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LETTERS TO THE EDITOR

ON THE EXISTENCE OF WEAK SOLUTIONS IN THE STUDY OF BEAMS

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1. Introduction

Consider a tapered beam of length ℓ, the width and depth which vary between $x = a$ and $x = b$, and whose ends are elastically restrained against rotation and translation and supported by a continuous elastic foundation. It is assumed that a load $q(x)$ causes a transverse deflection $u(x)$. The corresponding boundary value problem is given by

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(p(x)\frac{\mathrm{d}^2u(x)}{\mathrm{d}x^2}\right) + r(x)u(x) = q(x), \qquad \forall x \in I = (a, b), \tag{1}$$

$$r_1\frac{\mathrm{d}u(a)}{\mathrm{d}x} = p(a)\frac{\mathrm{d}^2u(a)}{\mathrm{d}x^2}, \qquad t_1(u)(a) = -\frac{\mathrm{d}}{\mathrm{d}x}\left(p(a)\frac{\mathrm{d}^2u(a)}{\mathrm{d}x^2}\right), \tag{2, 3}$$

$$\mathrm{r}_2\frac{\mathrm{d}u(b)}{\mathrm{d}x} = -p(b)\frac{\mathrm{d}^2u(b)}{\mathrm{d}x^2}, \qquad t_2u(b) = \frac{\mathrm{d}}{\mathrm{d}x}\left(p(b)\frac{\mathrm{d}^2u(b)}{\mathrm{d}x^2}\right), \tag{4, 5}$$

where $p(x)$ denotes the flexural rigidity, $r(x)$ the foundation modulus, and r_i and t_i the rotational and translational stiffness respectively.

In the case $0 < r_i < \infty$, $0 < t_i < \infty$, all the boundary conditions (2)–(5) are unstable [1]. Consequently the space V, of those functions from the Sobolev space $H^2(I)$, which satisfy the corresponding stable homogeneous boundary conditions, can be defined as $V = H^2(I)$. The boundary value problem above is transformed into one that leads to the concept of weak solution.

Let $q(x) \in C(\bar{I})$, $r(x) \in C(\bar{I})$, $p(x) \in C(\bar{I})$, $p'(x) \in C(\bar{I})$, $p''(x) \in C(\bar{I})$, $p(x) \geqslant p_0 > 0$ and $u(x) \in C^4(I)$, be the classical solution of the problem (1)–(5). If one takes an arbitrary function $v \in V$ and multiplies equation (1) by this function and integrates the result over I one obtains

$$\int_a^b (p(x)u''(x))''v(x)\,\mathrm{d}x + \int_a^b r(x)u(x)v(x)\,\mathrm{d}x = \int_a^b q(x)v(x)\,\mathrm{d}x. \tag{6}$$

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Integrating by parts the first term on the left hand side of equation (6) and taking into account the boundary conditions (2)–(5) one has

$$\int_a^b p(x)u''(x)v''(x)\,\mathrm{d}x + \int_a^b r(x)u(x)v(x)\,\mathrm{d}x + r_1u'(a)v'(a) + r_2u'(b)v'(b) + t_1u(a)v(a)$$

$$+ t_2u(b)v(b) = \int_a^b q(x)v(x)\,\mathrm{d}x, \qquad \forall v \in V = H^2(I). \tag{7}$$

The first two terms on the left hand side of equation (7) constitute the bilinear form $A(v, u)$ associated with the differential operator A, which applied to u gives the left hand side of equation (1). The other terms correspond to the boundary bilinear form $a(v, u)$. Now we are going to weaken the assumptions. Let $p \in L^\infty(I)$, $q \in L^2(I)$, $r \in L^\infty(I)$, and $B(v, u) = A(v, u) + a(v, u)$ continuous in V. The function u is called the weak solution of the boundary value problem given by equations (1)–(5) if

$$\begin{cases} u \in V = H^2(I), \\ B(v, u) = (v, q)_{L^2(I)} \quad \forall v \in V. \end{cases} \tag{8}$$

The continuity of the bilinear form $A(v, u)$ follows easily by applying the Schwarz inequality:

$$|A(v, u)| \leqslant \int_a^b |p(x)\|u''(x)\|v''(x)|\,\mathrm{d}x + \int_a^b |r(x)\|u(x)\|v(x)|\,\mathrm{d}x \leqslant \|p\|_{L^\infty(I)}\|u''\|_{L^2(I)}\|v''\|_{L^2(I)}$$

$$+ \|r\|_{L^\infty(I)}\|u\|_{L^2(I)}\|v\|_{L^2(I)} \leqslant C_1\|u\|_{H^2(I)}\|v\|_{H^2(I)}. \tag{9}$$

