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LETTERS TO THE EDITOR

COMMENTS ON "FAST EIGENVALUE SENSITIVITY CALCULATIONS FOR SPECIAL STRUCTURES OF SYSTEM MATRIX DERIVATIVES"

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Recently, in this journal an interesting paper was published [1] in which a general rank-1 matrix formula for eigenvalue sensitivity evaluation was presented. The principal aim of this letter is to make a positive comment on the above mentioned paper and to give a practical example emphasizing the practical value and importance of the sensitivity expressions derived in reference [1].

The mechanical system taken as an application example is shown in Figure 1. It consists of a n mass oscillator, damped by a single viscous damper of damping constant c acting at the pth mass. The $2n \times 2n$ system matrix is

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{D} \end{bmatrix}, \tag{1}$$

where $\mathbf{I}$ is the $n \times n$ unit matrix and the mass, stiffness and damping matrices are as in equation (2).

In order to obtain the sensitivities of the eigenvalues of the system with respect to some system parameter, the formulas in reference [1] are well applicable.

A fluid damper consists of a piston of length L, diameter D which has n' holes of diameter d; see Figure 2. The damping constant c of the damper can be shown to be [2]

$$c = (8\pi L\eta/n')(D/d)^4, \tag{3}$$

where η represents the viscosity of the damper oil. Let the system parameter with respect to which the eigenvalue sensitivity will be calculated be d: i.e., the diameter of the holes on the piston. This selection seems logically meaningful since, due to sedimentation of the oil, the influence of the variation of d on the eigencharacteristics of the system can have practical consequences.

According to formula (7) in reference [1] the first order sensitivity of the ith eigenvalue can be written as

$$\mathring{\lambda}_i := \partial\lambda_i/\partial d = \mathbf{v}_i^{\mathrm{T}}\mathring{\mathbf{A}}\mathbf{u}_i, \tag{4}$$

where $\mathbf{u}_i$ and $\mathbf{v}_i$ are defined as the right and left eigenvectors of the system matrix $\mathbf{A}$, normalized such that $\mathbf{v}_i^{\mathrm{T}}\mathbf{u}_i = 1$ and $\mathbf{v}_j^{\mathrm{T}}\mathbf{u}_i = 0$ for $j \neq i$. $\mathring{\mathbf{A}}$ is a $2n \times 2n$ matrix consisting of zeros as elements except the element at $(n + p, n + p)$—location which is simply $\alpha/m_p d^5$ where $\alpha = 32\pi L\eta D^4/n'$ is introduced. It is obvious that the matrix $\mathring{\mathbf{A}}$ described above is a rank-1 matrix. The formula (4) yields

$$\mathring{\lambda}_i(d) = (\alpha/m_p d^5)u_{i,n+p}v_{i,n+p}, \tag{5}$$

where $u_{i,n+p}$ and $v_{i,n+p}$ represent the $(n + p)$th elements of the eigenvectors $\mathbf{u}_i$ and $\mathbf{v}_i$ respectively.

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$$\mathbf{M} = \mathbf{diag}\,(m_i), \quad \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 & & & & \\ -k_2 & k_2 + k_3 & -k_3 & & \mathbf{0} & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ \mathbf{0} & & & -k_{n-1} & k_{n-1} + k_n & -k_n \\ & & & & -k_n & k_n \end{bmatrix} p, \qquad \mathbf{D} = \begin{bmatrix} & & & p & \\ \cdot & & & \cdot & \\ & \cdot & \mathbf{0} & \cdot & \\ & & \cdot & \cdot & \\ & \mathbf{0} & & \cdot & \\ \cdots & \cdots & \cdots & c & \\ & & & & \ddots \end{bmatrix}. \tag{2}$$

Hence, if the diameter d of the holes is changed by an amount of Δd, the eigenvalues of the modified system can be given in the first approximation as

$$\lambda_i(d + \Delta d) \approx \lambda_i(d) + \dot{\lambda}_i \, \Delta d. \tag{6}$$

