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ACOUSTIC EIGENFREQUENCIES IN CONCENTRIC SPHEROIDAL–SPHERICAL CAVITIES: CALCULATION BY SHAPE PERTURBATION

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The acoustic eigenfrequencies f_{nsm} in concentric spheroidal–spherical cavities are determined analytically, for both Dirichlet and Neumann boundary conditions, by a shape perturbation method. Two types of cavities are examined, one with spheroidal outer and spherical inner boundary and inversely for the other. The analytical determination is possible in the case of small $h = d/(2R_2)$, $(h \ll 1)$, where d is the interfocal distance of the spheroidal boundary and $2R_2$ the length of its rotation axis. In this case exact, closed form expressions are obtained for the expansion coefficients $g^{(2)}_{nsm}$ and $g^{(4)}_{nsm}$ in the resulting relation $f_{nsm}(h) = f_{ns}(0)\,[1 + h^2 g^{(2)}_{nsm} + h^4 g^{(4)}_{nsm} + O(h^6)]$. Analogous expressions are obtained with the use of the parameter $v = 1 - (R_2/R'_2)^2$, $(|v| \ll 1)$ where $2R'_2$ is the length of the other axis of the spheroidal boundary. There is no need for using any spheroidal wave functions and the expansion formulas connecting them with the concentric spherical ones. The pressure field is expressed in terms of spherical wave functions, while the equation of the spheroidal boundary is given in terms of the spherical co-ordinates. Numerical results are given for various values of the parameters.

1. INTRODUCTION

The motivation for solving the present problems can be found in reference [1], where the same problems were solved by the use of spheroidal wave functions, so it will not be repeated here.

The acoustic cavities, shown in Figures 1 and 2, are examined for both Dirichlet and Neumann boundary conditions. In Figure 1 the inner boundary is spherical with radius R_1, while the outer concentric one is prolate spheroidal, with major and minor semiaxes R_2 and R'_2, respectively, and interfocal distance d. In Figure 2 the positions of the inner and the outer boundaries of Figure 1 are interchanged. Both cavities are perturbations of the concentric spherical one, with radii R_1 and R_2. Only the prolate spheroidal boundaries are shown, but the oblate ones are considered simultaneously. The length of the rotation axis in each case is $2R_2$, while that of the other axis is $2R'_2$.

The acoustic eigenfrequencies, in the former cavities, are determined by a special analytical shape perturbation method. No spheroidal wave functions are used and consequently nor are the expansion formulas connecting them with the concentric spherical ones. The pressure field is expressed in a series of spherical wave functions only. The equation of the spheroidal boundary is given in terms of the spherical co-ordinates r and θ. After satisfying the boundary conditions, one obtains an infinite determinantal equation for the evaluation of the eigenfrequencies of the former cavities. In the special case of small $h = d/(2R_2)$, $(h \ll 1)$ one is led to an exact evaluation, up to the order h^4, for the elements

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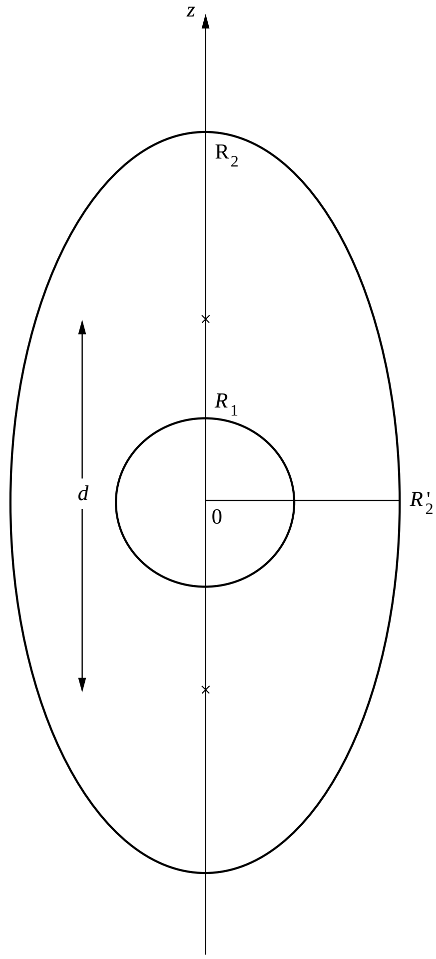

Figure 1. Geometry of the spheroidal–spherical cavity.

of the infinite determinant and, finally, for the determinant itself. It is then possible, after lengthy manipulation, to obtain the eigenfrequencies in the form $f_{nsm}(h) = f_{ns}(0)\,[1 + h^2 g^{(2)}_{nsm} + h^4 g^{(4)}_{nsm} + O(h^6)]$. The expansion coefficients $g^{(2)}_{nsm}$ and $g^{(4)}_{nsm}$ are independent of h and are given by exact, closed form expressions, while $f_{ns}(0)$ are the eigenfrequencies of the corresponding concentric spherical cavity with $h = 0$.

The main advantage of such an analytical solution lies in its general validity for each small value of h and for all modes, while all numerical techniques require repetition of the evaluation for each different h and, in general, their accuracy deteriorates quickly for higher order modes.

Analogous expansions are obtained with the use of the parameter $v = 1 - (R_2/R_2')^2$, $(|v| \ll 1)$.

The method can also be applied in the corresponding exterior (scattering) problems.

The case of Dirichlet boundary conditions is examined in section 2, while that of Neumann boundary conditions is examined in section 3. Finally, in section 4, some numerical results are shown, accompanied by discussion and comments.

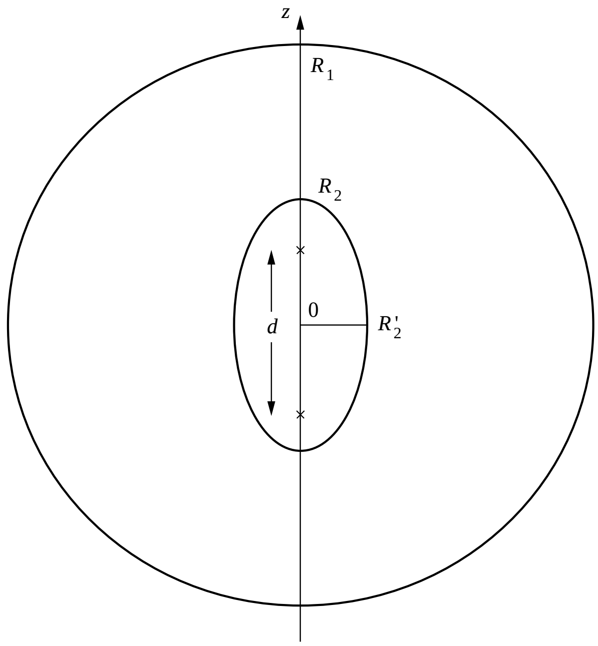

Figure 2. Geometry of the spherical–spheroidal cavity.

2. DIRICHLET BOUNDARY CONDITIONS

The cavities of Figures 1 and 2 can be treated simultaneously. Let p be the acoustic pressure field inside the cavity. This field satisfies the scalar Helmholtz equation. Its expression satisfying also the homogeneous Dirichlet boundary condition $p = 0$ at the spherical boundary $r = R_1$ is

$$p = \sum_{n=0}^{\infty} \sum_{m=0}^{n} [\mathrm{j}_n(kr) - \mathrm{n}_n(kr)\mathrm{j}_n(x_1)/\mathrm{n}_n(x_1)]\mathrm{P}_n^m(\cos\theta)\,[A_{nm}\cos m\varphi + B_{nm}\sin m\varphi],$$

$$x_1 = kR_1. \tag{1}$$

In equation (1), r, θ, φ are the spherical co-ordinates with respect to O, j_n and n_n are the spherical Bessel functions of the first and the second kind, respectively, P_n^m is the associated Legendre function of the first kind and k is the resonance wavenumber.

In order to satisfy the remaining boundary condition $p = 0$ at the spheroidal surface, one can express the equation of this surface in terms of r and θ [2] as

$$r = R_2/\sqrt{1 - v\sin^2\theta}, \tag{2}$$

where

$$v = 1 - \left(\frac{R_2}{R_2'}\right)^2 = 1 - \frac{R_2^2}{R_2^2 \mp d^2/4} = \mp\frac{d^2/4}{R_2^2 \mp d^2/4} = \mp\frac{h^2}{1 \mp h^2} = \mp h^2 - h^4 + O(h^6),$$

$$h = d/(2R_2). \tag{3}$$

So,

$$v^2 = h^4 + O(h^6). \tag{4}$$

The upper/lower signs in equation (3) correspond to the prolate ($v < 0$)/oblate ($v > 0$) spheroidal boundary.

By expanding equation (2) into power series in h and keeping terms up to the order h^4, one obtains

$$r = R_2\left[1 \mp \frac{h^2}{2}\sin^2\theta + h^4\left(-\frac{\sin^2\theta}{2} + \frac{3\sin^4\theta}{8}\right) + O(h^6)\right]. \tag{5}$$

By using equation (5) one obtains the following expansion ($x_2 = kR_2$):

$$\mathrm{j}_n(kr) = \mathrm{j}_n(x_2) \mp \frac{h^2}{2} x_2\,\mathrm{j}_n'(x_2)\sin^2\theta + h^4\left\{-\frac{x_2}{2}\mathrm{j}_n'(x_2)\sin^2\theta + \tfrac{1}{8}[3x_2\,\mathrm{j}_n'(x_2) + x_2^2\,\mathrm{j}_n''(x_2)]\sin^4\theta\right\} + O(h^6), \tag{6}$$

and a similar one for $\mathrm{n}_n(kr)$, where the primes denote derivatives with respect to the argument.

