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VIBRATION OF MULTI-SPAN NON-UNIFORM BEAMS UNDER MOVING LOADS BY USING MODIFIED BEAM VIBRATION FUNCTIONS

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Based on Hamilton's principle, the vibration of a multi-span non-uniform beam subjected to a moving load is analysed by using modified beam vibration functions as the assumed modes. The modified beam vibration functions satisfy the zero deflection conditions at all the intermediate point supports as well as the boundary conditions at the two ends of the beam. Numerical results are presented for both uniform and non-uniform beams under moving loads of various velocities. Examples show that this method converges very quickly and good results are obtained.

1. INTRODUCTION

There are many engineering structures such as bridges, car-parks and jetties which are subjected to moving loads. The vibration of structures due to moving loads has been analysed extensively in recent years. Many methods have been presented for its prediction, but only the notable ones are cited here. Fryba [1] presented in his monograph various analytical solutions for vibration problems of simple and continuous beams under moving loads. The modal superposition method was later employed by Hayashikawa and Watanabe [2] and Wu and Dai [3] to analyse similar problems. However, the former [2] used the "eigen stiffness matrix method", while the latter [3] used the transfer matrix method for the initial evaluation of natural frequencies and vibration modes. More recently Cai *et al*. [4] applied the U-transformation and modal superposition method and presented an exact solution for the dynamic response of an infinite uniform beam resting on periodic roller supports and subjected to a moving load.

The finite element method has also been used. The governing partial differential equations are first changed into a series of ordinary differential equations by spatial discretization. Then either modal superposition or step-by-step integration in time domain is applied to solve these equations [5–11]. Later Henchi *et al*. [12] used exact dynamic stiffness elements under the framework of finite element approximation to study the dynamic response of multi-span structures under a convoy of moving loads.

Recently Lee [13] utilized Hamilton's principle to solve the dynamic response of a beam with intermediate point constraints subjected to a moving load. He used the vibration modes of a simply-supported beam as the assumed modes. As these assumed modes do not satisfy the zero deflection conditions at the intermediate point constraints, he modelled the intermediate point constraints as very stiff linear springs and inevitably some errors were involved. In this paper, the modified beam vibration functions are used as the assumed modes. As all the zero deflection conditions at the intermediate point supports

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as well as the boundary conditions at the two ends of the beam are satisfied, quicker convergence and more accurate results can be expected.

2. ASSUMPTIONS AND FORMULATIONS

A continuous linear elastic Bernoulli–Euler beam with $(Q + 1)$ point supports subjected to N moving loads is shown in Figure 1.

The loads $\{P_s, s = 1, 2, \dots, N\}$ move as a group at a prescribed velocity $v(t)$ along the axial direction from left to right. The locations of the loads and the vibrations of the beam

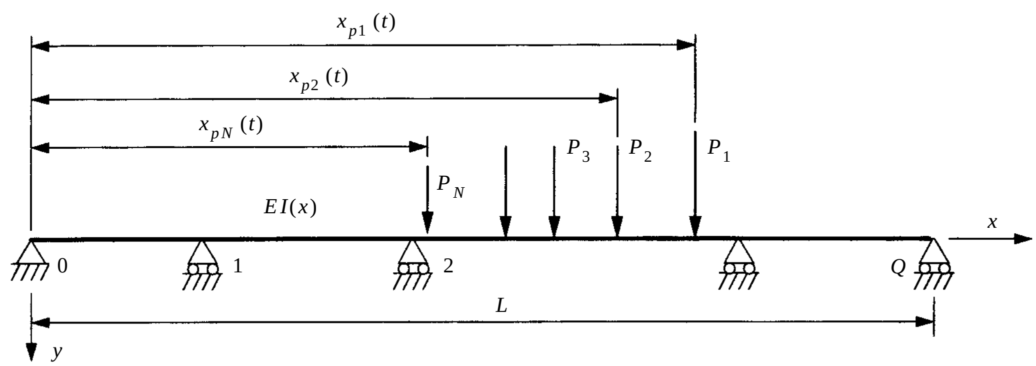

Figure 1. A continuous beam with $(Q - 1)$ intermediate point supports under N moving loads.

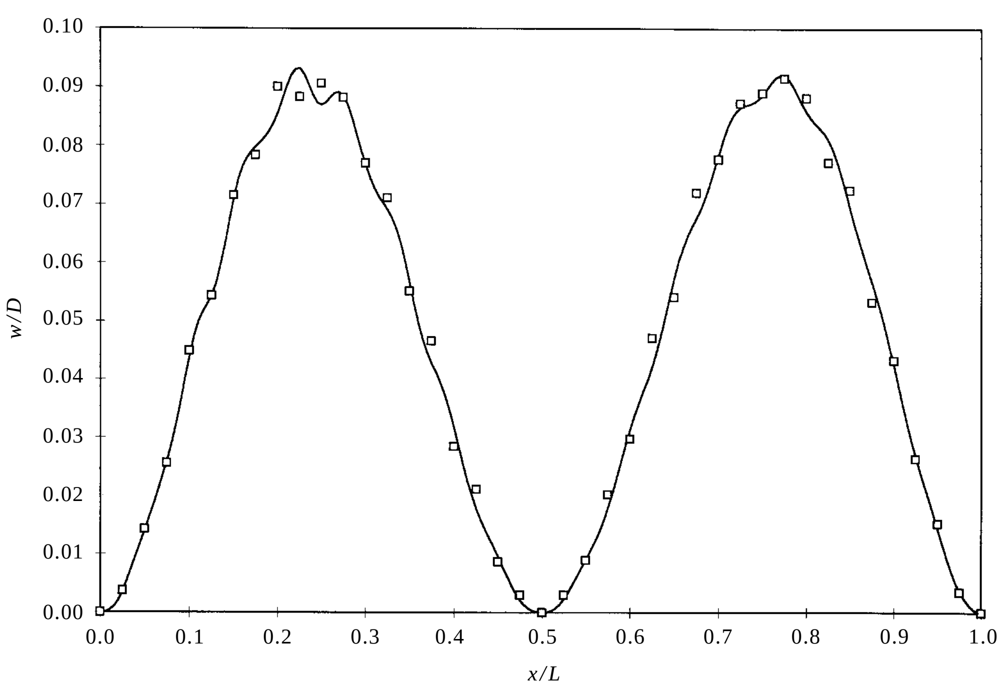

Figure 2. Deflection under the moving load at $v = 17{\cdot}3$ m/s ($a = 0{\cdot}111$): ——, present; □□□, reference [13].

