary rise function and studied the free vibrations approximately. A moderately thick clamped beam with a sinusoidal rise function is studied by Singh and Ali [8]. Finally, Yamaki and Mori [9] analysed a clamped buckled beam by considering the first three symmetric modes and used a combination of Galerkin and Harmonic Balance methods.

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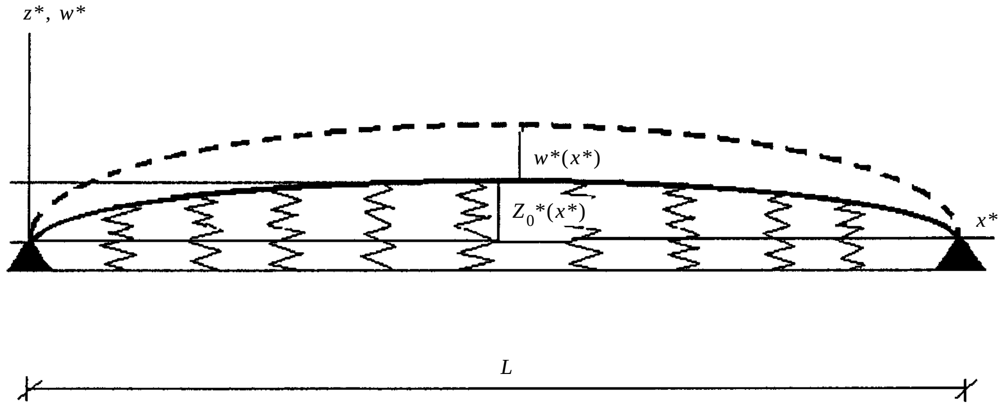

Figure 1. A simply supported slightly curved beam resting on a non-linear elastic foundation.

This study, is concerned with a simply supported slightly curved beam resting on an elastic foundation with cubic non-linearities. The equations of motion due to Rehfield [7] are modified by adding damping, forcing and non-linear elastic foundation terms. The initial curvature of the beam is not due to buckling; rather, the beam is considered to be fabricated with slight curvature. The equations of motion are solved by the method of multiple scales, a perturbation technique. With the curvature function assumed to be of order 1 and the amplitude of vibrations to be of order ϵ, the amplitude and phase modulation equations are derived. Free vibrations and forced vibrations with damping are investigated in detail. The effects of the elastic foundation, axial stretching and curvature on the vibrations of the beam are analysed. It is found that the non-linearities due to curvature are of softening type whereas those of elastic foundation are of hardening type.

2. EQUATIONS OF MOTION

The system considered is a simply supported slightly curved beam resting on a non-linear elastic foundation as shown in Figure 1. For the beam shown, A is the cross-section of the beam, I is the moment of inertia of the beam cross-section with respect to the neutral axis, ρ is the density, w^* is the transverse displacement, $Z_0^*(x^*)$ is the arbitrary initial rise function, k_1 and k_2 are the linear and non-linear coefficients of the elastic foundation, respectively, and L is the projected length of the beam (a list of notation is given in the Appendix). Following Rehfield [7], one can write the equations of motion as

$$\rho A\ddot{w}^* + EIw^{*iv} + \mu^*\dot{w}^* + k_1 w^* + k_2 w^{*3} = \frac{EA}{L}\int_0^L (Z_0^{*\prime} w^{*\prime} + \tfrac{1}{2} w^{*\prime 2})\,\mathrm{d}x^*(w^{*\prime\prime} + Z_0^{*\prime\prime}) + F^*\cos\Omega^* t^* \quad (1)$$

Here damping, forcing and non-linear elastic foundation terms are included. x^* is the spatial variable along the projected length and t^* denotes time. $()'$ represents derivatives with respect to the spatial variable and $(\cdot)$ represents derivatives with respect to time.

For convenience, the equations are made dimensionless by defining

$$x = x^*/L, \qquad t = (r/L^2)\sqrt{E/\rho}\,t^*, \qquad w = w^*/r, \qquad Z_0 = Z_0^*/r, \quad (2)$$

where r is the radius of gyration of the beam cross-section. Substituting dimensionless quantities (2) into equation (1), one obtains the dimensionless form of the equation:

$$\ddot{w} + w^{iv} + 2\bar{\mu}\dot{w} + \alpha_1 w + \alpha_2 w^3 = \bar{F}\cos\Omega t + \int_0^1 (Z_0' w' + \tfrac{1}{2} w'^2)\,\mathrm{d}x(Z_0'' + w''). \qquad (3)$$

Here new dimensionless parameters are defined as follows:

$$2\bar{\mu} = \mu^* L^2/A\sqrt{\rho E}, \qquad \bar{F} = F^* L^4/EIr, \qquad \Omega = \Omega^*(L^2/r)\sqrt{\rho/E},$$
$$\alpha_1 = k_1 L^4/EI, \quad \alpha_2 = k_2 L^4/EA. \qquad (4)$$

The boundary conditions for the problem are

$$w(0, t) = w''(0, t) = w(1, t) = w''(1, t) = 0. \qquad (5)$$

3. ANALYTICAL SOLUTION

To seek approximate analytical solutions of equation (3) subject to boundary conditions (5), the method of multiple scales (a perturbation technique) [10] is used. This method is applied directly to the partial differential system (direct-perturbation method). The common method of discretizing the equations first and then applying perturbations yields less accurate results for finite mode truncations and higher order perturbation schemes [11–18]. When the eigenvalues are not orthogonal, the direct-perturbation method is still applicable. In contrast, a transformation to another form for the equations is necessary for the discretization perturbation method [19]. Solutions are assumed to be of the form

$$w(x, t, \epsilon) = \epsilon w_1(x, T_0, T_1, T_2) + \epsilon^2 w_2(T_0, T_1, T_2) + \epsilon^3 w_3(x, T_0, T_1, T_2) + \ldots, \qquad (6)$$

where ϵ is a small parameter indicating that the amplitudes of vibrations are small (weakly non-linear system) and $T_0 = t$, $T_1 = \epsilon t$, and $T_2 = \epsilon^2 t$ are the usual fast and slow time scales in the multiple scales method. The primary resonance case is considered and it is further assumed that $Z_0(x)$ is or order one: that is,

$$\bar{F} = \epsilon^3 F, \qquad \bar{\mu} = \epsilon^2 \mu, \qquad Z_0 \sim O(1). \qquad (7)$$

