Journal of Sound and Vibration (1998) **212**(3), 547–557

BOUNDS ANALYSIS OF A BEAM BASED ON THE CONVEX MODEL OF UNCERTAIN FOUNDATION

L. Frýba

Institute of Theoretical and Applied Mechanics, Academy of Sciences of the Czech Republic, Prague, Czech Republic

and

N. Yoshikawa*

Institute of Industrial Science, The University of Tokyo, Tokyo, Japan

(*Received* 25 *February* 1997, *and in final form* 22 *December* 1997)

A methodology of bounds analysis based on the convex set of uncertainty is presented. An infinitely long beam on an uncertain foundation is subjected to a moving force with constant speed. The steady state vibration is analysed in the context of the finite element method. Uncertainty variables representing the uncertain fluctuation of foundation stiffness is confined within a convex hull given as a hyperellipse. The fluctuation of dynamic responses of the beam, that is, the vertical deflection and the bending moment are approximated to a first-order with respect to the uncertainty variables. The upper and lower bounds of the response fluctuations are identified on the boundary of the convex model by means of the Lagrange multiplier method. Numerical analysis demonstrates the validity of the proposed formulation for various constant speeds of the moving force. The analysis is summarized in terms of the worst response, which is the maximum absolute value of the response within the identified bounds.

1. INTRODUCTION

A railway track is frequently idealized by an infinitely long beam on an uncertain foundation, which governs the uncertain response of a railway track [1–3]. The uncertainty in a structural system has been dealt with by the stochastic methodology based on the probabilistic theory [4]. The stochastic analysis of a dynamic railway track is attempted by Frýba [5] by using approximate analytical methods, and that of the steady state vibration was carried out by the authors [2] by means of the stochastic finite element method.

If one can obtain an accurate stochastic property of the uncertainty, such as the probabilistic density function, the stochastic methodology will give useful and reliable information concerning the structural integrity. However, the identification of the stochastic property is a difficult task, since the sources of the uncertainty are very complex in reality [6, 7]. Even in case that one can completely identify the stochastic property, there should be a trade-off between the information used for the analysis and the computational labour required for the analysis, since the labour increases in accordance with the amount of used information and becomes tremendous for real structures.

Recently, a few methods coping with uncertainty in structural systems have been newly proposed based on quasi-probabilistic theories, where a part of the stochastic property is neglected for the reason of incomplete identification or for the purpose of mitigation of

*Visiting Institute of Theoretical and Applied Mechanics, Prague.

0022–460X/98/180547 + 11 $25.00/0/sv971482

the computational labour [6–10]. In these newly proposed methods, attention is only paid to the boundary of the fluctuation of the uncertainty, so that the fluctuation of the uncertain parameters is confined within a convex region [6–8, 10] or is handled by interval algebra [9].

In this study, a method is proposed to identify the bounds of uncertain fluctuation of a structural response based on a convex set of uncertainty variables applied to the problem of the dynamic railway track on an uncertain foundation. The structural system is discretized by finite elements developed by the authors [2], and uncertainty variables are assigned to the foundation stiffness of all the elements. The fluctuation of uncertainty variables is confined within a convex hull represented by a hyperellipse.

The uncertain change of structural responses, that is, the vertical deflection and the bending moment of the beam are approximated to a first-order by means of the finite element sensitivity analysis with respect to the uncertainty variables. The bounds of the response fluctuation are identified by means of the Lagrange multiplier method within the boundary of the convex model of the uncertainty variables. Numerical analysis for several constant speeds of the moving force is demonstrated to show the validity of the proposed method. The result of the bounds analysis is summarized in terms of the worst response, that is, the maximum absolute value within the estimated bounds.

2. FINITE ELEMENT DISCRETIZATION WITH UNCERTAINTY VARIABLES

The Bernoulli–Euler differential equation (1) represents the equation of motion of an infinitely long beam on a Winkler foundation subjected to a single moving force with constant speed (see Figure 1),

$$EI\frac{\partial^4 v(x,t)}{\partial x^4} + \mu\frac{\partial^2 v(x,t)}{\partial t^2} + 2\mu\omega_b\frac{\partial v(x,t)}{\partial t} + k(x)v(x,t) = \delta(x-ct)f, \tag{1}$$

where EI is the bending stiffness, $v(x, t)$ is the vertical deflection of the beam, x is the fixed co-ordinate along beam, t represents the time, μ is the mass per unit length, ω_b is the coefficient of damping of Kelvin–Voigt type, $k(x)$ is the stiffness of the Winkler foundation, $\delta(\cdot)$ is the Dirac delta function, c is the speed of moving force and f is the vertical force.