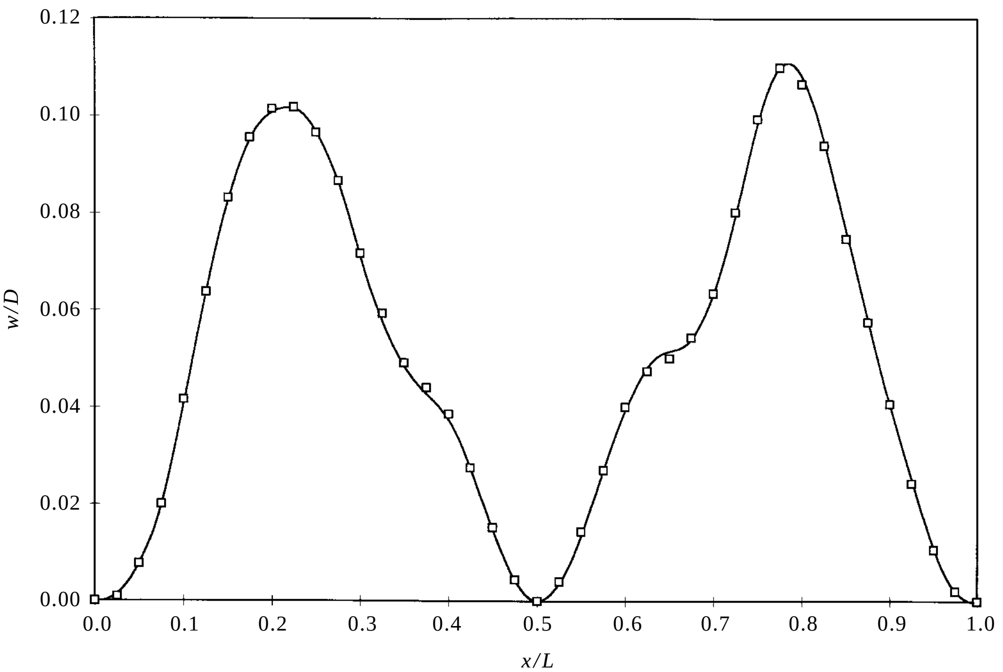

Figure 3. Deflection under the moving load at $v = 77{\cdot}8$ m/s ($a = 0{\cdot}5$): ——, present; □□□□, reference [13].

are denoted by $\{x_{ps}(t),\ s = 1, 2, \ldots, N\}$ and $w(x, t)$, respectively. The kinetic energy $\bar{V}$, the bending energy $\bar{U}$ and the work done by external force $\bar{W}$ are, respectively,

$$\bar{V} = \tfrac{1}{2}\int_0^L \rho A(x)\left[\frac{\partial w(x, t)}{\partial t}\right]^2 \mathrm{d}x, \tag{1}$$

$$\bar{U} = \tfrac{1}{2}\int_0^L EI(x)\left[\frac{\partial^2 w(x, t)}{\partial x^2}\right]^2 \mathrm{d}x, \tag{2}$$

$$\bar{W} = \sum_{s=1}^{N} P_s w[x_{ps}(t), t][u(t - \tau_s^1) - u(t - \tau_s^2)], \tag{3}$$

where ρ is the density, E is the Young's modulus, $A(x)$ is the area, $I(x)$ is the moment of inertia of the cross-section, τ_s^1 and τ_s^2 are the times when the load P_s just comes onto and leaves the beam respectively, and $u(t)$ is the unit step function defined as

$$u(t) = \begin{cases} 1, & t \geqslant 0 \\ 0, & t < 0 \end{cases}. \tag{4}$$

By separation of variables, the vibration of the beam $w(x, t)$ can be expressed as

$$w(x, t) = \sum_{i=1}^{n} q_i(t) X_i(x), \tag{5}$$

where $\{X_i(x), i = 1, 2, \ldots, n\}$ are the assumed vibration modes which satisfy the boundary conditions *a priori* and $\{q_i(t), i = 1, 2, \ldots, n\}$ are generalized co-ordinates. The velocity of vibration and the curvature of the beam are, respectively,

$$\frac{\partial w(x, t)}{\partial t} = \sum_{i=1}^{n} \dot{q}_i(t) X_i(x), \tag{6}$$

$$\frac{\partial^2 w(x, t)}{\partial x^2} = \sum_{i=1}^{n} q_i(t) X_i''(x). \tag{7}$$

Substituting equations (6), (7) and (5) into equations (1), (2) and (3), respectively, one obtains

$$\bar{V} = \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_0^L \rho A(x) \dot{q}_i(t) X_i(x) \dot{q}_j(t) X_j(x)\, \mathrm{d}x = \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \dot{q}_i(t) m_{ij} \dot{q}_j(t), \tag{8}$$

$$\bar{U} = \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_0^L EI(x) q_i(t) X_i''(x) q_j(t) X_j''(x)\, \mathrm{d}x = \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} q_i(t) k_{ij} q_j(t), \tag{9}$$

$$\bar{W} = \sum_{s=1}^{N} \sum_{i=1}^{n} P_s q_i(t) X_i[x_{ps}(t)][u(t - \tau_s^1) - u(t - \tau_s^2)], \tag{10}$$