On the other hand, the bilinear boundary form is given by

$$a(v, u) = r_1u'(a)v'(a) + r_2u'(b)v'(b) + t_1u(a)v(a) + t_2u(b)v(b).$$

Then one has

$$|a(v, u)| \leqslant r_1\|u'\|_{L^\infty(I)}\|v'\|_{L^\infty(I)} + r_2\|u'\|_{L^\infty(I)}\|v'\|_{L^\infty(I)} + t_1\|u\|_{L^\infty(I)}\|v\|_{L^\infty(I)}$$
$$+ t_2\|u\|_{L^\infty(I)}\|v\|_{L^\infty(I)}.$$

Since there exists a constant C such that [2] $\|u\|_{L^\infty(I)} \leqslant C\|u\|_{H^1(I)}\ \forall u \in H^1(I)$, one has

$$|a(v, u)| \leqslant C(r_1\|u\|_{H^2(I)}\|v\|_{H^2(I)} + r_2\|u\|_{H^2(I)}\|v\|_{H^2(I)} + t_1\|u\|_{H^2(I)}\|v\|_{H^2(I)}$$
$$+ t_2\|u\|_{H^2(I)}\|v\|_{H^2(I)}). \tag{10}$$

From equations (9) and (10) it has been proven that $B(v, u)$ is continuous on V, i.e., there exists a constant C_2 such that

$$|B(v, u)| \leqslant C_2\|v\|_{H^2(I)}\|u\|_{H^2(I)} \quad \forall u, v \in V = H^2(I).$$

If the bilinear form $B(v, u)$ is also V-elliptic, then the given problem has exactly one weak solution u [1]. Let us proceed with the proof of the V-ellipticity of $B(v, u)$. Denoting $C_3 = \min\{p_0, t_1, t_2\}$ one has

$$B(v, v) \geqslant p_0\int_a^b v(x)''^2\,\mathrm{d}x + t_1v(a)^2 + t_2v(b)^2 \geqslant C_3\left[\int_a^b v(x)''^2\,\mathrm{d}x + v(a)^2 + v(b)^2\right].$$

By applying Friedrich's inequality [1],

$$\|u\|_{H^2(I)} \leqslant K\left[\int_a^b v(x)''^2\,\mathrm{d}x + v(a)^2 + v(b)^2\right],$$

one has $B(v, v) \geqslant C_3/k\|v\|^2_{H^2(I)}$, $v \in V = H^2(I)$; then $B(v, u)$ is V-elliptic.

In the case where all the r_i and t_i are equal to zero and $r(x) \neq 0$ the properties of $B(v, u)$ remain valid, but if also $r(x) \equiv 0$ then $B(v, u)$ is not V-elliptic anymore.

2. APPLICATION OF DIRECT VARIATIONAL METHODS TO THE CONSTRUCTION OF APPROXIMATIONS OF THE WEAK SOLUTION

It was proved that the bilinear form $B(v, u)$ is continuous and V-elliptic. Since it is also symmetric, the function $u(x)$ is the weak solution of equation (8), if and only if it minimizes, in the space V, the functional [1].

$$I(v) = B(v, v) - 2(v, q)_{L^2(I)} \quad \forall v \in V.$$

So the Ritz method can be applied. Accordingly $u(x)$ is approximated by

$$u_n(x) = \sum_{i=1}^{n} c_{ni} v_i(x).$$

where the $v_i(x)$ are elements of a base in V.

The coefficient c_{ni} are determined by the condition $I(u_n) = \min$. This procedure leads to the following system of linear equations:

$$\sum_{j=1}^{n} c_{nj} B(v_i, v_j) = (v_i, q)_{L^2(I)}, \qquad i = 1, 2, 3, \ldots, n. \tag{11}$$

It is well known that when using the Ritz method, one chooses a sequence of functions v_i which constitutes a base in the before mentioned space V. Since only the homogeneous stable boundary conditions are included in V, there is no need to subject the functions v_i to the natural boundary conditions. It is the case when $0 < r_i < \infty$, $0 < t_i < \infty$. If any of the coefficients r_i and t_i are equal to 0 or ∞, stable boundary conditions appear. For instance, let us consider the problem of deformation of a rectangular cross-section beam under uniform loading, with the left end ($x = a = 0$) rigidly clamped and the right end ($x = b = \ell$) elastically restrained against rotation. In this case we must define

$$V = \{v \in H^2(I), v(0) = 0, v'(0) = 0, v(\ell) = 0\}.$$

If the non-dimensional variable x/ℓ is used and $v(x) = (1 - x)x^{i+1}$, $i = 1, 2, \ldots,$ is adopted, one obtains

$$B(v_i, v_j) = \frac{ij(i+1)(j+1)}{i+j-1} + \frac{i(i+1)(j+1)(j+2)}{i+j} - \frac{(i+1)(i+2)j(j+1)}{i+j}$$
$$+ \frac{(i+1)(i+2)(j+1)(j+2)}{i+j+1} + r_2,$$

$$(v_i, q)_{L^2(I)} = q\left(\frac{1}{i+2} + \frac{1}{i+3}\right).$$

The solution of system (11) for different values of the rotational coefficient r_2 leads to the numerical values which correspond to the exact solution:

$$u(x) = \frac{ql^4}{24EI}\left(x^4 - 2\,\frac{r_2+5}{r_2+4}\,x^3 + \frac{r_2+6}{r_2+4}\,x^2\right).$$