One can next substitute these expansions into equation (1) and use the orthogonal properties of the associated Legendre functions [3, 4] and the trigonometric functions, concluding finally to the following infinite set of linear homogeneous equations for the expansion coefficients A_{nm} (or B_{nm}), up to the order h^4:

$$\alpha_{n-4,n}A_{n-4,m} + \alpha_{n-2,n}A_{n-2,m} + \alpha_{nn}A_{nm} + \alpha_{n+2,n}A_{n+2,m} + \alpha_{n+4,n}A_{n+4,m} = 0, \qquad n \geqslant m. \tag{7}$$

The third subscript m is omitted from the various α's in equation (7), for simplicity. Their expressions are given subsequently in equations (8) and (A1)–(A5) of Appendix A. As is evident from equation (1), the subscripts of A's (and B's) are always non-negative and the first subscript is equal or greater than the second; i.e., than m. In the opposite case A's (and B's) are equal to zero and so disappear. The same is valid also for the corresponding α's.

If m has the same/opposite parity with n, i.e., $n - m$ is even/odd, the first subscript of the α's in equation (7) starts from the minimum value $m/m + 1$ and continues with the values $m + 2/m + 3$, $m + 4/m + 5$, etc. So, the set (7) separates into two distinct subsets, one with n even and the other with n odd. Setting each one of the determinants Δ of the coefficients α, in these subsets, equal to zero, one obtains two determinantal equations for the evaluation of the eigenfrequencies. Insofar as they show the same general form, they are treated simultaneously, under the symbol Δ.

For small values of h one can set, up to the order h^4,

$$\alpha_{nn} = D_{nn}^{(0)} + h^2 D_{nn}^{(2)} + h^4 D_{nn}^{(4)} + O(h^6),$$
$$\alpha_{n\pm 2,n} = h^2 D_{n\pm 2,n}^{(2)} + h^4 D_{n\pm 2,n}^{(4)} + O(h^6), \qquad \alpha_{n\pm 4,n} = h^4 D_{n\pm 4,n}^{(4)} + O(h^6). \tag{8}$$

Exact expressions for the various D's used in the calculations are given in equations (A1)–(A5) of Appendix A. It should be noted here that α's and also D's are different from the corresponding ones in reference [1], as can be seen by a simple comparison.

The relations (8) allow a closed form evaluation of the determinant Δ, up to the order h^4, in steps exactly the same as with ones appearing in reference [1], which will not be repeated here.

The resonance wavenumbers $k(h)$, as well as $x_2(h) = k(h)R_2$ also have expansions of the form

$$k(h) = k^{(0)} + h^2 k^{(2)} + h^4 k^{(4)} + O(h^6), \qquad k^{(0)} \equiv k^0, \tag{9}$$

$$x_2(h) = k(h)R_2 = x_2^{(0)} + h^2 x_2^{(2)} + h^4 x_2^{(4)} + O(h^6), \qquad x_2^{(\rho)} = k^{(\rho)} R_2, \qquad \rho = 0, 2, 4, \quad (10)$$

where k^0 and $x_2^{(0)} \equiv x_2^0$ correspond to the concentric spherical cavity with radii R_1 and R_2 ($h = 0$).

The expressions for $x_2^{(2)}$ and $x_2^{(4)}$ in terms of the D's are exactly the same as in reference [1], where the same problems were solved by the use of spheroidal functions, and, although the various D's are different here, the final results for $x_2^{(2)}$ and $x_2^{(4)}$ are identical here to those there, as is expected for the same problems, and are given by equations (23)–(26) of reference [1]. This provides a very good check for their correctness.

It is evident that equation (10) can be written in the form $x_2(h) = x_2^0 [1 + h^2 g^{(2)} + h^4 g^{(4)} + O(h^6)]$, where

$$g^{(2)} = x_2^{(2)}/x_2^0, \qquad g^{(4)} = x_2^{(4)}/x_2^0, \quad (11)$$

and $x_2^0 = (x_2^0)_{ns}$, $n = 0, 1, 2, \ldots,$ $s = 1, 2, 3, \ldots$ are the successive positive roots of the equation

$$\mathrm{j}_n(x_1^0)/\mathrm{n}_n(x_1^0) = \mathrm{j}_n(x_2^0)/\mathrm{n}_n(x_2^0), \qquad x_1^0 = x_2^0/\tau, \qquad \tau = R_2/R_1, \quad (12)$$

giving the resonance wavenumbers of the concentric spherical cavity with radii R_1 and R_2. So, the eigenfrequencies $f(h) = x_2(h)c/(2\pi R_2)$ in the cavities of Figures 1 and 2, with c the sound speed, are given by the expression

$$f_{nsm}(h) = f_{ns}(0)[1 + h^2 g_{nsm}^{(2)} + h^4 g_{nsm}^{(4)} + O(h^6)], \qquad n = 0, 1, 2, \ldots,$$
$$s = 1, 2, 3, \ldots, \qquad m = 0, 1, 2, \ldots, n. \quad (13)$$

In equation (13) $f_{ns}(0) = (x_2^0)_{ns}\, c/(2\pi R_2)$ are the eigenfrequencies in the concentric spherical cavity, each one with multiplicity $2n + 1$ ($f_{ns}(0)$ splits into one eigenfrequency $f_{nso}(h)$, of multiplicity 1 (for $\cos o\varphi$ in equation (1)) and n eigenfrequencies $f_{nsm}(h)$, $1 \leqslant m \leqslant n$, of multiplicity 2, (for $\cos m\varphi$ and $\sin m\varphi$ in equation (1))), while $f_{nsm}(h) = f(h)$ and $g_{nsm}^{(2),(4)} = g^{(2),(4)}[(x_2^{(2),(4)})_{nsm} = x_2^{(2),(4)}]$.

The same problems can be solved, from the beginning, by using the eccentricity parameter v instead of h. In this case the expansion of the general quantity y with respect to v is

$$y = y(v) = y^0 + v y_v^{(1)} + v^2 y_v^{(2)} + O(v^3), \quad (14)$$

while its expansion with respect to h is

$$y = y(h) = y^0 + h^2 y_h^{(2)} + h^4 y_h^{(4)} + O(h^6). \quad (15)$$

It is evident that y^0 is the same in both cases, because it corresponds to the concentric spherical cavity ($h = v = 0$). The remaining expansion coefficients in equations (14) and (15) are connected by simple relations. One can express h^2 and h^4 in terms of v and v^2, by using equations (3) and (4), as

$$h^2 = \mp(v + v^2) + O(v^3), \qquad h^4 = v^2 + O(v^3). \quad (16)$$

The upper/lower sign in equations (16) corresponds to the prolate/oblate spheroidal boundary.

Substituting equations (16) into equation (15) yields

$$y = y^0 \mp (v + v^2) y_h^{(2)} + v^2 y_h^{(4)} + O(v^3) = y^0 \mp v y_h^{(2)} + v^2[\mp y_h^{(2)} + y_h^{(4)}] + O(v^3). \quad (17)$$

From equations (14) and (17) one obtains simple relations

$$y_v^{(1)} = \mp y_h^{(2)}, \qquad y_v^{(2)} = \mp y_h^{(2)} + y_h^{(4)}, \quad (18)$$

which are unique for both the prolate and the oblate boundary (v includes the sign), because the $y_h^{(2)}$ simply change their signs in these two cases [1].

The expansions of equations (5) and (6), respectively, with respect to v are

$$r = R_2\left[1 + \frac{v}{2}\sin^2\theta + \frac{3v^2}{8}\sin^4\theta + O(v^3)\right], \tag{19}$$

$$\mathrm{j}_n(kr) = \mathrm{j}_n(x_2) + \frac{v}{2}x_2\,\mathrm{j}_n'(x_2)\sin^2\theta + \frac{v^2}{8}[3x_2\,\mathrm{j}_n'(x_2) + x_2^2\,\mathrm{j}_n''(x_2)]\sin^4\theta + O(v^3), \tag{20}$$

while $\mathrm{n}_n(kr)$ has an expansion similar to equation (20).

The rest of the steps are exactly analogous with the corresponding ones in the solution with the eccentricity parameter h. In any case the expansion (15) of the general quantity y is easily replaced by equation (14), with the use of equations (18). So one finally obtains the expressions for $x_{2,v}^{(1)}$ and $x_{2,v}^{(2)}$ $[x_2(v) = k(v)R_2 = x_2^0 + vx_{2,v}^{(1)} + v^2x_{2,v}^{(2)} + O(v^3)]$, which are given in Appendix B, and next for the eigenfrequencies

$$f_{nsm}(v) = f_{ns}(0)\,[1 + vg_{nsm,v}^{(1)} + v^2g_{nsm,v}^{(2)} + O(v^3)], \qquad n = 0, 1, 2, \ldots,$$
$$s = 1, 2, 3, \ldots, \qquad m = 0, 1, 2, \ldots, n, \tag{21}$$

where

$$g_{nsm,v}^{(1)} = (x_{2,v}^{(1)})_{nsm}/(x_2^0)_{ns}, \qquad g_{nsm,v}^{(2)} = (x_{2,v}^{(2)})_{nsm}/(x_2^0)_{ns}. \tag{22}$$

It is clear that one can start, equivalently, the analysis by interchanging j_n and n_n in equation (1). The same can be done in each one of the formulas following equation (1). This was verified numerically for various values of the parameters. Some more remarks about this interchange, which are referred to in reference [1], will not be repeated here.