Assume that only the foundation stiffness $k(x)$ varies uncertainly along the beam as illustrated in Figure 1, and the uncertainty is represented in the form of equation (2) by introducing an uncertainty variable $\varepsilon(x)$, such that

$$k(x) = k^*[1 + \varepsilon(x)]. \tag{2}$$

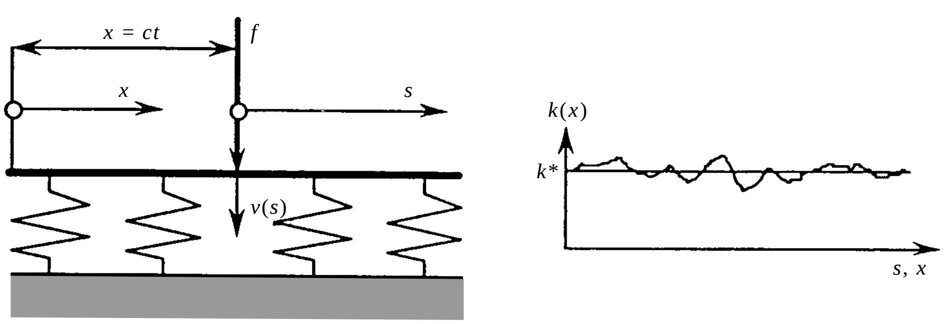

Figure 1. Railway track idealized by infinitely long beam on Winkler foundation with uncertain foundation stiffness.

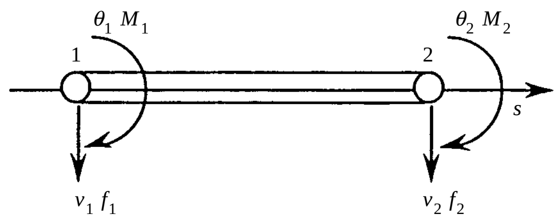

Figure 2. Finite element discretization of a beam on Winkler foundation.

The asterisk means a nominal value hereafter. In this study, the stochastic property is neglected and attention paid only to the boundary of the uncertain fluctuation of $\varepsilon(x)$ and an ellipsoidal convex model is employed to bound the fluctuation [11]. The nominal value of $\varepsilon(x)$ is set equal to zero in the following formulation.

The governing equation (1) is transformed into the equation of steady state vibration by the introduction of the normalized coordinate s, which moves with the force f at constant speed c, irrespective of initial condition [5],

$$s = \lambda(x - ct), \tag{3}$$

where

$$\lambda = [k^*/(4EI)]^{1/4}. \tag{4}$$

Assuming that the steady state vibration depends only on the normalized co-ordinate s, the vertical deflection $v(x, t)$ and bending moment $M(x, t)$ are normalized by the static vertical deflection v_0 and bending moment M_0 at the loading point as

$$v(x, t) = v_0 v(s), \tag{5}$$

$$M(x, t) = -EI\frac{\partial^2 v(x, t)}{\partial x^2} = M_0 M(s), \tag{6}$$

where

$$v_0 = f/(8\lambda^3 EI) = f\lambda/(2k^*), \tag{7}$$

$$M_0 = f/(4\lambda). \tag{8}$$

The governing equation of steady state vibration on the normalized co-ordinate s is derived as equation (9) after the substitution of equations (3), (4), (5) and (7) into the basic equation (1),

$$\frac{\mathrm{d}^4 v(s)}{\mathrm{d}s^4} + 4\alpha^2\frac{\mathrm{d}^2 v(s)}{\mathrm{d}s^2} - 8\alpha\beta\frac{\mathrm{d}v(s)}{\mathrm{d}s} + 4[1 + \varepsilon(s)]v(s) = 8\delta(s), \tag{9}$$

where α is speed parameter and β damping parameter defined by the following equations,