Note that, excitation amplitude and damping are reordered so that their effects balance the cubic non-linearities. Derivatives with respect to time are written as follows:

$$\mathrm{d}/\mathrm{d}t = D_0 + \epsilon D_1 + \epsilon^2 D_2 + \ldots, \qquad \mathrm{d}^2/\mathrm{d}t^2 = D_0^2 + 2\epsilon D_0 D_1 + \epsilon^2(D_1^2 + 2D_0 D_2) + \ldots \qquad (8)$$

In these equations $D_n = \partial/\partial T_n$. Substituting expressions (6)–(8) into equation (3) and separating each order of ϵ, one obtains the following:

order ϵ,

$$D_0^2 w_1 + w_1^{iv} + \alpha_1 w_1 - Z_0'' \int_0^1 Z_0' w_1' \,\mathrm{d}x = 0; \qquad (9)$$

order ϵ^2,

$$D_0^2 w_2 + w_2^{iv} + \alpha_1 w_2 - Z_0'' \int_0^1 Z_0' w_2' \,\mathrm{d}x = \tfrac{1}{2} Z_0'' \int_0^1 w_1'^2 \,\mathrm{d}x + w_1'' \int_0^1 Z_0' w_1' \,\mathrm{d}x - 2D_0 D_1 w_1; \quad (10)$$

order ϵ^3,

$$D_0^2 w_3 + w_3^{iv} + \alpha_1 w_3 - Z_0'' \int_0^1 Z_0' w_3' \,\mathrm{d}x = w_1'' \int_0^1 (Z_0' w_2' + \tfrac{1}{2} w_1'^2) \,\mathrm{d}x + w_2'' \int_0^1 Z_0' w_1' \,\mathrm{d}x$$

$$+ Z_0'' \int_0^1 w_1' w_2' \,\mathrm{d}x + F \cos \Omega T_0 - 2D_0 D_1 w_2 - 2\mu D_0 w_1 - (2D_0 D_2 + D_1^2) w_1 - \alpha_2 w_1^3. \tag{11}$$

At order ϵ, the solution may be represented by

$$w_1(x, T_0, T_1, T_2) = \{A(T_1, T_2)\,\mathrm{e}^{\mathrm{i}\omega T_0} + cc\} Y(x), \tag{12}$$

where *cc* denotes the complex conjugates of the preceeding terms. The mode shapes satisfy the following differential system:

$$Y^{iv} - \beta^4 Y - Z_0'' \int_0^1 Z_0' Y' \,\mathrm{d}x = 0, \qquad Y(0) = Y''(0) = Y(1) = Y''(1) = 0. \tag{13, 14}$$

Here β^4 is defined to be

$$\beta^4 = \omega^2 - \alpha_1. \tag{15}$$

Defining

$$b = \int_0^1 Z_0' Y' \,\mathrm{d}x \tag{16}$$

one has

$$Y^{iv} - \beta^4 Y - b Z_0'' = 0. \tag{17}$$

By choosing a sinusoidal curvature function

$$Z_0 = \sin \pi x \tag{18}$$

the solutions can be obtained for two different cases. If $b = 0$, the solutions are

$$Y = C \sin n\pi x, \qquad \beta = n\pi, \qquad n = 2, 3, 4, \ldots . \tag{19}$$

If $b \neq 0$ then the solution is

$$Y = C \sin \pi x, \qquad \beta = \sqrt[4]{3/2}\,\pi. \tag{20}$$

From the solvability condition at order ϵ^2 (see details of finding solvability conditions in reference [10]) one obtains

$$D_1 A = 0, \qquad A = A(T_2). \tag{21}$$

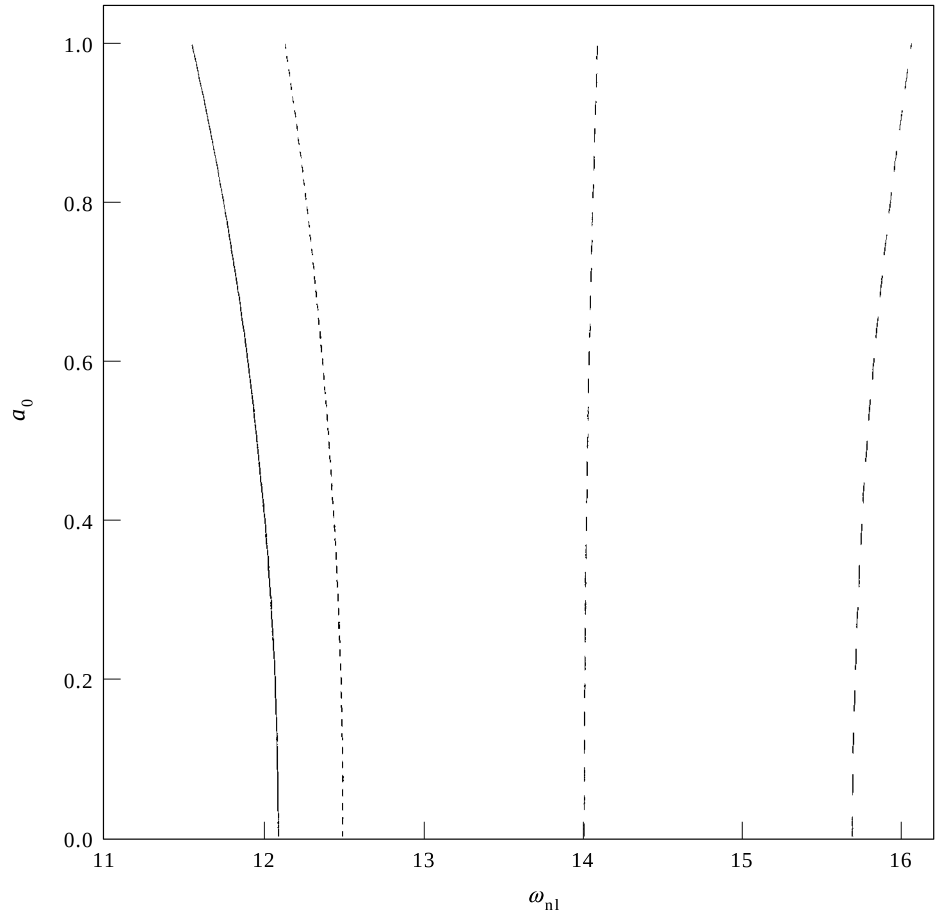

Figure 2. Non-linear frequency versus amplitude for the first mode. $\alpha_2 = 10$: $\alpha_1 = 0$ (—); $\alpha_1 = 10$ (- - -); $\alpha_1 = 50$ (– – –); $\alpha_1 = 100$ (— — —).