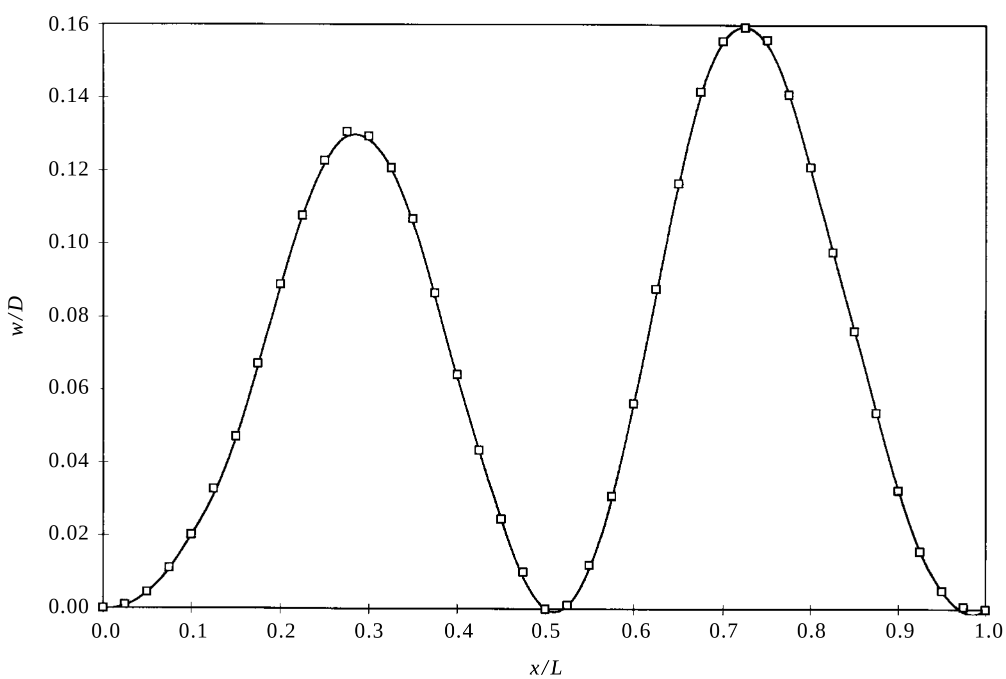

Figure 4. Deflection under the moving load at v = 171 m/s (a = 1·1): ——, present; □□□□, reference [13].

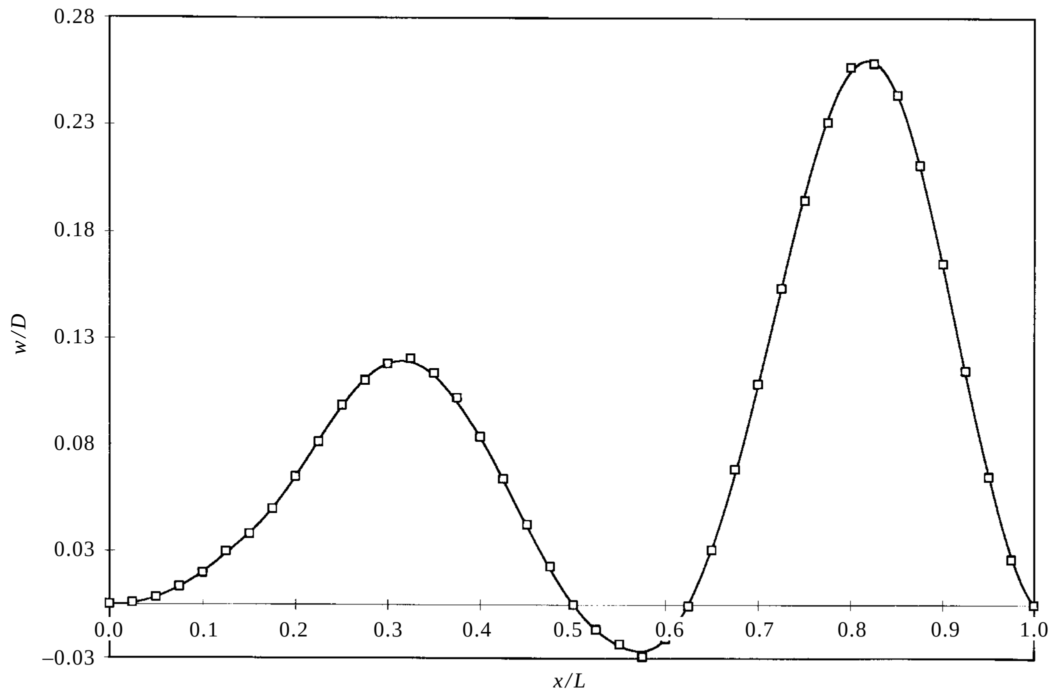

Figure 5. Deflection under the moving load at $v = 233$ m/s ($a = 1{\cdot}5$): ——, present; □□□□, reference [13].

where

$$m_{ij} = \int_0^L \rho A(x) X_i(x) X_j(x)\, \mathrm{d}x, \tag{11}$$

$$k_{ij} = \int_0^L EI(x) X_i''(x) X_j''(x)\, \mathrm{d}x, \tag{12}$$

are generalized mass and stiffness matrices, respectively.