3. NEUMANN BOUNDARY CONDITIONS

In this case the expansion for p, corresponding to equation (1) and satisfying the boundary condition $\partial p/\partial r = 0$ at $r = R_1$, is

$$p = \sum_{n=0}^{\infty}\sum_{m=0}^{n}[\mathrm{j}_n(kr) - \mathrm{n}_n(kr)\mathrm{j}_n'(x_1)/\mathrm{n}_n'(x_1)]\mathrm{P}_n^m(\cos\theta)\,[A_{nm}\cos m\varphi + B_{nm}\sin m\varphi]. \tag{23}$$

In order to satisfy the remaining boundary condition $\hat{\mathbf{u}}\cdot\boldsymbol{\nabla}p = 0$ at the spheroidal surface, where $\hat{\mathbf{u}}$ is the normal unit vector there, one can use the expansion formulas

$$\hat{\mathbf{u}} = \left(1 - \frac{h^4}{8}\sin^2 2\theta\right)\hat{\mathbf{u}}', \qquad \hat{\mathbf{u}}' = \hat{\mathbf{r}} + \frac{h^2}{2}\sin 2\theta(\pm 1 + h^2\cos^2\theta)\hat{\boldsymbol{\theta}} + O(h^6). \tag{24}$$

So, this boundary condition takes the form

$$\hat{\mathbf{u}}\cdot\boldsymbol{\nabla}p = \hat{\mathbf{u}}'\cdot\boldsymbol{\nabla}p = \frac{\partial p}{\partial r} + \frac{h^2}{2}\sin 2\theta(\pm 1 + h^2\cos^2\theta)\frac{1}{r}\frac{\partial p}{\partial\theta} = 0. \tag{25}$$

One next substitutes equation (23) into equation (25), to obtain

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left\{[\mathrm{j}_n'(kr)-\mathrm{n}_n'(kr)\mathrm{j}_n'(x_1)/\mathrm{n}_n'(x_1)]\mathrm{P}_n^m(\cos\theta)+\frac{h^2}{2}\sin 2\theta(\pm 1+h^2\cos^2\theta)\right.$$
$$\left.\times\frac{1}{kr}[\mathrm{j}_n(kr)-\mathrm{n}_n(kr)\mathrm{j}_n'(x_1)/\mathrm{n}_n'(x_1)]\frac{\mathrm{dP}_n^m(\cos\theta)}{\mathrm{d}\theta}\right\}[A_{nm}\cos m\varphi+B_{nm}\sin m\varphi]=0. \tag{26}$$

By using equation (5) one obtains expansions similar to equation (6) for $\mathrm{j}_n'(kr)$ and $\mathrm{n}_n'(kr)$, respectively, with the only difference that one more prime is added in each of the Bessel functions met there. Also, one obtains the expansion

$$\frac{\mathrm{j}_n(kr)}{kr}=\frac{\mathrm{j}_n(x_2)}{x_2}\mp\frac{h^2}{2}\left[-\frac{\mathrm{j}_n(x_2)}{x_2}+\mathrm{j}_n'(x_2)\right]\sin^2\theta+\frac{h^4}{2}\left\{\left[\frac{\mathrm{j}_n(x_2)}{x_2}-\mathrm{j}_n'(x_2)\right]\sin^2\theta\right.$$
$$\left.+\frac{1}{4}\left[-\frac{\mathrm{j}_n(x_2)}{x_2}+\mathrm{j}_n'(x_2)+x_2\,\mathrm{j}_n''(x_2)\right]\sin^4\theta\right\}+O(h^6), \tag{27}$$

and a similar one for $\mathrm{n}_n(kr)/(kr)$.

One can substitute the former expansions into equation (26) and use the orthogonal properties of the associated Legendre and the trigonometric functions. So one again obtains equations (7) and (8), but with different expressions for the various D's, which are given in equations (A9)–(A13) of Appendix A. Next, following the same steps as in the

TABLE 1

Simple spheroidal cavity—Dirichlet conditions

	n	m	s = 1	2	3	4	5
$(x_2^0)_{ns}$	0		3·14159	6·28319	9·42478	12·56637	15·70796
	1		4·49341	7·72525	10·90412	14·06619	17·22075
	2		5·76346	9·09501	12·32294	15·51460	18·68904
	3		6·98793	10·41712	13·69802	16·92362	20·12181
	4		8·18256	11·70491	15·03966	18·30126	21·52542
$g_{nsm}^{(2)}$	0	0	−0·06022	0·15910	0·52464	1·03640	1·69437
	1	0	−0·02792	0·10747	0·31051	0·58123	0·91962
		1	−0·08528	0·00498	0·14034	0·32082	0·54641
	2	0	−0·29784	−0·54906	−0·89993	−1·35083	−1·90187
		1	−0·05108	0·04513	0·17951	0·35220	0·56323
		2	−0·09697	−0·04886	0·01833	0·10467	0·21019
	3	0	−0·20595	−0·30425	−0·43457	−0·59726	−0·79240
		1	−0·15696	−0·19769	−0·25168	−0·31908	−0·39993
		2	−0·06640	0·00058	0·08938	0·20024	0·33321
		3	−0·10358	−0·07381	−0·03435	0·01492	0·07402
	4	0	−0·18457	−0·24674	−0·32590	−0·42241	−0·53638
		1	−0·16442	−0·20611	−0·25919	−0·32390	−0·40032
		2	−0·11624	−0·10875	−0·09921	−0·08757	−0·07383
		3	−0·07690	−0·02832	0·03353	0·10895	0·19800
		4	−0·10780	−0·08756	−0·06179	−0·03036	0·00674

Dirichlet case, one finally concludes with equations (11) and (13), where $x_2^0 = (x_2^0)_{ns}$ are the successive positive roots of the equation

$$j_n'(x_1^0)/n_n'(x_1^0) = j_n'(x_2^0)/n_n'(x_2^0), \qquad x_1^0 = x_2^0/\tau, \tag{28}$$

while $x_2^{(2)} = (x_2^{(2)})_{nsm}$ and $x_2^{(4)} = (x_2^{(4)})_{nsm}$ are identical as in reference [1] (equations (35)–(38) there).

One can also solve the same problems with the use of v instead of h. In any case the general expansion (15) is replaced by equation (14), with the use of equation (18). So, in the first of equations (24) h^4 is replaced by v^2, in the second of equations (24), as well as

TABLE 2

Simple spheroidal cavity—Neumann conditions

			s				
	n	*m*	1	2	3	4	5
$(x_2^0)_{ns}$	0		0	4·49341	7·72525	10·90412	14·06619
	1		2·08158	5·94037	9·20584	12·40444	15·57924
	2		3·34209	7·28993	10·61386	13·84611	17·04290
	3		4·51410	8·58375	11·97273	15·24451	18·46815
	4		5·64670	9·84045	13·29556	16·60935	19·86242
$g_{nsm}^{(1)}$	0	0	—	−0·33333	−0·33333	−0·33333	−0·33333
	1	0	−0·02854	−0·18798	−0·19517	−0·19737	−0·19834
		1	−0·48573	−0·40601	−0·40242	−0·40132	−0·40083
	2	0	−0·18283	−0·23203	−0·23542	−0·23656	−0·23709
		1	−0·25808	−0·28268	−0·28437	−0·28495	−0·28521
		2	−0·48384	−0·43463	−0·43125	−0·43011	−0·42958
	3	0	−0·21261	−0·24012	−0·24241	−0·24323	−0·24363
		1	−0·24279	−0·26342	−0·26514	−0·26576	−0·26606
		2	−0·33333	−0·33333	−0·33333	−0·33333	−0·33333
		3	−0·48424	−0·44985	−0·44698	−0·44596	−0·44546
	4	0	−0·22490	−0·24337	−0·24510	−0·24574	−0·24606
		1	−0·24116	−0·25687	−0·25833	−0·25888	−0·25915
		2	−0·28996	−0·29735	−0·29804	−0·29830	−0·29842
		3	−0·37129	−0·36482	−0·36422	−0·36399	−0·36388
		4	−0·48514	−0·45928	−0·45686	−0·45597	−0·45552
$g_{nsm}^{(2)}$	0	0	—	0·03845	0·33096	0·76963	1·35450
	1	0	−0·02470	0·03921	0·20575	0·44181	0·74600
		1	−0·12979	−0·04454	0·06889	0·22699	0·43012
	2	0	−0·23554	−0·41756	−0·71770	−1·11830	−1·61914
		1	−0·05327	−0·00261	0·11088	0·26380	0·45538
		2	−0·12880	−0·07577	−0·01733	0·05970	0·15576
	3	0	−0·17455	−0·25306	−0·36714	−0·51358	−0·69247
		1	−0·13813	−0·17565	−0·22335	−0·28418	−0·35837
		2	−0·07568	−0·03253	0·04427	0·14372	0·26545
		3	−0·12767	−0·09094	−0·05553	−0·01088	0·04345
	4	0	−0·16208	−0·21456	−0·28510	−0·37290	−0·47812
		1	−0·14702	−0·18417	−0·23164	−0·29059	−0·36118
		2	−0·11213	−0·11154	−0·10359	−0·09322	−0·08063
		3	−0·08825	−0·05239	0·00217	0·07058	0·15270
		4	−0·12681	−0·09957	−0·07581	−0·04694	−0·01254