A solution can be written at this order of the form

$$w_2 = (A^2\,e^{2i\omega T_0} + cc)\phi_1(x) + 2A\bar{A}\phi_2(x). \tag{22}$$

If one normalizes the eigenfunctions at order ϵ by requiring $\int_0^1 Y^2\,dx = 1$, one has

$$Y(x) = \sqrt{2}\sin n\pi x. \tag{23}$$

Substituting equation (22) into equation (10) yields

$$\phi_1^{iv} - (4\omega^2 - \alpha_1)\phi_1 - Z_0'' \int_0^1 Z_0'\,\phi_1'\,dx = \tfrac{1}{2} Z_0'' \int_0^1 Y'^2\,dx + Y'' \int_0^1 Z_0'\,Y'\,dx,$$

$$\phi_2^{iv} + \alpha_1\,\phi_2 - Z_0'' \int_0^1 Z_0'\,\phi_2'\,dx = \tfrac{1}{2} Z_0'' \int_0^1 Y'^2\,dx + Y'' \int_0^1 Z_0'\,Y'\,dx,$$

$$\phi_i(0) = 0, \qquad \phi_i(1) = 0, \qquad \phi_i''(0) = 0, \qquad \phi_i''(1) = 0, \qquad i = 1, 2. \tag{24}$$

For the case $b = 0 (n \neq 1)$ the mode shapes at this order are

$$\phi_1 = \frac{n^2\pi^4}{(8n^4 - 3)\pi^4 + 6\alpha_1} \sin \pi x, \qquad \phi_2 = -\frac{n^2\pi^4}{3\pi^4 + 2\alpha_1} \sin \pi x, \qquad n = 2, 3, \ldots, \tag{25}$$

and for case $b \neq 0 (n = 1)$

$$\phi_1 = \frac{\pi^4}{3\pi^4 + 2\alpha_1} \sin \pi x, \qquad \phi_2 = -\frac{3\pi^4}{3\pi^4 + 2\alpha_1} \sin \pi x, \qquad n = 1. \tag{26}$$

The solution at order ϵ^3 is written as

$$w_3 (x, T_0, T_2) = \varphi(x, T_2)\, e^{i\omega T_0} + W(x, T_0, T_2) + cc. \tag{27}$$

The excitation frequency is taken as

$$\Omega = \omega + \epsilon^2 \sigma. \tag{28}$$

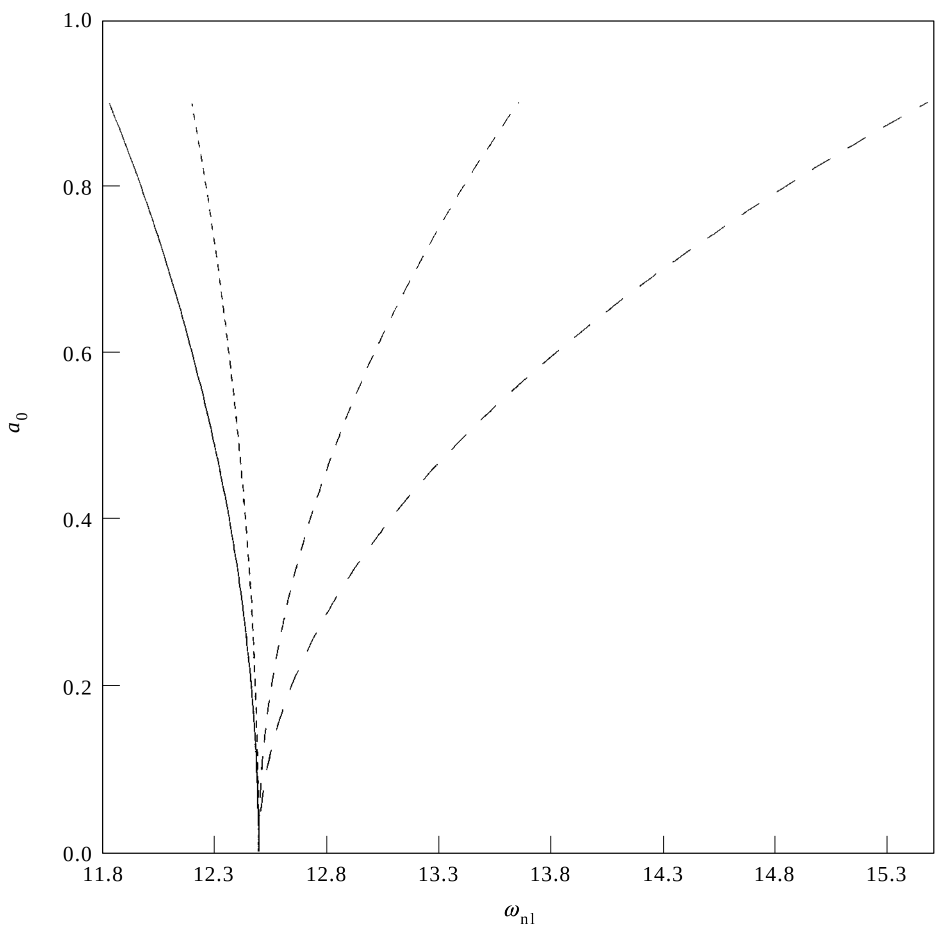

Figure 3. Non-linear frequency versus amplitude for the first mode. $\alpha_1 = 10$: $\alpha_2 = 0$ (—); $\alpha_2 = 10$ (- - -); $\alpha_2 = 50$ (– – –); $\alpha_2 = 100$ (— — —).