The Lagrangian function $\bar{L}$ for the beam is $\bar{V} - (\bar{U} - \bar{W})$ where $(\bar{U} - \bar{W})$ is the total potential energy, and the Euler-Lagrange equation is, therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \bar{L}}{\partial \dot{q}_i}\right) - \frac{\partial \bar{L}}{\partial q_i} = 0. \tag{13}$$

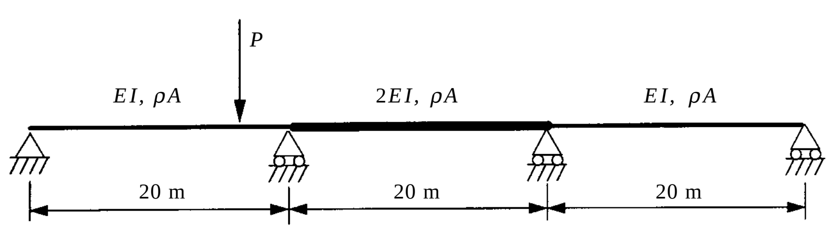

Figure 6. A three-span continuous stepped beam under a single moving load.

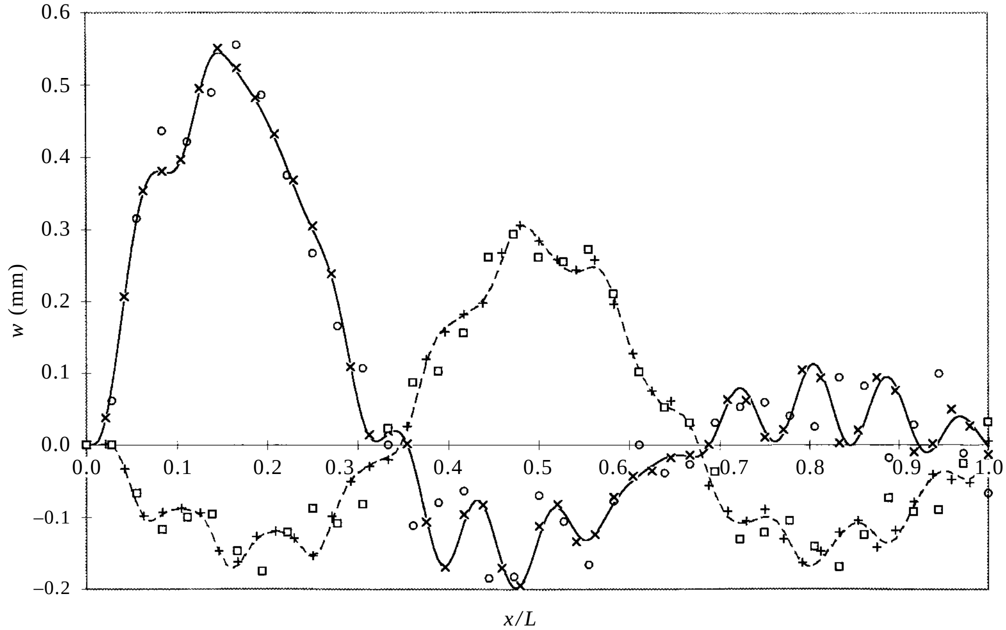

Figure 7. Deflections at mid-span positions: ——, present—span 1; – – –, present—span 2; ×, reference [16]—span 1; +, reference [16]—span 2; ○, reference [2]—span 1; □, reference [2]—span 2.

Substituting equations (8)–(10) into equation (13), one has

$$\sum_{j=1}^{n} m_{ij}\ddot{q}_j(t) + \sum_{j=1}^{n} k_{ij}q_j(t) = \sum_{s=1}^{N} P_s X_i[x_{ps}(t)][u(t-\tau_s^1) - u(t-\tau_s^2)] \qquad i = 1, 2, \ldots, n. \tag{14}$$

Once the assumed vibration modes $X_i(x)$ are chosen, m_{ij} and k_{ij} can be easily obtained from numerical integration using Gaussian quadrature.

3. ASSUMED VIBRATION MODES

The assumed vibration modes essentially comprise the vibration modes of a single span beam, modified by cubic spline expressions. They can be written as

$$X_i(x) = \bar{X}_i(x) + \tilde{X}_i(x), \tag{15}$$

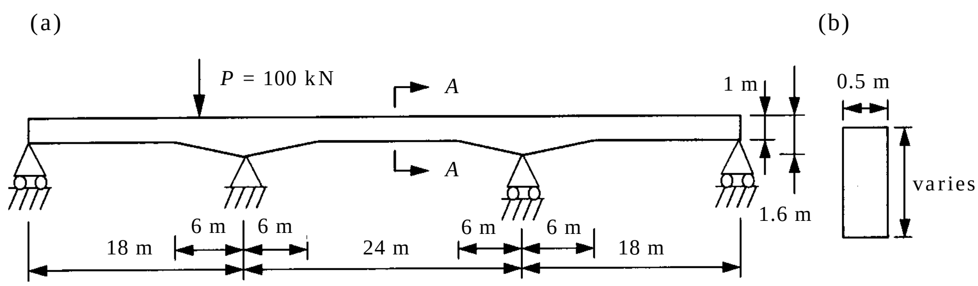

Figure 8. A three-span continuous haunched beam under a single moving load: (a) elevation; (b) section AA.