$$\alpha = c/c_{cr}, \tag{10}$$

$$\beta = \omega_b(\mu/k^*)^{1/2}. \tag{11}$$

c_{cr} denotes critical speed of the moving force [5],

$$c_{cr} = 2\lambda(EI/\mu)^{1/2}. \tag{12}$$

Discretizing the beam into N finite elements with length l, define nodal displacements v_i and θ_i, and nodal forces f_i and M_i as shown in Figure 2 in the context of the conventional finite element method. Applying the principal of virtual work to the governing equation (9) of steady state vibration, the stiffness equation for one element is obtained as

$$([\mathbf{K}] - 4\alpha^2[\mathbf{D}] + 8\alpha\beta[\mathbf{V}] + 4[1 + \varepsilon_n][\mathbf{R}])\{\mathbf{U}\} = \{\mathbf{F}\}, \tag{13}$$

where $\{\mathbf{U}\}$ and $\{\mathbf{F}\}$ consist of nodal displacements and nodal forces, respectively,

$$\{\mathbf{U}\} = \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}, \qquad \{\mathbf{F}\} = \begin{Bmatrix} f_1 \\ M_1 \\ f_2 \\ M_2 \end{Bmatrix}, \tag{14}$$

and $[\mathbf{K}]$, $[\mathbf{D}]$, $[\mathbf{V}]$ and $[\mathbf{R}]$ are called stiffness, dynamic, viscous and reaction matrices, respectively,

$$[\mathbf{K}] = \frac{2}{l^3}\begin{bmatrix} 6 & 3l & -6 & 3l \\ & 2l^2 & -3l & l^2 \\ & & 6 & -3l \\ \text{sym.} & & & 2l^2 \end{bmatrix}, \tag{15}$$

$$[\mathbf{D}] = \frac{1}{30l}\begin{bmatrix} 36 & 3l & -36 & 3l \\ & 4l^2 & -3l & -l^2 \\ & & 36 & -3l \\ \text{sym.} & & & 4l^2 \end{bmatrix}, \tag{16}$$

$$[\mathbf{V}] = \frac{1}{60}\begin{bmatrix} -30 & -6l & -30 & 6l \\ 6l & 0 & -6l & l^2 \\ 30 & 6l & 30 & -6l \\ -6l & l^2 & 6l & 0 \end{bmatrix}, \tag{17}$$

$$[\mathbf{R}] = \frac{l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ & 4l^2 & 13l & -3l^2 \\ & & 156 & -22l \\ \text{sym.} & & & 4l^2 \end{bmatrix}. \tag{18}$$

3. BOUNDS ANALYSIS BASED ON A CONVEX SET OF UNCERTAINTY

Conventional reliability analysis has been founded on the probabilistic variables spreading from minus infinity to infinity. In this study, the fluctuation of uncertainty variables are confined within a convex region and the probabilistic density of the inside is neglected. Uncertainty variables ε_n are assigned to the foundation stiffness of all the elements discretized by the manner stated in the previous section,

$$k_n = k^*(1 + \varepsilon_n), \tag{19}$$

where k^* is nominal value of the foundation stiffness, and the nominal value of ε_n is set equal to zero for all the elements. The fluctuation of the uncertainty variables is assumed to be bounded by a convex region. There is a possibility of using several kinds of convex models [11]. In this study, an ellipsoidal convex model given as,

$$\{\boldsymbol{\varepsilon}\}^T[\boldsymbol{\Omega}]\{\boldsymbol{\varepsilon}\} \leqslant 1, \tag{20}$$

is employed, where $\{\boldsymbol{\varepsilon}\}$ consists of the uncertainty variables ε_n. The symmetric and positive definite matrix $[\boldsymbol{\Omega}]$ is determined *a priori* so as to govern the extent and the boundary shape of the hyperellipse.