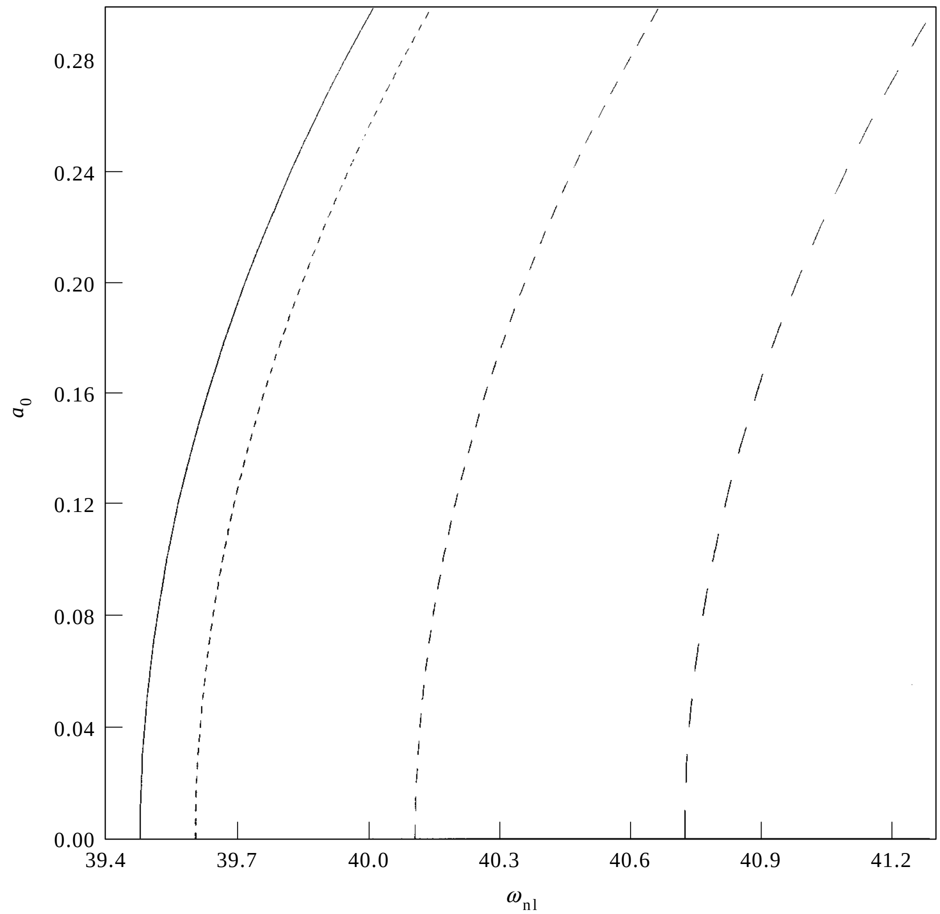

Figure 4. Non-linear frequency versus amplitude for the second mode. $\alpha_2 = 10$: $\alpha_1 = 0$ (—); $\alpha_1 = 10$ (- - -); $\alpha_1 = 50$ (– – –); $\alpha_1 = 100$ (— — —).

Here σ is a detuning parameter of $O(1)$, φ is the function for the secular terms and W is the function for the non-secular terms. Inserting expressions (28), (27), (22) and (12) into equations (11) and considering only the terms producing secularities, one has

$$\varphi^{iv} - (\omega^2 - \alpha_1)\varphi - Z_0'' \int_0^1 Z_0' \varphi' \, dx = -2i\omega Y(D_2 A + \mu A) + \frac{F}{2} e^{i\sigma T_2}$$

$$+ A^2 \bar{A}(-3\alpha_2 Y^3 + b_4 Y'' + 2b_5 Y'' + b(\phi_1'' + 2\phi_2'') + \tfrac{3}{2} n^2\pi^2 Y'' + b_2 Z_0'' + 2b_3 Z_0''), \quad (29)$$

$$\varphi(0) = 0, \qquad \varphi(1) = 0, \qquad \varphi''(0) = 0, \qquad \varphi''(1) = 0, \quad (30)$$

where

$$f = \int_0^1 FY \, dx, \qquad b = \int_0^1 Y' Z_0' \, dx, \qquad b_1 = \int_0^1 Y^4 \, dx, \qquad b_2 = \int_0^1 Y' \phi_1' \, dx,$$

$$b_3 = \int_0^1 Y' \phi_2' \, dx, \qquad b_4 = \int_0^1 Z_0' \phi_1' \, dx, \qquad b_5 = \int_0^1 Z_0' \phi_2' \, dx, \qquad \int_0^1 Y^2 \, dx = 1. \quad (31)$$

The homogeneous problem of equations (13) and (14) possesses a non-trivial solution. For the non-homogeneous problem of equations (29) and (30) to possess a solution, a solvability condition should be satisfied (see reference [10] for details of calculating this condition). For the present problem, the solvability condition requires

$$2\mathrm{i}\omega(\mu A + A') + \lambda A^2 \bar{A} - \tfrac{1}{2} f\, \mathrm{e}^{\mathrm{i}\sigma T_2} = 0, \tag{32}$$

where

$$\lambda = 3\alpha_2 b_1 + 2bb_2 + 4b_2 b_3 + \tfrac{3}{2} n^4\pi^4 + n^2\pi^2 b_4 + 2n^2\pi^2 b. \tag{33}$$

Equation (32) represents the modulations in the complex amplitudes. If one writes them in the polar form

$$A(T_2) = \tfrac{1}{2}\, a(T_2)\, \mathrm{e}^{\mathrm{i}\theta(T_2)}, \tag{34}$$

substitutes into equation (32) and separates real and imaginary parts, one finally obtains

$$\omega a\gamma' = a(\omega\sigma - \lambda a^2/8) + \tfrac{1}{2} f \cos\gamma, \qquad \omega a' = -\omega\mu a + \tfrac{1}{2} f \sin\gamma, \tag{35, 36}$$