3. THE EIGENVALUE PROBLEM

Let us consider a tapered beam generally restrained when it executes free transverse vibrations. The eigenvalue problem is given by

$$\frac{d^2}{dx^2}\left(p(x)\,\frac{d^2u(x)}{dx^2}\right) - \lambda u(x) = 0, \qquad \lambda = \rho A\omega^2,$$

$$r_1\,\frac{du(a)}{dx} = p(a)\,\frac{d^2u(a)}{dx^2}, \qquad t_1\,\frac{du(a)}{dx} = -\frac{d}{dx}\left(p(a)\,\frac{d^2u(a)}{dx^2}\right),$$

$$r_2\,\frac{du(b)}{dx} = -p(b)\,\frac{d^2u(b)}{dx^2}, \qquad t_2\,\frac{du(b)}{dx} = \frac{d}{dx}\left(p(b)\,\frac{d^2u(b)}{dx^2}\right).$$

In this case the problem of finding a number λ and a function u such that

$$\begin{cases} u\in V,\ u \neq 0, \\ B(v, u) - \lambda(v, u)_{L^2(I)} = 0, \quad \forall v\in V, \end{cases} \tag{12}$$

is the eigenvalue problem of the bilinear form $B(v, u)$. If it is symmetric, continuous and V-elliptic then it has a countable set of eigenvalues given by [1]

$$\lambda_1 = \min\left\{\frac{B(v, v)}{(v, v)_{L^2(I)}};\ v\in V,\ v \neq 0\right\},$$

$$\lambda_n = \min\left\{\frac{B(v, v)}{(v, v)_{L^2(I)}};\ v\in V,\ v \neq 0,\ (v, v_1)_{L^2} = \cdots = (v, v_n)_{L^2} = 0\right\}.$$

Let us introduce a new inner product in space V: $((v, u)) = B(v, u)\ \forall u, v\in V$. If the sequence $\{v_i(x)\}$ is a base in the space V with the inner product $((v, u))$, the Ritz method leads to the equation:

$$\begin{vmatrix} ((v_1, v_1)) - \lambda(v_1, v_1)_{L^2(I)} & \cdots\cdots\cdots & ((v_1, v_n)) - \lambda(v_1, v_{n1})_{L^2(I)} \\ ((v_n, v_1)) - \lambda(v_n, v_1)_{L^2(I)} & \cdots\cdots\cdots & ((v_n, v_n)) - \lambda(v_n, v_n)_{L^2(I)} \end{vmatrix} = 0. \tag{13}$$

Since it was proved that $B(v, u)$ is continuous, V-elliptic and symmetric approximate eigenvalues can be obtained from equation (13) when dealing with the dynamical behaviour of the beam considered above. Since the ends are elastically restrained against rotation and translation, the problems with classical end conditions are particular cases of the general problem considered here.

4. CONCLUSIONS

Differential equations which describe physical phenomena are often obtained from physical principles by means of the techniques of variational calculus. Certain functionals, (functional of elastic energy, functional of potential energy, etc.) are minimized. The necessary conditions for the existence of extremes of these functionals lead to Euler differential equations. Thus, there is a variational problem, equivalent to the boundary or eigenvalue problem considered [3, 6]. But the differential equation involves unnecessarily derivatives of higher order than the order of the derivatives included in the corresponding functional, which describes a certain type of energy, so it is more natural, from a physical point of view, to look for the weak solution of the given problem than to look for its classical solution [4–9]. The weak solution of a boundary or eigenvalue problem may be obtained under rather natural assumptions by variational methods.

The existence and uniqueness of the weak solutions of a boundary value problem and an eigenvalue problem, which correspond, respectively, to the statical and dynamical behaviour of a tapered beam with edges generally restrained has been demonstrated. The Ritz method has been employed by using polynomials as trial functions.

The use of the weak solution theory enables a substantial generalization of assumptions concerning, the smoothness of coefficients of the differential equation (1) and the continuity of the load q. Consequently, problems involving non-uniform beams such as stepped beams, discontinuous loads, intermediate supports, etc., can be considered.

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