In the context of bounds analysis, the problem is posed as follows: find the upper and lower bounds of the response of a structural system caused by the fluctuation of the uncertainty variables within a convex hull. The bounds of the vertical deflection and the bending moment of the idealized beam are estimated in this study. The estimation along the beam is carried out on the nodes for the deflection and on the mid-points of elements for the moment. In the following formulation, the deflection and the moment under the estimation are denoted by z and called structural response. For the bounds estimation, the change of structural response z is approximated to a first-order with respect to the uncertainty variables,

$$z = z^* + \sum_{n=1}^{N} z_n^I \varepsilon_n = z^* + \{\boldsymbol{\varepsilon}\}^T\{\mathbf{z}^I\}, \tag{21}$$

where z_n^I is the rate of change of z with respect to ε_n, and $\{\mathbf{z}^I\}$ consists of the rate of change. z^* is evaluated by the nominal analysis, in which the nominal value of ε_n, that is, $\varepsilon_n = 0$ is employed. The rate of change z_n^I is analysed by the finite element sensitivity analysis [12]. The maximum and minimum values of the approximated structural response take place on the boundary of the convex hull of equation (20), since equations (21) and (20) are linear and quadratic functions, respectively, with respect to the uncertainty variables. The search for both values is carried out by the Lagrange multiplier method employing the following functional Π,

$$\Pi = \{\boldsymbol{\varepsilon}\}^T\{\mathbf{z}^I\} + \xi(\{\boldsymbol{\varepsilon}\}^T[\boldsymbol{\Omega}]\{\boldsymbol{\varepsilon}\} - 1), \tag{22}$$

where ξ is the Lagrange multiplier. The stationary condition of Π with respect to the uncertainty variables and the Lagrange multiplier gives rise to the simultaneous quadratic equations:

$$\frac{\partial \Pi}{\partial \{\boldsymbol{\varepsilon}\}^T} = \{\mathbf{z}^I\} + 2\xi[\boldsymbol{\Omega}]\{\boldsymbol{\varepsilon}\} = \{\mathbf{0}\}, \tag{23}$$

$$\frac{\partial \Pi}{\partial \xi} = \{\boldsymbol{\varepsilon}\}^T[\boldsymbol{\Omega}]\{\boldsymbol{\varepsilon}\} - 1 = 0. \tag{24}$$

The solution of the above equations is as follows:

$$\{\boldsymbol{\varepsilon}\} = \pm \frac{\{\mathbf{r}\}}{\sqrt{\{\mathbf{r}\}^T[\boldsymbol{\Omega}]\{\mathbf{r}\}}}, \tag{25}$$

where $\{\mathbf{r}\}$ is an abbreviation vector given as

$$\{\mathbf{r}\} = [\boldsymbol{\Omega}]^{-1}\{\mathbf{z}^I\}. \tag{26}$$

The bounds of uncertain fluctuation of z are estimated by equation (27) within the sense of the first-order approximation:

$$z = z^* + \{\boldsymbol{\varepsilon}\}^T\{\mathbf{z}^I\}. \tag{27}$$

The above formulation is based on the first-order approximation of the structural response change with respect to the uncertainty variables. The validity of the proposed formulation is therefore limited within the case that the linearity of the response with respect to the uncertainty variables is not violated much.

When the convex model is given as a hypersphere, that is, the matrix $[\mathbf{\Omega}]$ is given as a multiplied identity matrix, the uncertainty variables for the bounds, which are evaluated by equations (25), become proportional to the rate of change of the response $\{\mathbf{z}^{\prime}\}$.

Making the observation point move along the beam, one can estimate the bounds of z throughout the system. After the estimation, the maximum absolute value of z within the estimated bounds is identified. The worst case is thus defined as the status which gives rise to the maximum absolute value.

4. NUMERICAL ANALYSIS

The proposed bounds analysis is carried out for various constant speeds of a moving force. The speed parameter α is changed from 0·0 to 2·0 with increments of 0·1. Damping parameter β is fixed to 0·5. Employing normalized co-ordinate s and applying the load at $s = 0{\cdot}0$, the system is analysed from $s = -15$ to $s = 15$ until $\alpha = 1{\cdot}0$, from $s = -30$ to $s = 30$ over $\alpha = 1{\cdot}0$ to realize the boundary condition of an infinitely long beam on the foundation. The length of one element l is set equal to 0·1 in every analysis. The convex model of uncertain fluctuation of the foundation stiffness is given as a hypersphere with radius 0·8. The observation points are set at all the nodes for the analysis of vertical deflection, and at mid-points of all the elements for the bending moment. In this case the value of a UIC 60 rail is used and the nominal value of the foundation stiffness taken as $k^* = 50{\cdot}0$ Mpa, the parameter λ of equation (3) and the critical speed c_{cr} become 1·18 l/m and 772 m/s, respectively.