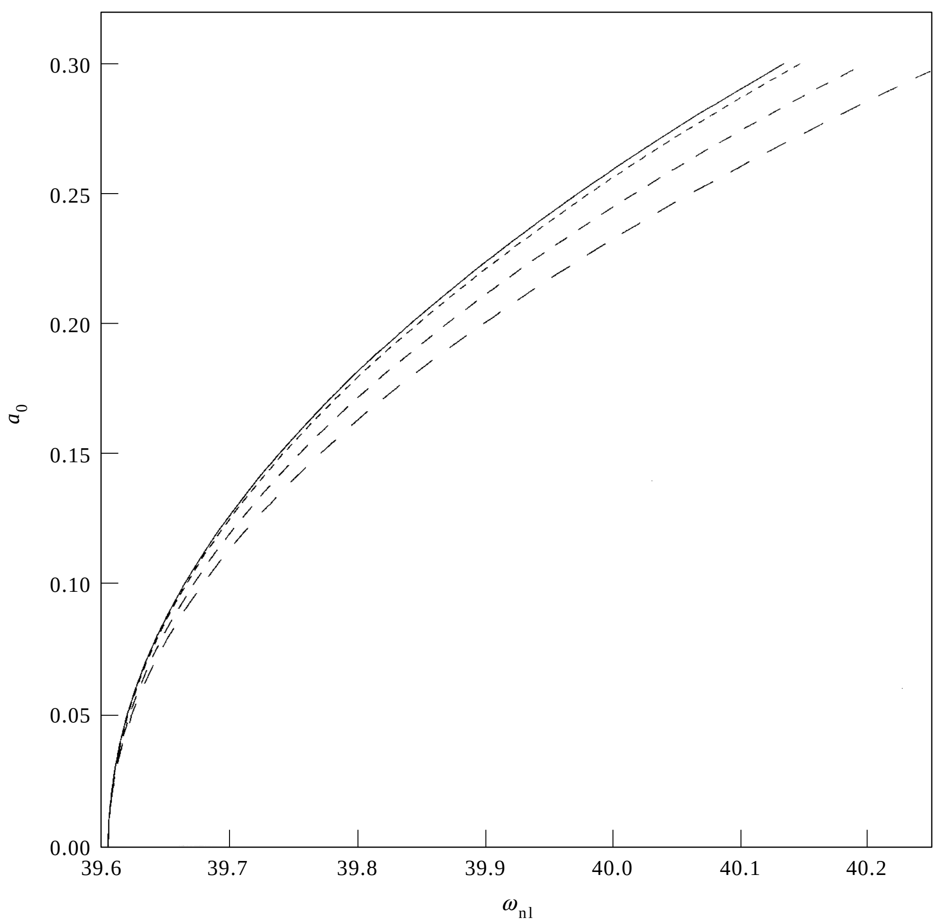

Figure 5. Non-linear frequency versus amplitude for the second mode. $\alpha_1 = 10$: $\alpha_2 = 0$ (—); $\alpha_2 = 10$ (- - -); $\alpha_2 = 50$ (– – –); $\alpha_2 = 100$ (— — —).

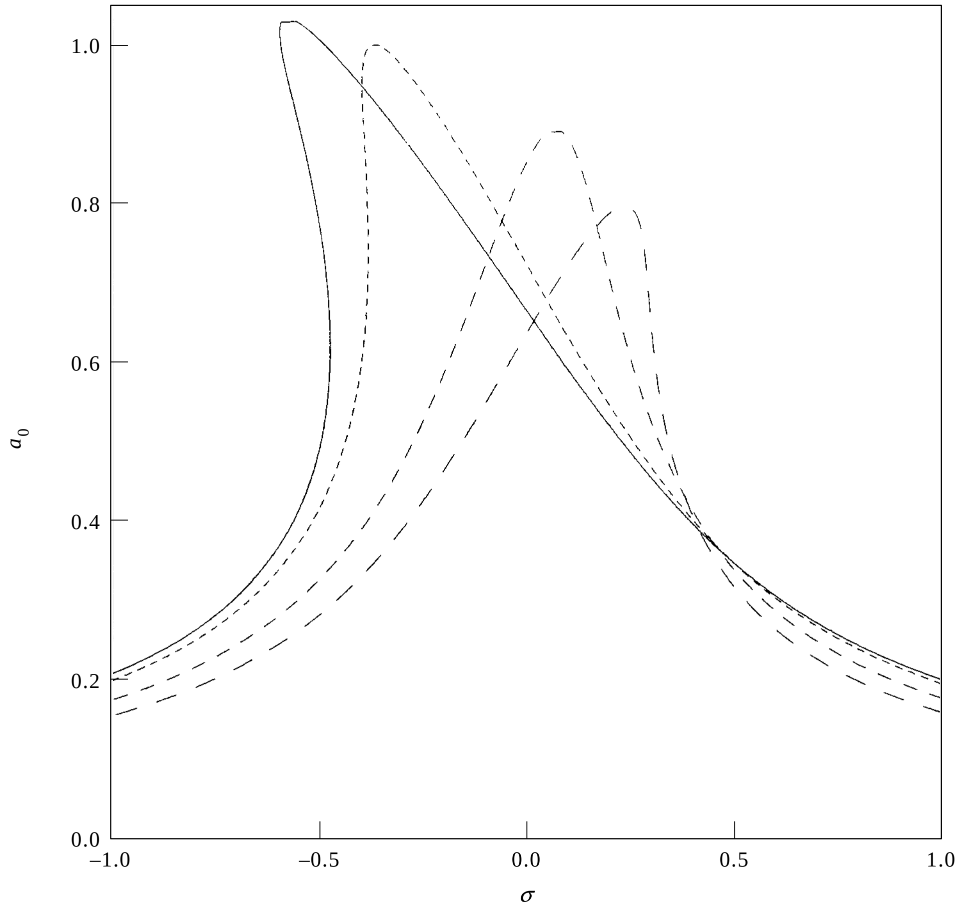

Figure 6. Frequency-response curves for the first mode. $\mu = 0{\cdot}2$, $f = 5$, $\alpha_2 = 10$: $\alpha_1 = 0$ (—); $\alpha_1 = 10$ (- - -); $\alpha_1 = 50$ (– – –); $\alpha_1 = 100$ (— — —).

where γ is defined to be

$$\gamma = \sigma T_2 - \theta, \tag{37}$$

The response is found by substituting equations (37), (34), (28), (23), (22) and (12) into equation (6), and is

$$w(x, t) = \epsilon a \cos(\Omega t - \gamma)\sqrt{2} \sin n\pi x + \epsilon^2 (a^2/2)(\cos[2(\Omega t - \gamma)]\phi_1(x) + \phi_2(x)) + O(\epsilon^3). \tag{38}$$

The amplitude a and the phase γ are now governed by equations (35) and (36).

4. NUMERICAL RESULTS

In this section, numerical results for free vibrations are first presented. Then forced vibrations with damping are considered.