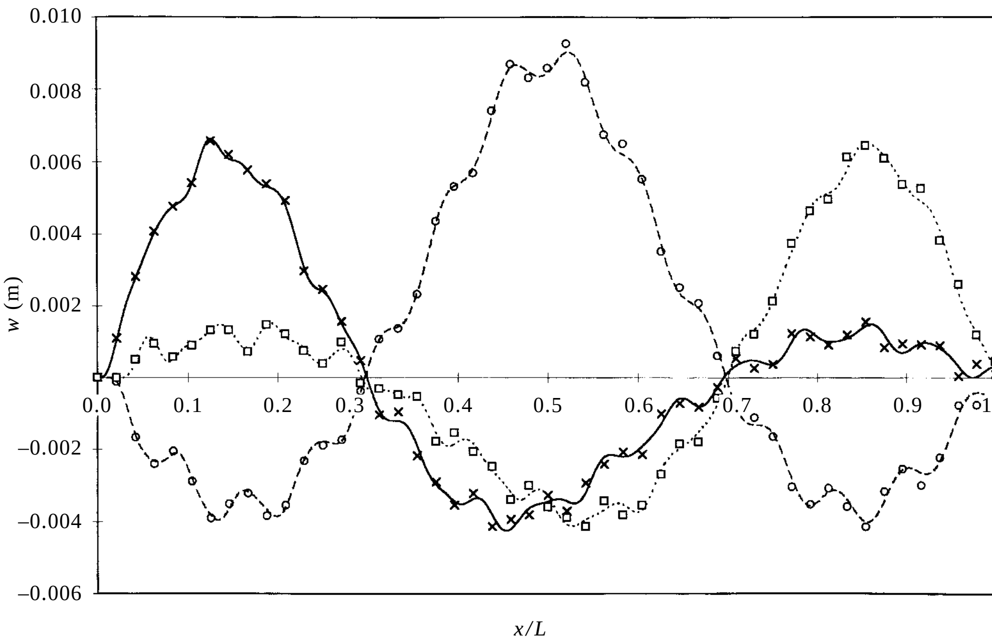

Figure 9. Deflections at mid-span positions: ——, present—span 1; – – –, present—span 2; · · · ·, present—span 3; ×, reference [16]—span 1; ○, reference [16]—span 2; □, reference [16]—span 3.

where $\{\bar{X}_i(x),\ i = 1, 2, \ldots, n\}$ are the vibration modes of a hypothetical prismatic beam of total length L with the same end supports but without the intermediate supports, and $\{\tilde{X}_i(x),\ i = 1, 2, \ldots, n\}$ are the augmenting cubic spline expressions which are so chosen that each $X_i(x)$ satisfies the boundary conditions at the two ends and the zero deflection conditions at the intermediate point supports. The vibration modes $\bar{X}_i(x)$ are Fourier sine series for a simply supported beam, and those for other end conditions are given in reference [14]. Note that cubic spline expressions are chosen instead of a higher order polynomial so that convergence is better.

For example, consider a continuous beam with simply-supported ends at x_0 and x_Q, and with intermediate point supports at $\{x = x_j,\ \ j = 1, 2, \ldots, Q-1\}$. The boundary conditions at the ends can be expressed in the form

$$X_i(x_0) = X_i(x_Q) = X_i''(x_0) = X_i''(x_Q) = 0, \tag{16}$$

which implies

$$\tilde{X}_i(x_0) = \tilde{X}_i(x_Q) = \tilde{X}_i''(x_0) = \tilde{X}_i''(x_Q) = 0. \tag{17}$$

For the intermediate point supports, one has

$$X_i(x_j) = \bar{X}_i(x_j) + \tilde{X}_i(x_j) = 0, \qquad j = 1, 2, \ldots, Q-1, \tag{18}$$

and hence

$$\tilde{X}_i(x_j) = -\bar{X}_i(x_j), \qquad j = 1, 2, \ldots, Q-1. \tag{19}$$

If the following coefficients are introduced

$$y_{ij} = \tilde{X}_i(x_j) = -\bar{X}_i(x_j), \quad \theta_{ij} = \tilde{X}_i'(x_j), \qquad i = 1, 2, \ldots, n; \quad j = 0, 1, \ldots, Q, \quad (20, 21)$$

the augmenting cubic spline expression $\tilde{X}_i(x)$ can be written as

$$\tilde{X}_i(x) = H_{1j}(\xi_j)y_{i(j-1)} + H_{2j}(\xi_j)\theta_{i(j-1)} + H_{3j}(\xi_j)y_{ij} + H_{4j}(\xi_j)\theta_{ij}, \qquad x \in [x_{j-1}, x_j],$$
$$j = 1, 2, \ldots, Q \quad (22)$$

where the Hermitian polynomials are given by

$$H_{1j}(\xi_j) = 1 - 3\xi_j^2 + 2\xi_j^3, \qquad H_{2j}(\xi_j) = l_j\xi_j(1-\xi_j)^2, \quad (23, 24)$$

$$H_{3j}(\xi_j) = 3\xi_j^2 - 2\xi_j^3, \qquad H_{4j}(\xi_j) = l_j\xi_j^2(\xi_j - 1), \quad (25, 26)$$

$$\xi_j = (x - x_{j-1})/l_j, \quad l_j = x_j - x_{j-1}, \qquad j = 1, 2, \ldots, Q. \quad (27, 28)$$

Note that the coefficients y_{ij} are known from the vibration modes $\bar{X}_i(x)$ but θ_{ij} are yet to be determined. The process is similar to interpolation by spline curves. Continuity of the second derivative at the intermediate support points then gives