The results of a bound analysis for the vertical deflection and the bending moment are shown in Figure 3 for $\alpha = 0{\cdot}0$, that is, the static case. The solid and broken lines correspond to the results for the deflection and the moment, respectively. The thick line indicates the nominal value, and the thin lines give the estimated bounds in both results. Theoretically

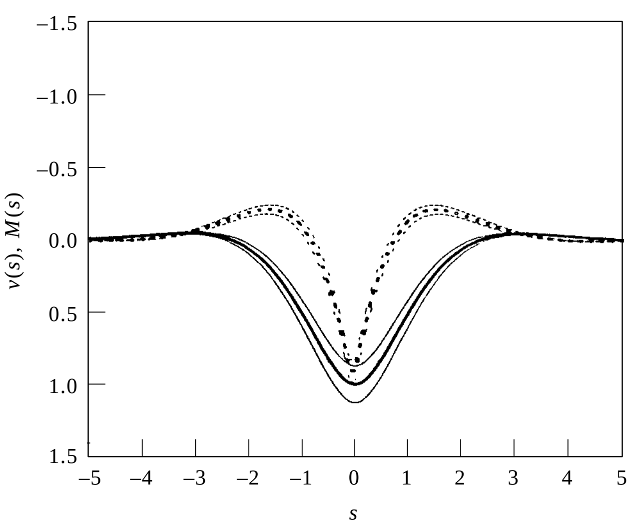

Figure 3. Result of bounds analysis: $\alpha = 0{\cdot}0$, $\beta = 0{\cdot}5$. ——, Vertical deflection; - - -, bending moment. Thick line for nominal value and thin lines for bounds.

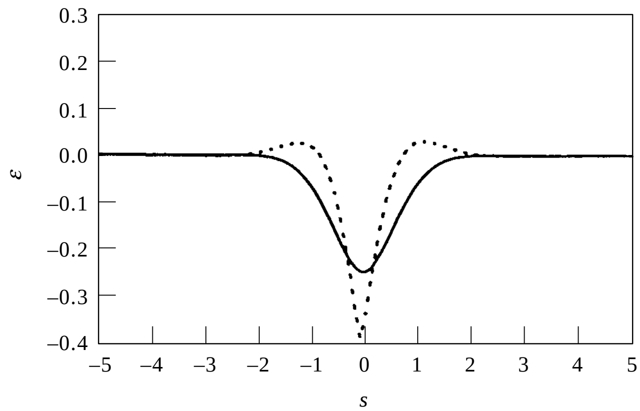

Figure 4. Uncertainty variables raising the worst case: $\alpha = 0{\cdot}0$, $\beta = 0{\cdot}5$. ——, Vertical deflection; - - -, bending moment.

speaking, nominal vertical deflection and nominal bending moment at loading point $s = 0{\cdot}0$ must be equal to 1·0 [5]. The result of the finite element analysis for vertical deflection coincides with the theoretical analysis. On the other hand, nominal bending moment at the loading point looks smaller than 1·0 in Figure 3, since the evaluation is done at the mid-point of the element, which is located apart from the loading point by $l/2$. The interval between upper and lower bounds is larger for the vertical deflection than for the bending moment in this static problem. The worst cases, which give rise to the maximum absolute value within the evaluated bounds, occur at loading point both for the vertical deflection and the bending moment. The uncertainty variables raising these worst cases are illustrated in Figure 4. The symmetry with respect to the loading point is kept at the nominal value, the bounds and the uncertainty variables of the worst case for both the vertical deflection and the bending moment.