4.1. FREE VIBRATIONS

One can begin by calculating the natural frequencies from equation (15),

$$\omega = \sqrt{\beta^4 + \alpha_1}, \tag{39}$$

TABLE 1

The first five natural frequencies corresponding to different linear coefficients of the elastic foundation

α_1	ω_1	ω_2	ω_3	ω_4	ω_5
0	12·0877	39·4783	88·8264	157·9137	246·7401
10	12·4945	39·6048	88·8827	157·9453	246·7604
50	14·0041	40·1066	89·1074	158·0719	246·8414
100	15·6880	40·7252	89·3876	158·2210	246·9427
500	25·4188	45·3711	91·5979	158·4890	247·7513

and substituting for β from equations (19) and (20), yields

$$\omega = \sqrt{\tfrac{3}{2}\pi^4 + \alpha_1}, \qquad n = 1, \qquad b \neq 0, \qquad \omega = \sqrt{n^4\pi^4 + \alpha_1}, \qquad n \neq 1, \qquad b = 0, \quad (40, 41)$$

for different α_1 (linear dimensionless coefficient of the elastic foundation) values. The first five frequencies are given in Table 1 for $\alpha_1 = 0$, 10, 50, 100 and 500. Next the non-linear frequency corrections to these linear ones, which are amplitude dependent, are calculated.

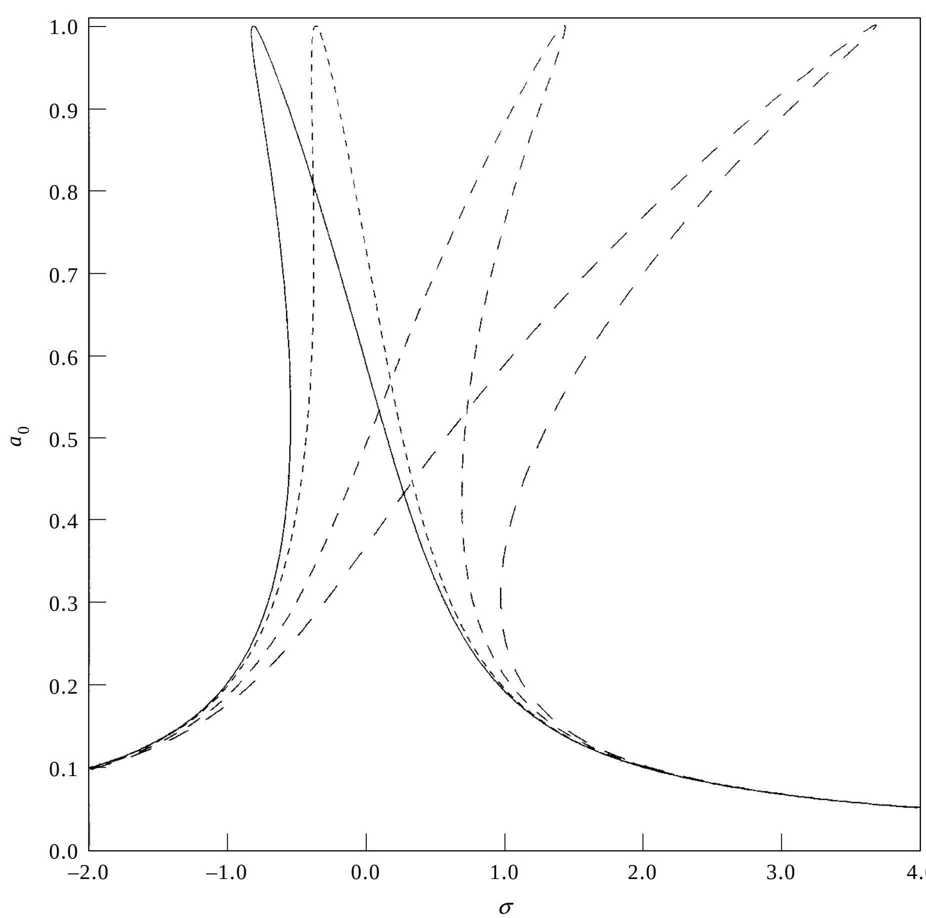

Figure 7. Frequency-response curves for the first mode. $\alpha_1 = 10$, $\mu = 0{\cdot}2$, $f = 5$: $\alpha_2 = 0$: (—); $\alpha_2 = 10$ (- - -); $\alpha_2 = 50$ (– – –); $\alpha_2 = 100$ (— — —).