in equations (25) and (26), the quantity $h^2 \sin 2\theta(\pm 1 + h^2 \cos^2\theta)$ is replaced by $-v \sin 2\theta(1 + v \sin^2\theta)$, while equation (27) is replaced by

$$\frac{\mathrm{j}_n(kr)}{kr} = \frac{\mathrm{j}_n(x_2)}{x_2} + \frac{v}{2}\left[-\frac{\mathrm{j}_n(x_2)}{x_2} + \mathrm{j}_n'(x_2)\right]\sin^2\theta + \frac{v^2}{8}\left[-\frac{\mathrm{j}_n(x_2)}{x_2} + \mathrm{j}_n'(x_2) + x_2\,\mathrm{j}_n''(x_2)\right]\sin^4\theta + O(v^3), \quad (29)$$

and a similar expansion is valid for $\mathrm{n}_n(kr)/(kr)$. The rest of the steps are exactly analogous with the corresponding ones in the solution with the parameter h, if one keeps in mind the remarks for the Dirichlet case. So, one finally obtains again the expansions (21) and (22), with $x_{2,v}^{(1)}$ and $x_{2,v}^{(2)}$ given in Appendix B.

Also in the Neumann case, the various formulas are again valid when interchanging j_n, and n_n, with some remarks referred to in reference [1] and omitted here. This was verified numerically for various values of the parameters.

4. NUMERICAL RESULTS AND DISCUSSION

As far as numerical results for g's corresponding to the eccentricity parameter h, these are given in reference [1]; here, results are given for g's corresponding to the eccentricity parameter v. For reasons of simplicity, the subscript v is omitted from these g's.

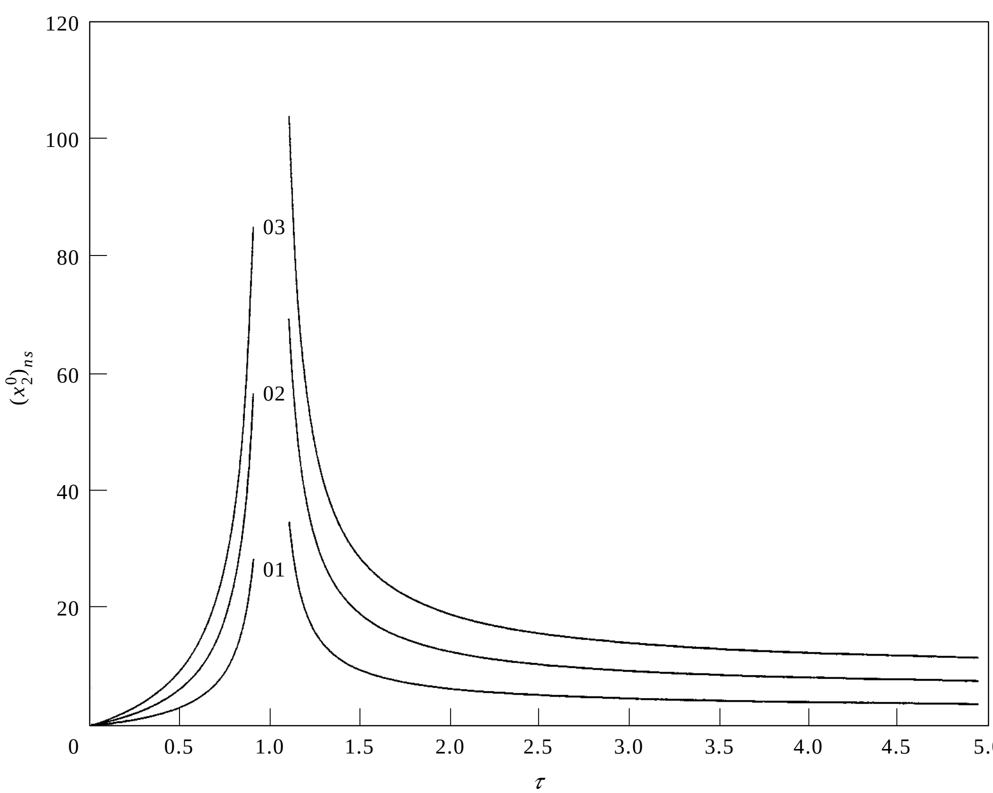

Figure 3. Eigenfrequencies in concentric spherical cavities—Dirichlet conditions.

Tables 1–4 of reference [1] are referred to in the cavities of Figures 1 and 2. No results are given there for the simple spheroidal cavity. For this reason such results are given here, in Tables 1 and 2.

In Table 1, the roots $(x_0^2)_{ns}$ $(n = 0\text{–}4,\ s = 1\text{–}5)$, for "the equation" $\mathrm{j}_n\,(x_2^0) = 0$ [6], as well as the corresponding values of $g_{nsm}^{(2)}$ are given, for a simple spheroidal cavity with Dirichlet conditions. The values of $g_{nsm}^{(1)}$ in this case are independent of s and equal to $-F < 0$, as it is seen easily from equations (22), (B5) from Appendix B and (A7) from Appendix A, while their calculation is very simple. In order to avoid repetition of these values for each different s, they are omitted from Table 1 and are given here. So, $g_{0s0}^{(1)} = -1/3$, $g_{1s0}^{(1)} = -1/5$, $g_{1s1}^{(1)} = -2/5$, $g_{2s0}^{(1)} = -0{\cdot}23810$, $g_{2s1}^{(1)} = -0{\cdot}28571$, $g_{2s2}^{(1)} = -0{\cdot}42857$, $g_{3s0}^{(1)} = -0{\cdot}24444$, $g_{3s1}^{(1)} = -0{\cdot}26667$, $g_{3s2}^{(1)} = -1/3$ $g_{3s3}^{(1)} = -0{\cdot}44444$, $g_{4s0}^{(1)} = -0{\cdot}24675$, $g_{4s1}^{(1)} = -0{\cdot}25974$, $g_{4s2}^{(1)} = -0{\cdot}29870$, $g_{4s3}^{(1)} = -0{\cdot}36364$, $g_{4s4}^{(1)} = -0{\cdot}45455$.

In Table 2, the roots $(x_2^0)_{ns}$ $(n = 0\text{–}4,\ s = 1\text{–}5)$ of "the equation" $\mathrm{j}_n'\,(x_2^0)$ and the corresponding values of $g_{nsm}^{(1)}$ and $g_{nsm}^{(2)}$ are given, for a simple spheroidal cavity with Neumann conditions. The value $(x_2^0)_{01} = 0$ corresponds to the smallest eigenvalue $k^0 = 0$, with constant eigenfunction, of the Helmholtz equation under Neumann conditions. As $(x_2^0)_{01} = 0$, also $f_{01}\,(0) = 0$ and $f_{010}\,(v) = 0$, so the values of $g_{010}^{(1)}$ and $g_{010}^{(2)}$ do not matter.

From the former tables and other available results, it is evident that $(x_2^0)_{ns}$ $(n \geqslant 0,\ s \geqslant 1)$ and so $f_{ns}\,(0)$ for Neumann conditions are smaller than the corresponding ones for Dirichlet conditions. The same is valid for $f_{nsm}\,(v)$, as can be easily proved for the results given in those tables, in the case that $|v| \ll 1$. The last remark is also true for any value of v [7].