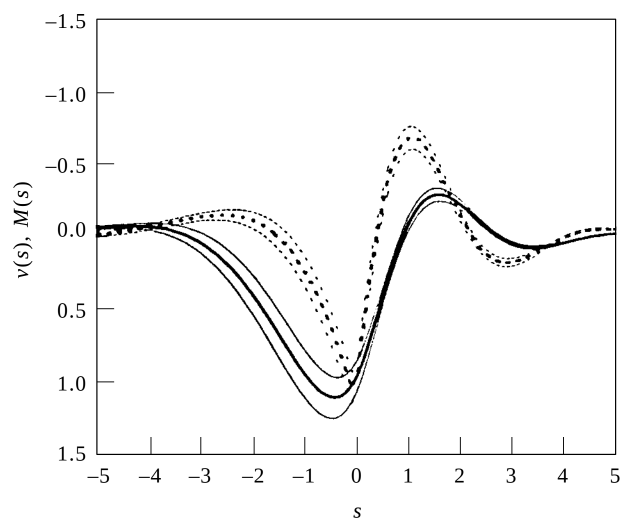

Figure 5. Result of bounds analysis: $\alpha = 0{\cdot}8$, $\beta = 0{\cdot}5$. ——, Vertical deflection; - - -, bending moment. Thick line for nominal value and thin lines for bounds.

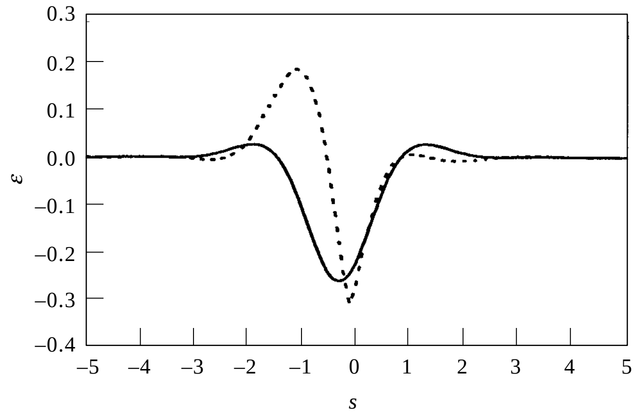

Figure 6. Uncertainty variables raising the worst case: $\alpha = 0{\cdot}8$, $\beta = 0{\cdot}5$. ——, Vertical deflection; - - -, bending moment.

The maximum nominal value, the interval of bounds and consequently the worst value increase with raising speed until $\alpha = 0{\cdot}8$ both for the vertical deflection and the bending moment. The bounds of the vertical deflection and the bending moment are illustrated in Figure 5 for $\alpha = 0{\cdot}8$. The interval of the bounds is larger for the deflection than that for the moment also in this case. Different from the result of the static case, the worst deflection occurs behind the loading point in dynamic problems, whereas the worst moment occurs at the loading point until $\alpha = 1{\cdot}0$. The uncertainty variables raising the worst cases are indicated in Figure 6. The symmetry of the uncertainty variables is not violated so much for the deflection as for the moment.

The maximum nominal value and the interval between bounds behind the loading point decrease when the speed rises above $\alpha = 0{\cdot}8$ both for the deflection and the moment. Above $\alpha = 1{\cdot}0$, the worst moment occurs before the loading point and maximizes at $\alpha = 1{\cdot}2$. The results of the bounds analyses for the deflection and the moment at $\alpha = 1{\cdot}2$ are indicated

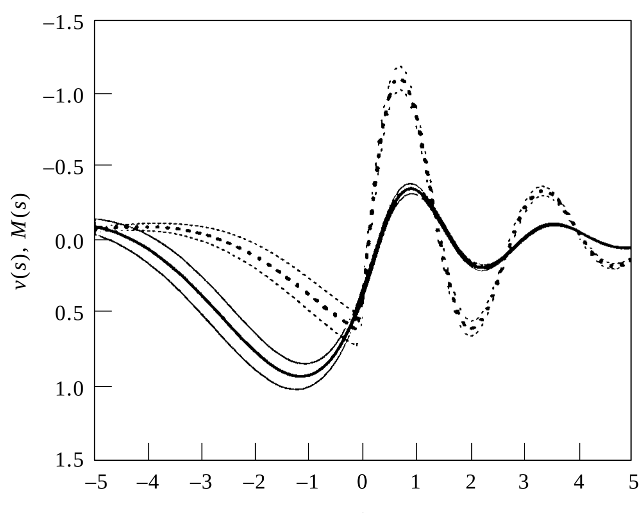

Figure 7. Result of bounds analysis: $\alpha = 1{\cdot}2$, $\beta = 0{\cdot}5$. ——, Vertical deflection; - - -, bending moment. Thick line for nominal value and thin lines for bounds.