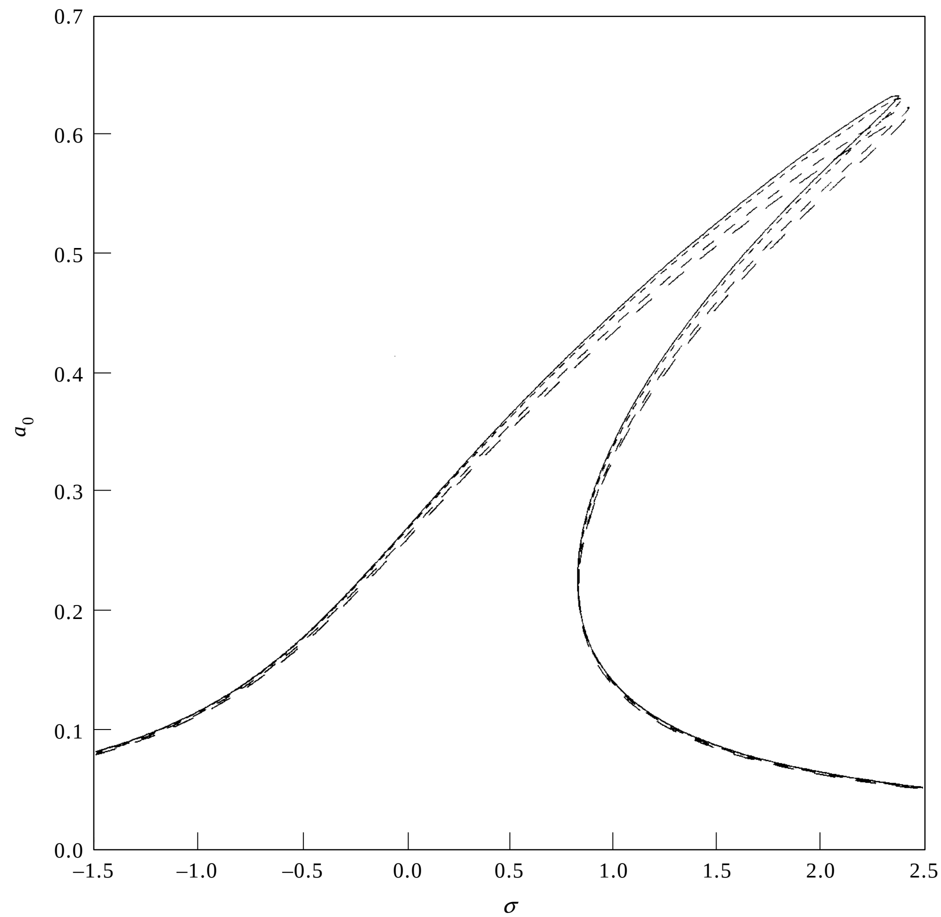

Figure 8. Frequency-response curves for the second mode. $\alpha_2 = 10$, $\mu = 0{\cdot}2$, $f = 10$: $\alpha_1 = 0$: (—); $\alpha_1 = 10$ (- - -); $\alpha_1 = 50$ (– – –); $\alpha_1 = 100$ (— — —).

Returning to equations (35) and (36), one takes $\sigma = 0$, $\mu = 0$, $f = 0$, $\gamma = -\theta$. From equation (36) one obtains $a = a_0$, a constant amplitude. Substituting this further into equation (35) yields

$$\theta' = \lambda a_0^2/8\omega. \tag{42}$$

The non-linear frequency is

$$\omega_{nl} = \omega + \theta' = \omega + \lambda a_0^2/8\omega. \tag{43}$$

In Tables 2 and 3, λ values are given for the cases $n = 1$ and $n = 2$, respectively, corresponding to linear and non-linear elastic foundation coefficients. From the tables, for the first mode ($n = 1$), for sufficiently low values, softening behaviour can be observed (negative λ), whereas for the second mode ($n = 2$) only hardening behaviour can be observed. In Table 4, for $n = 1$ and $n = 2$, critical values of α_1 and α_2 making $\lambda = 0$ are given. These values represent the transition from softening behaviour to hardening behaviour. Note that for the second mode, the non-linear elastic coefficient should be of softening type to observe overall softening behaviour.

In Figures 2 and 3, the non-linear frequencies versus amplitudes are shown for the first mode ($n = 1$). One can observe a transition from softening behaviour to hardening

TABLE 2

λ values corresponding to α_1 and α_2 values for $n = 1$

$n = 1$	$\alpha_1 = 0$	$\alpha_1 = 10$	$\alpha_1 = 50$	$\alpha_1 = 100$	$\alpha_1 = 500$
$\alpha_2 = 0$	−97·409	−81·810	−35·322	1·538	91·043
$\alpha_2 = 10$	−52·409	−36·810	9·678	46·538	136·043
$\alpha_2 = 50$	127·591	143·190	189·678	226·538	316·043
$\alpha_2 = 100$	352·591	368·190	414·678	451·538	541·043
$\alpha_2 = 500$	2152·591	2168·190	2214·678	2251·538	2341·043

TABLE 3

λ values corresponding to α_1 and α_2 values for $n = 2$

$n = 2$	$\alpha_1 = 0$	$\alpha_1 = 10$	$\alpha_1 = 50$	$\alpha_1 = 100$	$\alpha_1 = 500$
$\alpha_2 = 0$	1824·531	1857·779	1956·834	2035·326	2225·329
$\alpha_2 = 10$	1869·531	1902·779	2001·834	2080·326	2270·329
$\alpha_2 = 50$	2049·531	2082·779	2181·834	2260·326	2450·329
$\alpha_2 = 100$	2274·531	2307·779	2406·834	2485·326	2675·329
$\alpha_2 = 500$	4074·531	4107·779	4206·834	4285·326	4475·329

TABLE 4

α_1 and α_2 values making $\lambda = 0$

		$\alpha_1 = 0$	$\alpha_1 = 10$	$\alpha_1 = 50$	$\alpha_1 = 100$	$\alpha_1 = 500$
$n = 1$	α_2	21·646	18·180	7·849	−0·342	−20·232
$n = 2$	α_2	−405·451	−412·839	−434·852	−452·294	−494·517

behaviour; the frequencies decrease with amplitude in softening behaviour and increase with amplitude in hardening behaviour. Figure 2 shows the comparison of non-linear frequencies for various α_1 values. From this figure, as α_1 increases, the non-linear frequencies increase. In Figure 3, α_1 is fixed and α_2 is increased. The non-linear frequencies increase in this case. In Figures 4 and 5, the non-linear frequency versus amplitudes are shown for the second mode ($n = 2$). One can observe a hardening behaviour: that is, the frequencies increase with amplitude. In Figure 4 one observes that as α_1 increases the non-linear frequencies increase. In Figure 5, α_1 is fixed and α_2 is increased, and again it can be seen that the non-linear frequencies increase.

4.2. FORCED VIBRATIONS WITH DAMPING

To consider forced vibrations with damping one returns again to the amplitude and phase modulation equations given in equations (35) and (36), but now searches for the steady state periodic solutions. Requiring that $a' = \gamma' = 0$ and eliminating γ_n between the equations yields

$$\sigma = (\lambda a^2/8\omega) \pm \sqrt{\tfrac{1}{4}(f^2/a^2\omega^2) - \mu^2}. \tag{44}$$

In Figures 6 and 7, for the first mode, the frequency-response curves are shown. σ is defined in equation (28) and represents the nearness of the external excitation frequency to the natural frequency. In Figure 6, the variation of σ with amplitude for various α_1