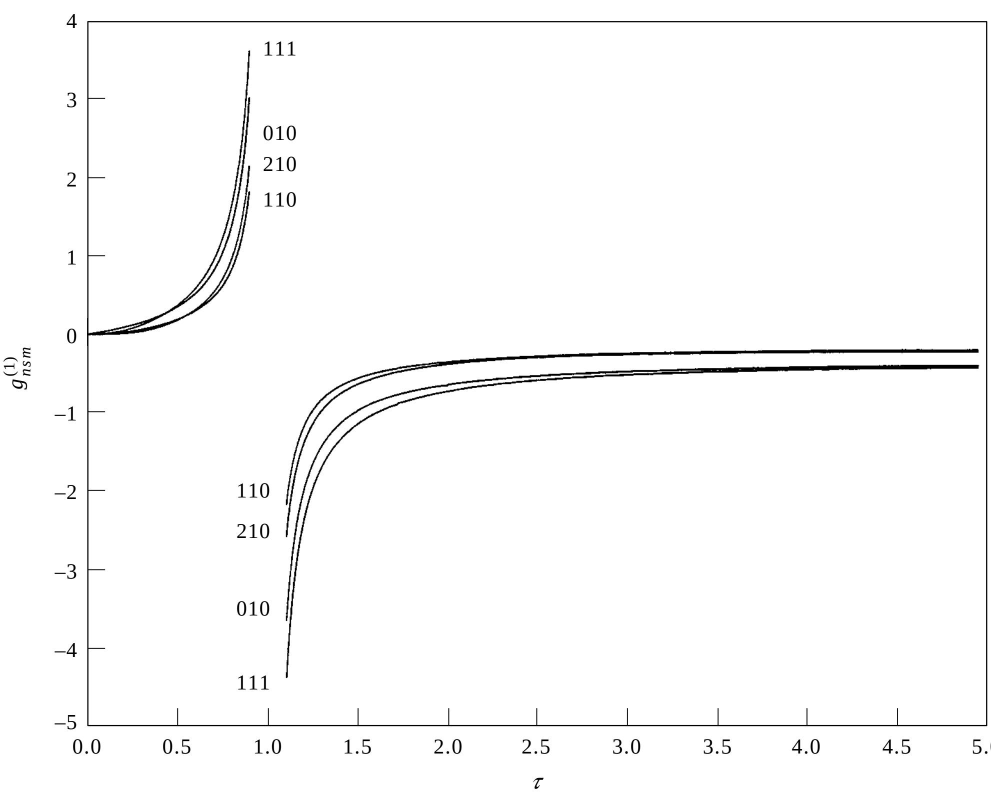

Figure 4. First order expansion coefficients for eigenfrequencies in concentric spheroidal–spherical cavities—Dirichlet conditions.

The values of $g_{nsm}^{(1)}$ are in each case negative. This was explained before the Dirichlet conditions and can be proved also for the Neumann conditions (for $(nsm \neq 010)$), with the use of equations (22), (B8), (B11), (A7) and (A15). For this purpose, one can first observe from Table 2, and many other available results, that $(x_2^0)^2 - n(n+1) > 2n$ (with the exception of $(x_2^0)_{01} = 0$, which does not matter). So, if $M \leqslant 0$ the proof is self-evident, while if $M > 0$, $(x_2^0)^2 - n(n+1)$ is replaced by its smaller $2n$ and after some manipulation the proof is obtained. It can also be proved that $-1/2 \leqslant g_{nsm}^{(1)} \leqslant -1/5$ for Dirichlet conditions and $-1/2 < g_{nsm}^{(1)} < 0$ for Neumann conditions.

With formula (21) in mind, the former remarks mean that up to the order v, the eigenfrequencies of the simple spheroidal cavity increase/decrease for $v < 0$ (prolate cavity)/$v > 0$ (oblate cavity) in comparison with those of the corresponding spherical cavity. In the case of the Dirichlet conditions, this observation is in agreement with the well-known monotonicity theorem for the eigenvalues of the Laplacian under these conditions [7]. According to this theorem, the eigenvalues increase/decrease when the domain gets smaller/larger. So, for a simple spheroidal cavity with fixed R_2 and variable R_2', this is valid for $v < 0/v > 0$ and so the eigenfrequencies increase/decrease, in comparison with those of the corresponding spherical cavity with radius R_2. Also, for the cavities of Figures 1 and 2 with fixed R_1 and R_2 and variable R_2', the eigenfrequencies increase/decrease, in comparison with those of the corresponding concentric spherical cavity with radii R_1 and R_2 $(\tau = R_2/R_1)$, if $v < 0, \tau > 1$ or $v > 0, \tau < 1/v < 0, \tau < 1$ or $v > 0$, $\tau > 1$. The former theorem holds for any value of v and not only for $|v|$ small, as is the

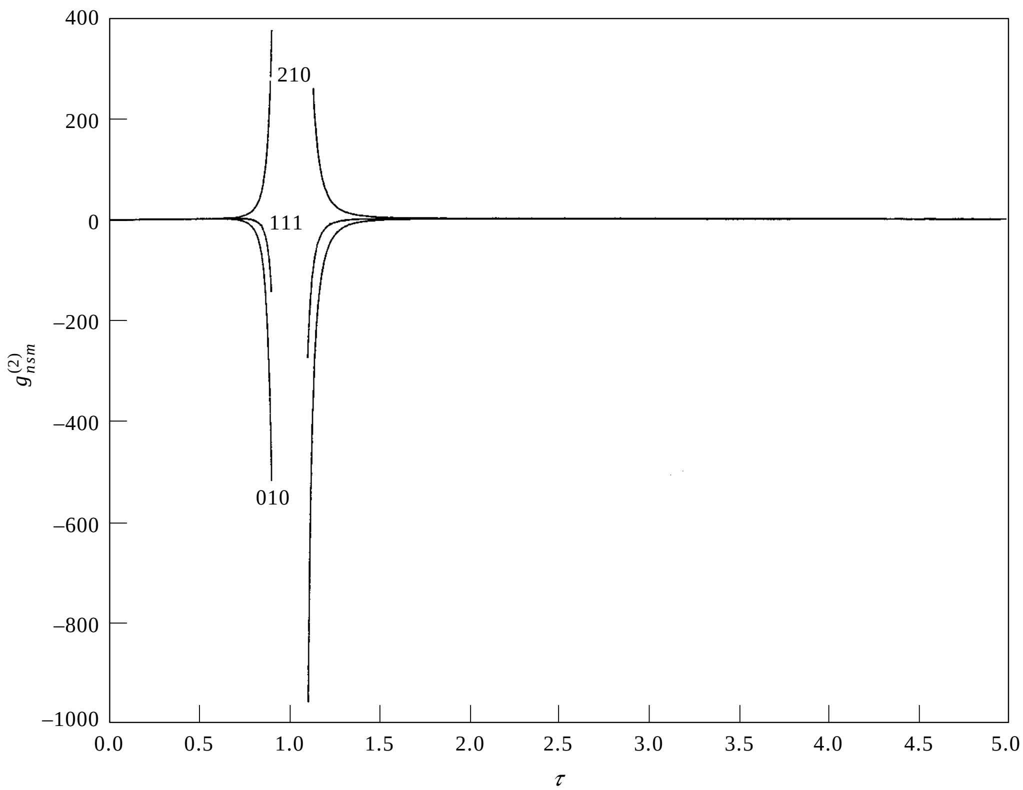

Figure 5. Second order expansion coefficients for eigenfrequencies in concentric spheroidal–spherical cavities—Dirichlet conditions.

case examined here. Yet another consequence of the monotonicity is that $g^{(1)}_{nsm} < 0$ if $\tau > 1$ and $g^{(1)}_{nsm} > 0$ if $\tau < 1$ (for the cavities of Figures 1 and 2). The monotonicity theorem and so the former relations do not hold for Neumann conditions [7].

A general remark on the $g^{(1)}_{nsm}$ for the Neumann conditions is that, for $s > 2$, their values stabilize and become almost independent of s (for the Dirichlet conditions they are exactly independent of s, for each $s \geqslant 1$). This remark agrees with the results of references [1, 5] and is readily explained by the use of equations (B8), (B11), (A7) and (A15), because for $s \to \infty$, $(x^0_2)_{ns} \to \infty$ [6], so $g^{(1)}_{nsm} \to -F$ (equal to $g^{(1)}_{nsm}$ for Dirichlet conditions). In the special case where $n = m = 0$, or $n = 3$, $m = 2$, there results from equation (A15) that $M = 0$, so $g^{(1)}_{nsm} = -F = -1/3$ from equation (A7), independent of s (see Table 2).

Figure 3 is a plot of the roots $(x^0_2)_{ns}$ ($n = 0$, $s = 1$–3) of equation (12) versus τ, for a concentric spherical cavity with radii R_1 and R_2 and for Dirichlet conditions. The roots for $n = 1$–3, $s = 1$–3 are very close to the corresponding ones for $n = 0$, so their plots are almost the same as the ones shown in this figure. For $\tau \to 1$, $(x^0_2)_{ns} \to \infty$, as is expected for $R_1 \to R_2$.

In Figures 4 and 5 are plotted $g^{(1)}_{nsm}$ and $g^{(2)}_{nsm}$, respectively, versus τ, for concentric spheroidal–spherical cavities with Dirichlet conditions. In Figure 4 one sees that $g^{(1)}_{nsm} > 0$ for $\tau < 1$ and $g^{(1)}_{nsm} < 0$ for $\tau > 1$, in agreement with the former remarks. For $\tau \to 1$, $|g^{(1),(2)}_{nsm}| \to \infty$.

In Figure 6 the roots $(x^0_2)_{ns}$ ($n = 0$, 1 and $s = 1$–3) of equation (28) are plotted versus τ, for a concentric spherical cavity with radii R_1 and R_2 and for Neumann conditions. We have set $(x^0_2)_{01} = 0$, as in Table 2, corresponding to $k^0 = 0$ and to a constant eigenfunction.