The second order sensitivity formula (18) of reference [1] for a rank-1 matrix $\dot{\mathbf{A}}$,

$$\ddot{\lambda}_i = \mathbf{v}_i^{\mathrm{T}} \ddot{\mathbf{A}} \mathbf{u}_i + 2 \sum_{j \neq i} \frac{\dot{\lambda}_i \dot{\lambda}_j}{\lambda_i - \lambda_j}, \tag{7}$$

yields

$$\ddot{\lambda}_i(d) = u_{i,n+p} v_{i,n+p} \left[\frac{-5\alpha}{m_p d^6} + 2\left(\frac{\alpha}{m_p d^5}\right)^2 \sum_{j \neq i} \frac{u_{j,n+p} v_{j,n+p}}{\lambda_i - \lambda_j} \right]. \tag{8}$$

Therefore, the modified eigenvalues can be obtained, in the second approximation via

$$\lambda_i(d + \Delta d) \approx \lambda_i(d) + \dot{\lambda}_i(d) \, \Delta d + \ddot{\lambda}_i(d)(\Delta d)^2/2. \tag{9}$$

The calculation of the third order approximation is not considered since it will not have an appreciable effect on the results. Therefore, it is not evaluated.

Consider the simple but instructive example in Figure 3 with only two degrees of freedom, where $m_1 = 100$ kg, $m_2 = 50$ kg, $k_1 = 10\,000$ N/m, $k_2 = 1500$ N/m,

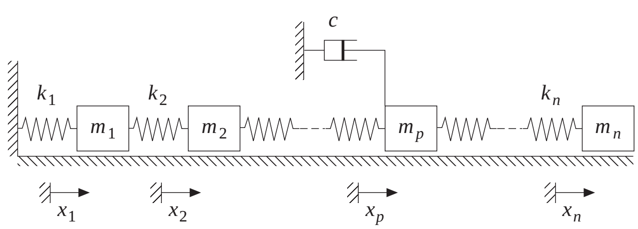

Figure 1. n-mass oscillator viscously damped by a fluid damper acting at the pth mass.

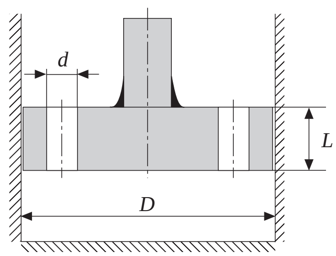

Figure 2. Sketch of a fluid damper.

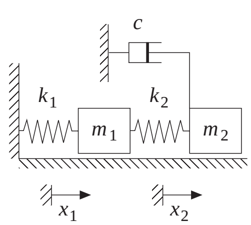

Figure 3. Two-mass oscillator viscously damped at the second mass.

TABLE 1

Exact and approximate eigenvalues of the sample system

	$\lambda_{1,2}$	λ_3	λ_4
From **A**	$-0{\cdot}09758187 \pm 10{\cdot}77516143i$	$-13{\cdot}74638985$	$-1{\cdot}87953064$
From (6)	$-0{\cdot}09773129 \pm 10{\cdot}77516259i$	$-13{\cdot}74637626$	$-1{\cdot}87952658$
From (9)	$-0{\cdot}09773121 \pm 10{\cdot}77516257i$	$-13{\cdot}74639016$	$-1{\cdot}87953063$

$c = 790$ N/m/s, $L = 0{\cdot}020$ m, $n' = 4$, $d = 0{\cdot}003$ m, $\eta = 0{\cdot}19891259391505$ kg/ms (in order to have an integer c-value).