$$\tilde{X}_i''(x_j - 0) = \tilde{X}_i''(x_j + 0), \quad (29)$$

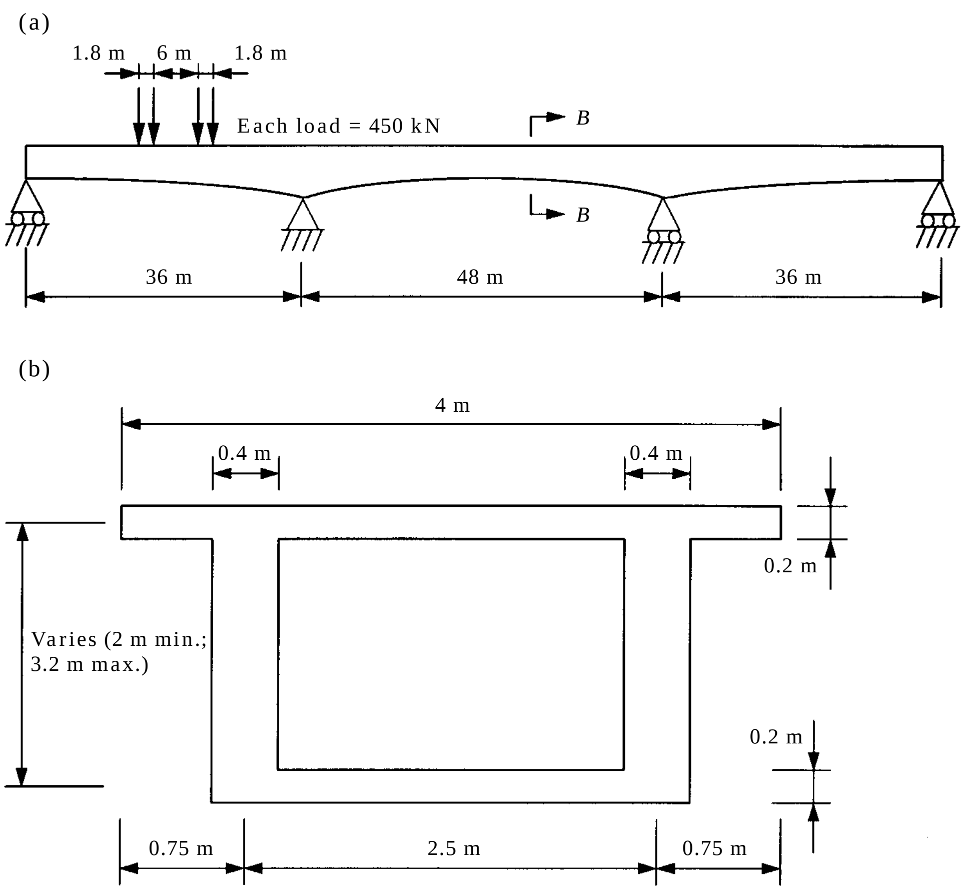

Figure 10. A three-span continuous bridge of parabolic soffit under a moving vehicle of four axles: (a) elevation; (b) section *BB*.

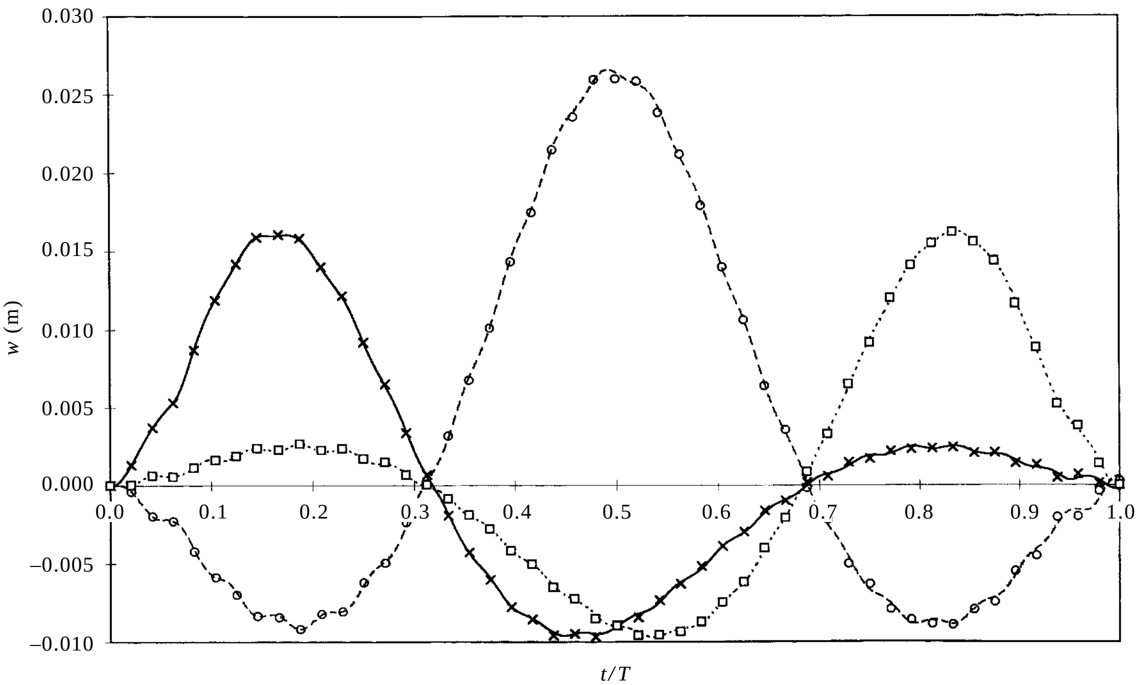

Figure 11. Deflections at mid-span positions: ——, present—span 1; –––, present—span 2; · · · ·, present—span 3; ×, reference [16]—span 1; ○, reference [16]—span 2; □, reference [16]—span 3.

and hence,

$$l_{j+1}\theta_{i(j-1)} + 2(l_j + l_{j+1})\theta_{ij} + l_j\theta_{i(j+1)} = 3\left(\frac{l_{j+1}}{l_j}(y_{ij} - y_{i(j-1)}) + \frac{l_j}{l_{j+1}}(y_{i(j+1)} - y_{ij})\right),$$

$$j = 1, 2, \ldots, Q-1. \tag{30}$$

At the two ends of the beam, the boundary conditions are

$$\tilde{X}_i''(x_0) = \tilde{X}_i''(x_Q) = 0, \tag{31}$$

which can be expressed as

$$2\theta_{i0} + \theta_{i1} = 3\,\frac{y_{i1} - y_{i0}}{l_1}, \qquad \theta_{i(Q-1)} + 2\theta_{iQ} = 3\,\frac{y_{iQ} - y_{i(Q-1)}}{l_Q}. \qquad (32, 33)$$