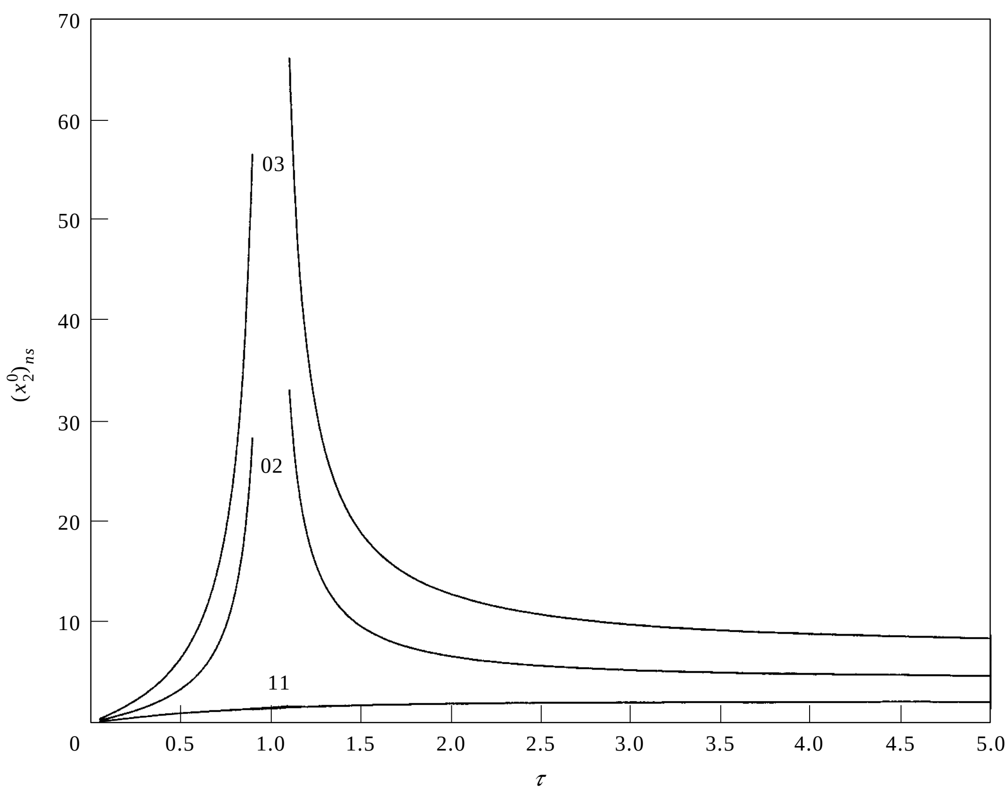

Figure 6. Eigenfrequencies in concentric spherical cavities—Neumann conditions.

(In Tables 3 and 4 of reference [1] this degenerate root was omitted, with the consequence that s in reference [1], for $n = 0$, corresponds to $s + 1$ here.) So, also in this case, $f_{01}(0) = 0$ and $f_{010}(v) = 0$ (the values of $g^{(1)}_{010}$ and $g^{(2)}_{010}$ do not matter), while $(x^0_2)_{ns}$ $(n \geqslant 0,\ s \geqslant 1)$, $f_{ns}(0)$ and $f_{nsm}(v)$ for Neumann conditions, are smaller than the corresponding ones for Dirichlet conditions [7]. The roots for $ns = 12$, 22 are very close to the ones for $ns = 02$, while those for $ns = 13$, 23 are very close to the ones for $ns = 03$, so their plots are almost the same as those shown in Figure 6. For $\tau \to 1$, $(x^0_2)_{ns} \to \infty$, with the exception of the cases where $s = 1$. In these last cases $(x^0_2)_{n1}$ remains finite and continuous for $\tau \to 1$. This can be proved from equation (28), by setting it in the form $\mathrm{j}'_n(x^0_1)\mathrm{n}'_n(\tau x^0_1) - \mathrm{j}'_n(\tau x^0_1)\mathrm{n}'_n(x^0_1) = 0$, where $\tau = 1 + \delta\tau$, and by keeping only the first order term in its Taylor expansion, as $\delta\tau \to 0$. So, the former equation is reduced to $\mathrm{j}'_n(x^0_1)\mathrm{n}''_n(x^0_1) - \mathrm{j}''_n(x^0_1)\mathrm{n}'_n(x^0_1) = 0$, the left side of which, by the substitution of $\mathrm{n}''_n(x^0_1)$ and $\mathrm{j}''_n(x^0_1)$ from the differential equation for the spherical Bessel functions and the use of the Wronskian for these functions [6], becomes equal to $[(x^0_1)^2 - n(n+1)]/(x^0_1)^4$. The first non-negative root of this last expression is $(x^0_1)_{n1} = \sqrt{n(n+1)}$, equal to $(x^0_2)_{n1}$ in this special case where $\delta\tau \to 0$ $(\tau \to 1)$. The rest of the roots tend to ∞ in this case, as is evident from the former expression. In Figure 6, the curve for $(x^0_2)_{11}$ is continuous for $\tau = 1$, with the value $(x^0_2)_{11} = \sqrt{2}$, according to the above result with $n = 1$.

Finally, in Figures 7 and 8, $g^{(1)}_{nsm}$ and $g^{(2)}_{nsm}$, respectively, are plotted versus τ, for concentric spheroidal–spherical cavities with Neumann conditions. For $\tau \to 1$, $|g^{(1),(2)}_{nsm}| \to \infty$.

For $\tau \to \infty$, i.e., for $R_1 \to 0$ in Figure 1, these plotted results tend to those for a simple spheroidal cavity, with major and minor semiaxis R_2 and R'_2, respectively.

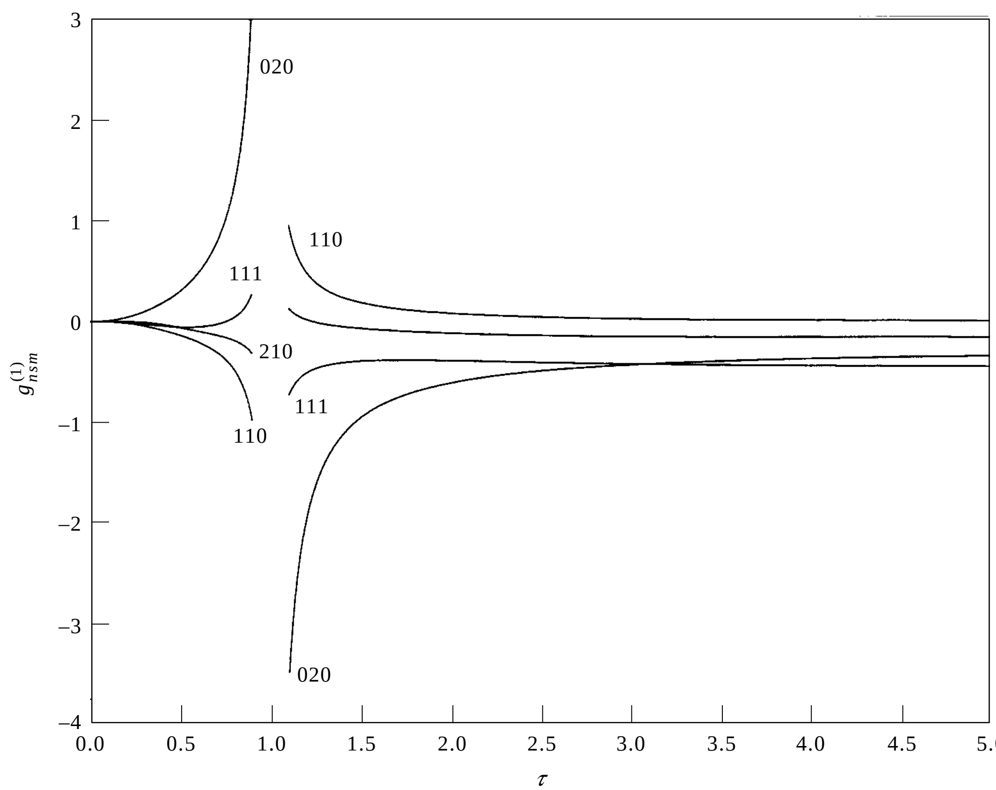

Figure 7. First order expansion coefficients for eigenfrequencies in concentric spheroidal–spherical cavities—Neumann conditions.