The eigenvalues of the system with the nominal diameter of holes $d = 0{\cdot}003$ m are obtained from matrix **A** in equation (1) via MATLAB as $\lambda_{1,2} = -0{\cdot}09765571 \pm 10{\cdot}77527053i$, $\lambda_3 = -13{\cdot}72183260$, $\lambda_4 = -1{\cdot}88285598$. The normalized eigenvectors $\mathbf{u} = [\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3\ \mathbf{u}_4]$ and $\mathbf{v} = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4]$ are

$$\mathbf{u}_1 = \begin{bmatrix} -0{\cdot}04077701 + 0{\cdot}08165328i \\ 0{\cdot}01443807 - 0{\cdot}00025003i \\ -0{\cdot}87585412 - 0{\cdot}44735720i \\ 0{\cdot}00128415 + 0{\cdot}15559857i \end{bmatrix}, \qquad \mathbf{u}_2 = \mathbf{u}_1^* \quad \text{(comp. conj.)},$$

$$\mathbf{u}_3 = \begin{bmatrix} 0{\cdot}00359039 \\ 0{\cdot}07259507 \\ -0{\cdot}04926680 \\ -0{\cdot}99613746 \end{bmatrix}, \qquad \mathbf{u}_4 = \begin{bmatrix} -0{\cdot}05888224 \\ -0{\cdot}46534695 \\ 0{\cdot}11086679 \\ 0{\cdot}87618129 \end{bmatrix},$$

$$\mathbf{v}_1 = \begin{bmatrix} -2{\cdot}52165876 + 4{\cdot}86024402i \\ 0{\cdot}27448027 - 0{\cdot}71461167i \\ -0{\cdot}44889760 - 0{\cdot}2380911i \\ 0{\cdot}03311582 - 0{\cdot}02278512i \end{bmatrix}, \qquad \mathbf{v}_2 = \mathbf{v}_1^*,$$

$$\mathbf{v}_3 = \begin{bmatrix} 1{\cdot}58745440 \\ -2{\cdot}43055039 \\ -0{\cdot}11568822 \\ -1{\cdot}16956430 \end{bmatrix}, \qquad \mathbf{v}_4 = \begin{bmatrix} 0{\cdot}08594704 \\ -2{\cdot}51030486 \\ -0{\cdot}04564717 \\ -0{\cdot}18037500 \end{bmatrix}.$$

Let one assume that the nominal diameter d is diminished by a small amount of $\Delta d = 0{\cdot}000001$ m, due to sedimentation. The exact eigenvalues of the so-modified system are calculated from **A** via MATLAB. Those values and approximate eigenvalues obtained via first and second order approximations as given in equations (6) and (9) are collected in Table 1. The comparison of the approximate values with the exact values indicate clearly that equations (6) and (9) give accurate approximations to the eigenvalues of the modified system without having to resolve the eigenvalue problem of the matrix **A**.

REFERENCES

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AUTHORS' REPLY

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The authors wish to thank Professor Gürgöze for his pertinent comments on their paper. Professor Gürgöze confirmed a belief which the authors have that the findings of the paper are well applicable to a wide class of engineering problems. While their main research applications were in electrical engineering, they tried a number of other mechanical and civil engineering applications as well with just as much success. Nevertheless, the work by Professor Gürgöze went beyond the presentation of just another application by providing a general "canonical form" of the system matrix **A** for which the derivatives with respect to parameters of interest always yield rank-1 matrices. Furthermore, Professor Gürgöze was able to derive the analytical forms of the dynamic mode (eigenvalue) sensitivities for a standard mechanical system.

Professor Gürgöze, however, did not provide an assessment of the computer time and memory requirements for the mechanical system considered, which would be of a great interest to the authors. This is because, in their testing of the novel eigenvalue sensitivity formulas on large systems, they noticed remarkable savings in CPU and memory requirement as compared to the conventional sensitivity formulas.

Professor Gürgöze also noted that the third order eigenvalue sensitivities were not needed in analyzing the mechanical system considered. The authors agree with this assessment with, however, a word of caution in generalizing such a conclusion. In many other applications, the third order sensitivities could have appreciable impact on the results, especially when relatively large parameter changes are being considered.