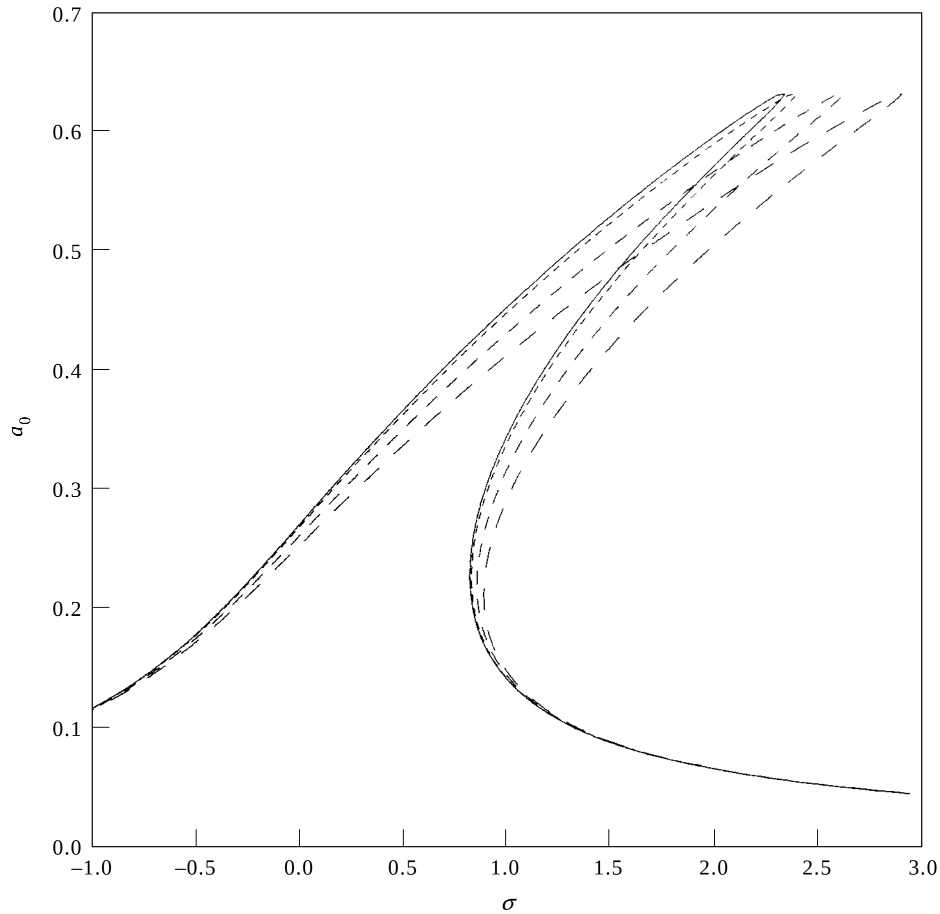

Figure 9. Frequency-response curves for the second mode. $\alpha_1 = 10$, $\mu = 0{\cdot}2$, $f = 10$: $\alpha_2 = 0$: (—); $\alpha_2 = 10$ (- - -); $\alpha_2 = 50$ (– – –); $\alpha_2 = 100$ (— — —).

values when α_2 is fixed is shown. As α_1 increases, the multivalued regions causing the well-known jump phenomena decrease. The maximum amplitudes follow the same trend also. For Figure 7, α_1 is fixed this time and α_2 is increased. α_2 has a direct contribution to the non-linearities and the multivalued regions increase considerably without an increase in the maximum amplitudes. A transition occurs from softening behaviour to hardening behaviour. In Figures 8 and 9, for $n = 2$, the frequency–response curves are shown. In Figure 8, α_2 is fixed and α_1 is increased. All curves show hardening behaviour. In Figure 9, α_1 is fixed and α_2 is increased. The multivalued regions increase considerably without an increase in the maximum amplitude.

Finally, the softening behaviour observed for the first mode in the absence of elastic foundation was also reported in Rehfield [7]. He also reported that when $r = \sqrt{3/\pi}$ (r defined in his paper) a transition from softening to hardening behaviour occurs. An equivalent r can be calculated for our case. We found that $r = 1/\sqrt{\pi}$. Therefore one is in the softening region in agreement with Rehfield [7].

Note that if the initial curvature function and the vibrations are chosen to be of the same order, then one is assuming that the curvature function is appreciably small: that is, the beam would have characteristics similar to those of a straight beam. Hence, one expects that the hardening behaviour of the straight beam would be retrieved for this type of ordering.

5. CONCLUDING REMARKS

A simply supported slightly curved beam resting on a non-linear elastic foundation has been considered. The end supports are immovable causing axial stretching during the vibrations. The non-linearities arise due to stretching, curvature and the non-linear elastic foundation. The equations of motion have been written for an arbitrary curvature function with damping and forcing terms included. Approximate analytical solutions have been sought by using the method of multiple scales, a peturbation technique. The non-linear frequencies and the frequency response curves have been drawn for different elastic foundation coefficients, a sinusoidal curvature function being assumed.

In agreement with the previous literature [7–9], softening behaviour due to the curvature function has been found for the first mode. However, the non-linear elastic foundation has a reverse effect (for a hardening foundation) and for sufficiently high foundation coefficients the softening behaviour may be suppressed by the hardening effects of the foundation. For the second mode the curvature effects are of hardening type only.

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APPENDIX: NOTATION

ρ	beam density
A	beam cross-section
w^*	displacement
E	Young's modulus
I	moment of inertia
μ^*	damping coefficient
k_1	linear coefficient of elastic foundation
k_2	non-linear coefficient of elastic foundation
L	projected length of beam
Z_0^*	curvature function
x^*	spatial variable
t^*	time
F^*	amplitude of excitation
Ω^*	frequency of excitation
x	dimensionless spatial variable
t	dimensionless time
r	radius of gyration
w	dimensionless displacement
Z_0	dimensionless curvature function
$\bar{\mu}$	dimensionless damping coefficient
α_1	dimensionless linear coefficient of elastic foundation
α_2	dimensionless non-linear coefficient of elastic foundation
$\bar{F}$	dimensionless amplitude of excitation
Ω	dimensionless frequency of excitation
ϵ	perturbation parameter
w_1	$O(\epsilon)$ solution
w_2	$O(\epsilon^2)$ solution
w_3	$O(\epsilon^3)$ solution
T_0	fast time scale
$T_{1,2}$	slow time scales
F	ordered amplitude of excitation
μ	ordered damping coefficient
D_0	derivative with respect to fast time scale
$D_{1,2}$	derivatives with respect to slow time scales
A	complex amplitude
ω	natural frequency
Y	mode shape
σ	detuning parameter
ϕ_1, ϕ_2	parts of solution w_2 related to secular terms
φ	part of solution w_3 related to secular terms
W	parts of solution w_3 related to non-secular terms
b	coefficients related to curvature function
f	coefficient related to excitation amplitude
a	real amplitude
γ	phase
ω_{nl}	non-linear frequencies