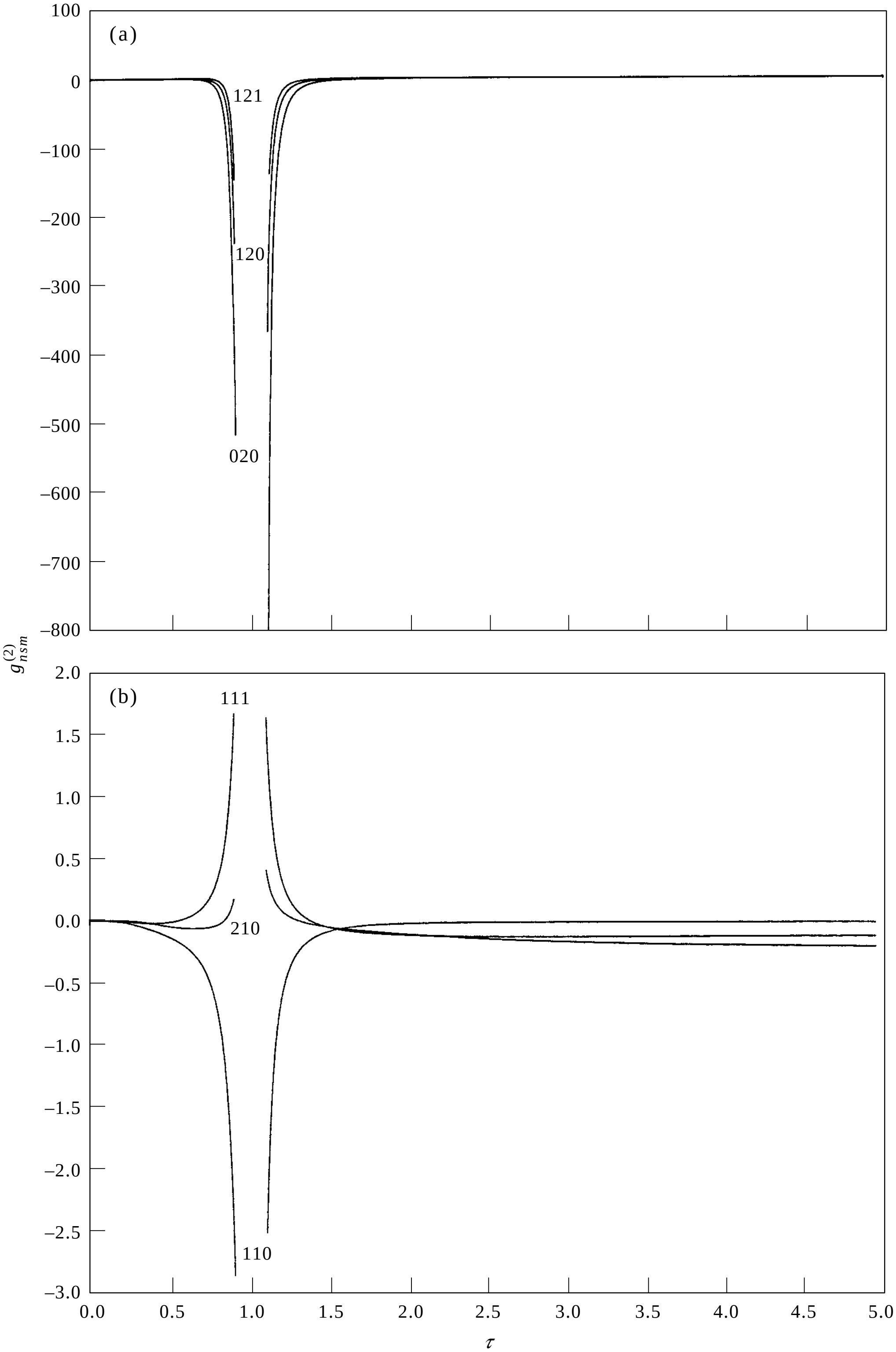

Figure 8. Second order expansion coefficients for eigenfrequencies in concentric spheroidal–spherical cavities—Neumann conditions.

For $\tau \to 0$, i.e., for $R_2 \to 0$ in Figure 2, all of these plotted results tend to zero, as is expected ($(x_2^0)_{ns} = k_{ns}^0 R_2 \to 0$, for $R_2 \to 0$, while $g_{nsm}^{(1),(2)} \to 0$ in this special case of a simple spherical cavity with radius R_1).

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APPENDIX A

The expressions for the various D's appearing in equation (8) and used in our calculations are the following.

A.1. DIRICHLET BOUNDARY CONDITIONS

$$D_{nn}^0 = u_{nn}(x_2, x_1), \qquad D_{nn}^{(2)} = \mp x_2 F u'_{nn}(x_2, x_1) \tag{A1, A2}$$

$$D_{nn}^{(4)} = x_2 G[3u'_{nn}(x_2, x_1) + x_2 u''_{nn}(x_2, x_1)] - x_2 F u'_{nn}(x_2, x_1), \tag{A3}$$

$$D_{n+2,n}^{(2)} = \pm x_2 \frac{(n+m+1)(n+m+2)}{2(2n+3)(2n+5)} u'_{n+2,n+2}(x_2, x_1), \tag{A4}$$

$$D_{n-2,n}^{(2)} = \pm x_2 \frac{(n-m-1)(n-m)}{2(2n-3)(2n-1)} u'_{n-2,n-2}(x_2, x_1), \tag{A5}$$

with

$$u_{vv}^i(x_2, x_1) = \mathrm{j}_v^i(x_2) - \mathrm{n}_v^i(x_2)\frac{\mathrm{j}_v(x_1)}{\mathrm{n}_v(x_1)}, \tag{A6}$$

where $i = 0$–2 denotes the number of primes over the corresponding symbols and

$$F = (n^2 + m^2 + n - 1)/(2n-1)(2n+3), \tag{A7}$$

$$G = \frac{(n+m+1)(n+m+2)(n+m+3)(n+m+4)}{8(2n+1)(2n+3)^2(2n+5)} + \frac{(n-m-1)(n-m)(n+m+1)(n+m+2)}{2(2n-1)^2(2n+3)^2} + \frac{(n-m-3)(n-m-2)(n-m-1)(n-m)}{8(2n-3)(2n-1)^2(2n+1)}. \tag{A8}$$

A.2. NEUMANN BOUNDARY CONDITIONS

$$D_{nn}^0 = x_2 p'_{nn}(x_2, x_1), \qquad D_{nn}^{(2)} = \mp x_2^2 F p''_{nn}(x_2, x_1) \mp M p_{nn}(x_2, x_1), \tag{A9, A10}$$

$$D_{nn}^{(4)} = x_2^2\, G[3p''_{nn}(x_2, x_1) + x_2\, p'''_{nn}(x_2, x_1)] + \frac{L}{2(2n+1)}[p_{nn}(x_2, x_1) + x_2\, p'_{nn}(x_2, x_1)]$$

$$- x_2^2\, Fp''_{nn}(x_2, x_1) - Mp_{nn}(x_2, x_1), \tag{A11}$$

$$D_{n+2,n}^{(2)} = \pm\frac{(n+m+1)(n+m+2)}{2(2n+3)(2n+5)}[x_2^2\, p''_{n+2,n+2}(x_2, x_1) - 2(n+3)p_{n+2,n+2}(x_2, x_1)], \tag{A12}$$

$$D_{n-2,n}^{(2)} = \pm\frac{(n-m-1)(n-m)}{2(2n-3)(2n-1)}[x_2^2\, p''_{n-2,n-2}(x_2, x_1) + 2(n-2)p_{n-2,n-2}(x_2, x_1)], \tag{A13}$$

with

$$p_{vv}^{i}(x_2, x_1) = \mathrm{j}_v^{i}(x_2) - \mathrm{n}_v^{i}(x_2)\frac{\mathrm{j}'_v(x_1)}{\mathrm{n}'_v(x_1)}, \tag{A14}$$

where $i = 0$–3 denotes the number of primes over the corresponding symbols and

$$M = \frac{1}{2n+1}\left[\frac{(n+1)(n^2-m^2)}{2n-1} - \frac{n[(n+1)^2-m^2]}{2n+3}\right], \tag{A15}$$

$$L = \frac{(n-m)(n+m+1)}{2n+1}\left[\frac{(n+1)(n+m)}{2n-1} + \frac{n(n-m+1)}{2n+3}\right]\left(\frac{n+m}{2n-1} - \frac{n-m+1}{2n+3}\right)$$

$$- \frac{n[(n+1)^2-m^2](n+m+2)(n+m+3)}{(2n+3)^2(2n+5)} + \frac{(n+1)(n-m-2)(n-m-1)(n^2-m^2)}{(2n-3)(2n-1)^2} \tag{A16}$$

In the former expressions the following integrals have been used:

$$I_1(n, n) = \int_0^{\pi} (\mathrm{P}_n^m)^2 \sin\theta \,\mathrm{d}\theta = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}, \tag{A17}$$

$$I_2(n, s) = \int_0^{\pi} \mathrm{P}_n^m\, \mathrm{P}_s^m \sin^3\theta \,\mathrm{d}\theta = \begin{cases} 2I_1(n, n)F, & s = n \\ -\dfrac{(n+m+1)(n+m+2)}{(2n+3)(2n+5)} I_1(n, n), & s = n+2 \\ -\dfrac{(n-m-1)(n-m)}{(2n-3)(2n-1)} I_1(n, n), & s = n-2 \end{cases}, \tag{A18}$$

$$I_3(n, n) = \int_0^{\pi} (\mathrm{P}_n^m)^2 \sin^5\theta \,\mathrm{d}\theta = 8I_1(n, n)G, \tag{A19}$$

$$I_4(n,s) = \int_0^\pi \frac{\mathrm{dP}_n^m}{\mathrm{d}\theta} \mathrm{P}_s^m \sin 2\theta \sin\theta \, \mathrm{d}\theta = \begin{cases} -2I_1(n,n)M, & s = n \\ \dfrac{2n(n+m+1)(n+m+2)}{(2n+3)(2n+5)} I_1(n,n), & s = n+2 \\ -\dfrac{2(n+1)(n-m-1)(n-m)}{(2n-3)(2n-1)} I_1(n,n), & s = n-2 \end{cases} \tag{A20}$$

$$I_5(n,n) = \int_0^\pi \frac{\mathrm{d}P_n^m}{\mathrm{d}\theta} \mathrm{P}_n^m \sin 2\theta \sin^3\theta \, \mathrm{d}\theta = -\frac{2}{2n+1} I_1(n,n)L. \tag{A21}$$

The various integrals of Legendre functions appearing in equations (A18)–(A21), are evaluated by using the recurrence relations and the orthogonal properties of these functions [3, 4, 6].