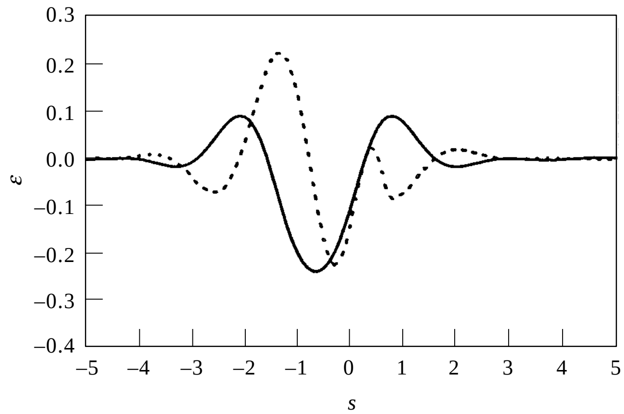

Figure 8. Uncertainty variables raising the worst case: $\alpha = 1{\cdot}2$, $\beta = 0{\cdot}5$. ——, Vertical deflection; - - -, bending moment.

in Figure 7. The interval between the bounds for the moment is larger than that for the deflection in this case. The distributions of the uncertainty variables, which raise the worst cases, are indicated in Figure 8. The symmetry of the distribution seems to be kept in the uncertainty variables for the worst deflection.

The results of the bounds analyses are summarized in Figures 9 and 10 for the deflection and the moment, respectively. In both figures, the solid line with blank squares indicates the worst value, that is, the maximum absolute value within the estimated bounds, and the solid line with blank circles corresponds to the nominal value. The worst moment occurs before loading point over $\alpha = 1{\cdot}0$. This change in location results in two maximum worst values in Figure 10. The nominal value and the upper bound at the point of load application is indicated by the broken line in Figure 10. The difference of the worst value

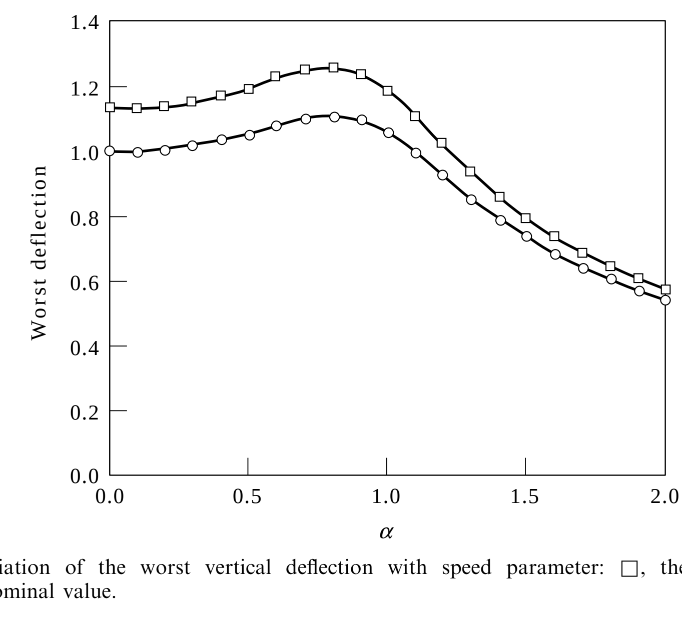

Figure 9. Variation of the worst vertical deflection with speed parameter: □, the worst value; ○, corresponding nominal value.

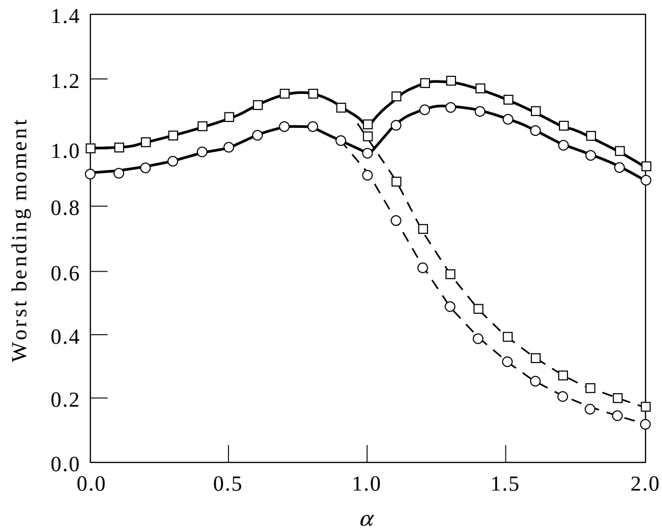

Figure 10. Variation of the worst bending moment with speed parameter: □, the worst value; ○, corresponding nominal value; ---, values at the loading point.

from the nominal value decreases more in the deflection than in the moment with the rising speed over $\alpha = 0{\cdot}8$.