Equations (30), (32) and (33) can be expressed in matrix form as

$$\begin{bmatrix} 2 & \alpha_0 & 0 & \cdots & 0 & 0 & 0 \\ \beta_1 & 2 & \alpha_1 & \cdots & 0 & 0 & 0 \\ 0 & \beta_2 & 2 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \beta_{Q-1} & 2 & \alpha_{Q-1} \\ 0 & 0 & 0 & \cdots & 0 & \beta_Q & 2 \end{bmatrix} \begin{Bmatrix} \theta_{i0} \\ \theta_{i1} \\ \theta_{i2} \\ \vdots \\ \theta_{i(Q-1)} \\ \theta_{iQ} \end{Bmatrix} = \begin{Bmatrix} B_{i0} \\ B_{i1} \\ B_{i2} \\ \vdots \\ B_{i(Q-1)} \\ B_{iQ} \end{Bmatrix}, \tag{34}$$

in which the first and the last rows represent the boundary conditions at the ends while the remaining ones stand for the continuity conditions at the intermediate point supports, and

$$\alpha_j = \frac{l_j}{l_j + l_{j+1}}, \quad \beta_j = \frac{l_{j+1}}{l_j + l_{j+1}}, \quad B_{ij} = \frac{3}{l_j + l_{j+1}}\left(\frac{y_{ij} - y_{i(j-1)}}{l_j} l_{j+1} + \frac{y_{i(j+1)} - y_{ij}}{l_{j+1}} l_j\right),$$

$$j = 1, 2, \ldots, Q-1, \tag{35–37}$$

$$\alpha_0 = 1, \qquad B_{i0} = 3\,\frac{y_{i1} - y_{i0}}{l_1}, \qquad \beta_Q = 1, \qquad B_{iQ} = 3\,\frac{y_{iQ} - y_{i(Q-1)}}{l_Q}. \tag{38–41}$$

Solving equation (34) gives the undetermined coefficients $\{\theta_{ij}, j = 0, 1, \ldots, Q\}$ and hence the augmenting spline expression $\tilde{X}_i(x)$.

If the boundary condition at the end is different, the above procedures should be modified, and details are given in Appendix B.

After the undetermined coefficients of the augmenting spline expressions have been found, the modified beam vibration functions defined by equation (15) are substituted into equations (11) and (12) to form the generalized mass and stiffness matrices. Equation (14) can then be solved by the Wilson-θ method [15].

4. RESULTS AND SIMULATIONS

A number of numerical examples are presented to demonstrate the versatility, accuracy and efficiency of the present method. The present results are compared with either published results where applicable or results obtained using finite element method. Good agreement is achieved. In the following examples, note that x is the distance between the first moving load and the left end of the beam unless otherwise stated.

4.1. EXAMPLE 1: A SYMMETRICAL TWO-SPAN CONTINUOUS BEAM UNDER A SINGLE MOVING LOAD [13]

A symmetrical two-span continuous prismatic beam with simply-supported ends is considered. The cross-sectional area A and the density ρ are, respectively, $1{\cdot}146 \times 10^{-3}$ m^2 and 7700 kg/m^3. The total length L is 1 m and Young's modulus E is 207 000 MPa. Following the convention of Lee [13], the prescribed axial velocity of the moving load is defined by a non-dimensional parameter given by $a = \pi v/L\omega_1$ where v is the axial velocity of the moving load and ω_1, defined as $(\pi/L)^2\sqrt{EI/\rho A}$, has a value of 488·7 rad/s. The deflection under the moving load w is normalized by the deflection D ($D = PL^3/48EI$).

The problem was solved with 12 terms of modified beam vibration functions and 200 equal time steps. Four axial velocities were investigated, i.e., $a = 0{\cdot}111$, $a = 0{\cdot}5$, $a = 1{\cdot}1$ and $a = 1{\cdot}5$. The normalized deflections under the moving load are shown in Figures 2–5 and compared with those given in reference [13]. Good agreement was observed. It is noted that, in the present method, the zero deflection conditions at the intermediate point supports are satisfied strictly.

4.2. EXAMPLE 2: A THREE-SPAN CONTINUOUS STEPPED BEAM UNDER A SINGLE MOVING LOAD [2]

The three-span continuous stepped beam under a single moving load, as shown in Figure 6, was solved by the present method using 12 terms and 240 equal time steps. The moving point load P is 9·8 kN and the total length L is 60 m. The beam has a constant mass per unit length ρA of 1000 kg/m but the central span has double the flexural rigidity of the side span EI which is $1{\cdot}96 \times 10^6$ kNm^2. The fundamental natural circular frequency ω_1 of the beam is 38·98 rad/s. The speed parameter α defined by $\alpha = v(\rho A/EI\omega_1^2)^{1/4}$ is taken to be 0·144. In order to demonstrate the dynamic response, the history curves for deflections at the centres of the first and second spans are shown in Figure 7, and compared to results from the finite element method [16] using 60 beam elements and 240 equal time steps, and those from reference [2]. Very good agreement is observed between the present results and those obtained from finite element methods, although the continuity of bending moment at the section of abrupt change in EI is not satisfied. It illustrates that the present method is also applicable to stepped beams. Some discrepancies are, however, noticed between those from reference [2] and the rest, but the peak values of all three methods do agree with one another.

4.3. EXAMPLE 3: A THREE-SPAN CONTINUOUS HAUNCHED BEAM UNDER A SINGLE MOVING LOAD

Figure 8 shows a three-span continuous haunched beam under a single moving load. The density ρ and Young's modulus E are, respectively, 2400 kg/m^3 and 30 000 MPa. The load P of 100 kN travels at a speed v of 17 m/s across the beam. The problem was solved by the present method using 12 terms and 480 equal time steps. The history curves for deflections at the centres of the three spans are shown in Figure 9, and compared to results from finite element methods [16] using 80 beam elements and 480 equal time steps. Very good agreement is observed.