APPENDIX B

The closed form expressions for $x_{2,v}^{(1)}$ and $x_{2,v}^{(2)}$ appearing in equations (21) and (22) are obtained after lengthy but straightforward calculations by using equation (12) and equations (A1)–(A8) from Appendix A, in the Dirichlet case, as well as equation (28) and equations (A9)–(A16) from the same Appendix, in the Neumann case. In both cases one also uses various recurrence relations and Wronskians for the spherical Bessel functions [6]. These expressions are the following:

B.1. DIRICHLET BOUNDARY CONDITIONS

$$x_{2,v}^{(1)} = -x_2^0 F[1 - \tau \, \mathrm{n}_n^2(x_2^0)/\mathrm{n}_n^2(x_1^0)]^{-1}, \tag{B1}$$

$$\begin{aligned} x_{2,v}^{(2)} = \Bigg\{ & Z + x_{2,v}^{(1)} \frac{\mathrm{n}_n^2(x_2^0)}{\mathrm{n}_n^2(x_1^0)} \left[\tau x_2^0 F \frac{\mathrm{n}_n'(x_2^0)}{\mathrm{n}_n(x_2^0)} + x_{2,v}^{(1)} \left[\tau \frac{\mathrm{n}_n'(x_2^0)}{\mathrm{n}_n(x_2^0)} - \frac{1}{x_1^0} - \frac{\mathrm{n}_n'(x_1^0)}{\mathrm{n}_n(x_1^0)} \right] \right] \\ & - \frac{\tau^3 \mathrm{n}_n(x_2^0)}{4(2n+1)\mathrm{n}_n(x_1^0)} \left[\frac{[(n+1)^2 - m^2][(n+2)^2 - m^2]}{(2n+3)^2(2n+5) w_{n+2,n+2}(x_2^0, x_1^0)} \right. \\ & \left. + \frac{[(n-1)^2 - m^2](n^2 - m^2)}{(2n-3)(2n-1)^2 w_{n-2,n-2}(x_2^0, x_1^0)} \right] \Bigg\} \left[1 - \tau \frac{\mathrm{n}_n^2(x_2^0)}{\mathrm{n}_n^2(x_1^0)} \right]^{-1}, \end{aligned} \tag{B2}$$

where

$$\begin{aligned} Z = {} & x_{2,v}^{(1)} F + \frac{(x_{2,v}^{(1)})^2}{x_2^0} - x_2^0 G \\ & + \frac{x_2^0}{4(2n+1)} \left[\frac{[(n+1)^2 - m^2][(n+2)^2 - m^2][(x_2^0)^2 - (n+3)(2n+3)]}{(2n+3)^3(2n+5)} \right. \\ & \left. - \frac{[(n-1)^2 - m^2](n^2 - m^2)[(x_2^0)^2 - (n-2)(2n-1)]}{(2n-3)(2n-1)^3} \right] \end{aligned} \tag{B3}$$

$$w_{qq}(x_2^0, x_1^0) = \mathrm{j}_q(x_2^0)\mathrm{n}_q(x_1^0) - \mathrm{n}_q(x_2^0)\mathrm{j}_q(x_1^0), \tag{B4}$$

while F and G are given in equations (A7) and (A8), respectively.

The first of equations (18) is immediately verified, by a simple comparison of equation (B1) with the corresponding equation in reference [1] (equation (23) there), while the second of equations (18) is also verified, after some manipulation, with the use of equation (B2) and the corresponding equations in reference [1] (equations (23)–(26) there). These verifications constitute a very convincing check for the correctness of our results.

By using the small argument formulas for the various Bessel functions [6] in equations (B1) and (B2), as $R_1 \to 0$ (for the cavity of Figure 1), one obtains the expressions for $x_{2,v}^{(1)}$ and $x_{2,v}^{(2)}$ in the case of a simple spheroidal cavity, with major and minor semiaxis R_2 and R_2', respectively (i.e., in the absence of the inner sphere). The same expressions were also obtained by the independent solution, from the beginning, of this last problem and are the following:

$$x_{2,v}^{(1)} = -x_2^0 F, \qquad x_{2,v}^{(2)} = Z. \tag{B5}$$

It should be noticed that formulas (B5) do not contain any Bessel functions, while x_2^0 there, are roots of "the equation" $\mathrm{j}_n(x_2^0) = 0$.

B.2. NEUMANN BOUNDARY CONDITIONS

$$x_{2,v}^{(1)} = -U\left[1 - \tau^3\left(\frac{\mathrm{n}_n'(x_2^0)}{\mathrm{n}_n'(x_1^0)}\right)^2 \frac{(x_1^0)^2 - n(n+1)}{(x_2^0) - n(n+1)}\right]^{-1}, \tag{B6}$$

$$\begin{aligned}
x_{2,v}^{(2)} = {} & \frac{1}{(x_2^0)^2 - n(n+1)}\left\{W + \tau^3\left(\frac{\mathrm{n}_n'(x_2^0)}{\mathrm{n}_n'(x_1^0)}\right)^2 \frac{x_{2,v}^{(1)}}{x_2^0}\left[\left[x_{2,v}^{(1)}\left(1 + x_2^0\frac{\mathrm{n}_n''(x_2^0)}{\mathrm{n}_n'(x_2^0)} - \mathrm{x}_1^0\frac{\mathrm{n}_n''(x_1^0)}{\mathrm{n}_n'(x_1^0)}\right)\right.\right.\right. \\
& \left.\left. + (x_2^0)^2\frac{\mathrm{n}_n''(x_2^0)}{\mathrm{n}_n'(x_2^0)}F + \frac{\mathrm{n}_n(x_2^0)}{\mathrm{n}_n'(x_2^0)}M\right][(x_1^0)^2 - n(n+1)] - x_{2,v}^{(1)}[(x_1^0)^2 - 2n(n+1)]\right] \\
& - \frac{\tau^5\mathrm{n}_n'(x_2^0)}{4(2n+1)(x_2^0)^2\mathrm{n}_n'(x_1^0)} \\
& \times\left[\frac{[(n+1)^2 - m^2][(n+2)^2 - m^2][(x_2^0)^2 - n(n+3)][(x_1^0)^2 - n(n+3)]}{(2n+3)^2(2n+5)w_{n+2,n+2}'(x_2^0, x_1^0)}\right. \\
& \left.\left. + \frac{[(n-1)^2 - m^2][(n^2 - m^2)][(x_2^0)^2 - (n-2)(n+1)][(x_1^0)^2 - (n-2)(n+1)]}{(2n-3)(2n-1)^2 w_{n-2,n-2}'(x_2^0, x_1^0)}\right]\right\} \\
& \times\left[1 - \tau^3\left(\frac{\mathrm{n}_n'(x_2^0)}{\mathrm{n}_n'(x_1^0)}\right)^2\frac{(x_1^0)^2 - n(n+1)}{(x_2^0)^2 - n(n+1)}\right]^{-1},
\end{aligned} \tag{B7}$$

where

$$U = x_2^0\left[F - \frac{M}{(x_2^0)^2 - n(n+1)}\right], \tag{B8}$$

$$W = -x_{2,v}^{(1)}\, n(n+1)\left[\frac{x_{2,v}^{(1)}}{x_2^0} + 2F\right] + \frac{x_2^0}{4(2n+1)}$$

$$\times \left[\frac{[(n+1)^2 - m^2]\,[(n+2)^2 - m^2]\,[(x_2^0)^2 - n(n+3)]\,[(x_2^0)^2 - (n+2)\,(2n+3)]}{(2n+3)^3(2n+5)}\right.$$

$$\left. - \frac{[(n-1)^2 - m^2]\,(n^2 - m^2)\,[(x_2^0)^2 - (n-2)\,(n+1)]\,[(x_2^0)^2 - (n-1)\,(2n-1)]}{(2n-3)\,(2n-1)^3}\right]$$

$$- x_2^0\,[(x_2^0)^2 + n(n+1)]G + \frac{x_2^0}{2(2n+1)}\,L, \tag{B9}$$

$$w'_{qq}\,(x_2^0, x_1^0) = \mathrm{j}'_q\,(x_2^0)\mathrm{n}'_q\,(x_1^0) - \mathrm{n}'_q\,(x_2^0)\mathrm{j}'_q\,(x_1^0), \tag{B10}$$

while M and L are given in equations (A15) and (A16), respectively.

The first of equations (18) is immediately verified, by a simple comparison of equation (B6) with the corresponding equation in reference [1] (equation (35) there), while the second of equations (18) is also verified, after lengthy manipulation, by using equation (B7) and the corresponding equations in [1] (equations (36)–(38) there). These verifications constitute a very convincing check for the correctness of our results.

Finally, following the same procedure as for the Dirichlet case, one can obtain the expressions for $x_{2,v}^{(1)}$ and $x_{2,v}^{(2)}$ in a simple spheroidal cavity with major and minor semiaxis R_2 and R'_2, respectively. These expressions, which are the following,

$$x_{2,v}^{(1)} = -U, \qquad x_{2,v}^{(2)} = \frac{W}{(x_2^0)^2 - n(n+1)}, \tag{B11}$$

do not contain any Bessel functions, while x_2^0 there are roots of "the equation" $\mathrm{j}'_n\,(x_2^0) = 0$.