5. CONCLUSIONS

A methodology to evaluate the uncertain dynamic response of an infinitely long beam on an uncertain foundation based on a convex set of uncertainty is presented. The beam is discretized by finite elements and subjected to a moving force with constant speed. The uncertainty variables are assigned to the foundation stiffness of the Winkler foundation, and the fluctuation of the uncertainty variables are confined within a convex region given by a hyperellipse. The change of dynamic response caused by the uncertainty variables is approximated to a first-order by means of a finite element sensitivity analysis. The bounds of the structural response caused by the fluctuation within the convex region of uncertainty variables are evaluated by means of the Lagrange multiplier method.

Numerical analyses are carried out for various constant speeds by changing the speed parameter from $\alpha = 0{\cdot}0$ to $\alpha = 2{\cdot}0$ with increments of $0{\cdot}1$. The results of the analyses are summarized in terms of the worst value, which is the maximum absolute value within the estimated bounds. The worst vertical deflection is maximized at one point with respect to the speed parameter. On the other hand the worst bending moment holds two peaks. The interval of the bounds decreases more for the deflection than for the moment when the speed is raised over $\alpha = 0{\cdot}8$.

REFERENCES

1. V. V. Bolotin 1982 *Application of the Theory of Probability and of the Theory of Reliability to the Analysis of Structures* (in Russian). Moscow: Strojizdat. See section 2·1.
2. L. Frýba, S. Nakagiri and N. Yoshikawa 1993 *Journal of Sound and Vibration* **163**, 31–45. Stochastic finite elements for a beam on a random foundation with uncertain damping under a moving force.

3. R. N. IYENGAR and O. R. JAISWAL 1995 *Solid Mechanics and Its Applications* **47**, *Proceedings of the IUTAM Symposium on Advances in Nonlinear Stochastic Mechanics* (A. Naess and S. Krenk, editors). 213–224. Dordecht: Kluwer Academic Publishers. Stochastic response of irregular tracks under moving vehicles.
4. Y. K. LIN and G. Q. CAI 1995 *Probabilistic Structural Dynamics*. Singapore: McGraw-Hill.
5. L. FRÝBA 1972 *Vibration of Solids and Structures Under Moving Loads*. Prague: Academia/Groningen: Noordhoff International Publishing.
6. Y. BEN-HAIM and I. ELISHAKOFF 1990 *Convex Models of Uncertainty in Applied Mechanics*. New York: Elsevier.
7. H. E. LINDBERG 1992 *ASME Journal of Applied Mechanics* **59**, 929–936. An evaluation of convex modeling for multimode dynamic buckling.
8. C. P. PANTELIDES 1995 *Proceedings of the American Society of Civil Engineers* **121**, 837–844. Buckling of elastic columns using convex model of uncertain spring.
9. H. U. KOYLUOGLU, A. S. CAKMAK and S. R. K. NIELSEN 1995 *Proceedings of the American Society of Civil Engineers* **121**, 1149–1157. Interval algebra to deal with pattern loading and structural uncertainties.
10. S. NAKAGIRI and N. YOSHIKAWA 1995 *Probabilistic Mechanics & Structural Reliability*, *Proceedings of the Seventh Specialty Conference*, *Engineering Mechanics and Structural Divs./ASCE*, 278–281. Finite element interval estimation by convex model.
11. Y. BEN-HAIM 1993 *ASME Journal of Applied Mechanics* **60**, 683–688. Convex models of uncertainty in radial pulse buckling of shells.
12. R. L. FOX 1965 *American Institute of Aeronautics and Astronautics Journal* **3**, 1517–1518. Constraint surface normals for structural synthesis.