4.4. EXAMPLE 4: A THREE-SPAN CONTINUOUS BOX GIRDER BRIDGE OF PARABOLIC SOFFIT UNDER A MOVING VEHICLE OF FOUR AXLES

Figure 10 shows a three-span continuous box girder bridge under a moving vehicle. The density ρ and Young's modulus E are, respectively, 2400 kg/m^3 and 30 000 MPa. The vehicle consists of four axles each of 450 kN and it travels at a speed v of 17 m/s across the bridge. The problem was solved by the present method using 12 terms and 480 equal time steps. The history curves for deflections at the centres of the three spans are shown in Figure 11, and compared with results from the finite element method [16] using 120 beam elements and 480 equal time steps. Note that t is the time elapsed since the first axle starts from the left end and T is the time taken for all axles of the vehicle to travel over the bridge. Very good agreement is again observed.

5. CONCLUSIONS

The modified beam vibration functions are developed for the analysis of multi-span beams under moving loads. Based on Hamilton's principle, the equation of motion in matrix form has been formulated. The modified beam vibration functions satisfy the zero deflection conditions at all the intermediate point supports as well as the boundary conditions at the two ends of the beam. Programming for the whole process is very easy. Numerical results are presented for both prismatic and non-prismatic beams under moving loads of various velocities, and they agree well with the available results. Numerical simulation shows that this method is versatile, accurate and efficient.

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APPENDIX A: NOTATION

$EI(x)$	flexural rigidity
$\{k_{ij},\ i, j = 1, 2, \ldots, n\}$	generalized stiffness matrix
L	overall length of beam
$\bar{L}$	Lagrangian function of the beam
$\{x_j,\ j = 0, 1, 2, \ldots, Q\}$	abscissa of the jth point support or beam end
$\{m_{ij},\ i, j = 1, 2, \ldots, n\}$	generalized mass matrix
N	number of loads in a load series
n	number of assumed vibration modes
$\{P_s,\ s = 1, 2, \ldots, N\}$	magnitudes of the load series
Q	$(Q - 1)$ is the total number of intermediate point supports
$\{q_i(t),\ i = 1, 2, \ldots, n\}$	generalized co-ordinates of the beam
$\bar{U}$	bending energy of the beam
$u(t)$	unit step function
$\bar{V}$	kinetic energy of the beam
$v(t)$	velocity of the load series
$\bar{W}$	work done by external loads
$w(x, t)$	deflection of the beam
$\{X_i(x),\ i = 1, 2, \ldots, n\}$	modified beam vibration functions
$\{\bar{X}_i(x),\ i = 1, 2, \ldots, n\}$	vibration modes of a hypothetical prismatic beam of length L with the same end support conditions

$\{\tilde{X}_i(x),\ i = 1, 2, \ldots, n\}$	augmenting cubic spline expressions
$\{x_{ps}(t),\ s = 1, 2, \ldots, N\}$	abscissa of the sth load P_s at time t
$\rho A(x)$	mass per unit length
$\{\tau_s^1,\ s = 1, 2, \ldots, N\}$	time when the load P_s just comes onto the beam
$\{\tau_s^2,\ s = 1, 2, \ldots, N\}$	time when the load P_s just leaves the beam

APPENDIX B: BOUNDARY CONDITIONS FOR THE DETERMINATION OF AUGMENTING CUBIC SPLINE EXPRESSIONS

The augmenting cubic spline expressions are effectively fixed by the coefficients y_{ij} and θ_{ij}. For the determination of coefficients θ_{ij} using equation (34), the coefficients y_{ij} are taken to be $-\bar{X}_i(x_j)$ and α_j, β_j and B_{ij} are given by equations (35–41) unless otherwise stated below.

B.1. BOUNDARY CONDITIONS AT THE LEFT END $(x = x_0)$

(1) Simply supported end $(X_i(x_0) = X_i''(x_0) = 0)$

$$\alpha_0 = 1, \qquad B_{i0} = 3\,\frac{y_{i1} - y_{i0}}{l_1}.$$

(2) Clamped end $(X_i(x_0) = X_i'(x_0) = 0)$

$$\alpha_0 = 0, \qquad B_{i0} = 0.$$

(3) Free end $(X_i''(x_0) = X_i'''(x_0) = 0)$

$$\alpha_0 = -2, \qquad \beta_{i0} = 0, \qquad \beta_1 = \frac{-2l_2}{l_1 + l_2}, \qquad B_{i1} = \frac{3l_1}{l_1 + l_2}\,\frac{y_{i2} - y_{i1}}{l_2}.$$

Note that the coefficient y_{i0} does not appear in equation (34) and it is subsequently determined by $y_{i0} = y_{i1} - l_1\theta_{i0}$.

B.2. BOUNDARY CONDITIONS AT THE RIGHT END $(x = x_Q)$

(1) Simply supported end $(X_i(x_Q) = X_i''(x_Q) = 0)$

$$\beta_Q = 1, \qquad B_{iQ} = 3\,\frac{y_{iQ} - y_{i(Q-1)}}{l_Q}.$$

(2) Clamped end $(X_i(x_Q) = X_i'(x_Q) = 0)$

$$\beta_Q = 0, \qquad B_{iQ} = 0.$$

(3) Free end $(X_i''(x_Q) = X_i'''(x_Q) = 0)$

$$\beta_Q = -2, \qquad B_{iQ} = 0, \qquad \alpha_{Q-1} = \frac{-2l_{Q-1}}{l_{Q-1} + l_Q}, \qquad B_{i(Q-1)} = \frac{3l_Q}{l_{Q-1} + l_Q}\,\frac{y_{i(Q-1)} - y_{i(Q-2)}}{l_{Q-1}}.$$

Note that the coefficient y_{iQ} does not appear in equation (34) and it is subsequently determined by $y_{iQ} = y_{i(Q-1)} + l_Q\theta_{iQ}$.