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THE INFLUENCE OF CONCENTRATED MASSES AND PASTERNAK SOIL ON THE FREE VIBRATIONS OF EULER BEAMS—EXACT SOLUTION

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The free vibration frequencies of a beam with flexible ends resting on Pasternak soil are determined in the presence of a concentrated mass at an arbitrary intermediate abscissa. The differential equation of motion is deduced and solved, and the resulting frequency equation gives the exact frequencies of the system. Some numerical examples and comparisons end the paper.

1. INTRODUCTION

The dynamic and stability analysis of Euler–Bernoulli and Timoshenko beams on Pasternak soil has been the subject of various recent investigations. A comprehensive review of various linear elastic soil models and of their physical meanings can be found in reference [1].

The simplest model is obviously given by the Winkler elastic soil, whose dynamic and stability behaviour has been thoroughly investigated both by approximate methods [2] and an exact approach [3–5], in the presence of flexible ends and stepped beam cross-section [3]. The effect of eccentric concentrated masses and of axial forces on the free vibration frequencies has been illustrated in reference [4]. Some foundation models and a finite element for the static analysis of an Euler–Bernoulli beam resting on a Winkler soil have been given in reference [6], whereas dynamic and stability analysis has been presented in references [7–9]. The same beam on a Pasternak two-parameter soil has been analysed in an exact way in references [10, 11], and the corresponding Timoshenko beam has been studied in reference [12]. A useful lower bound for frequencies and critical loads can be obained from reference [13], whereas an extension to a three-parameter Baratha–Levinson soil has been given in reference [14].

In this paper, the exact free vibration frequencies of a Euler beam on two-parameter elastic soil are calculated, in the presence of flexible ends and of a concentrated mass acting along the span at an arbitrary abscissa. Two different reference frames are introduced, with origins at the beam ends, and the solutions of the differential equation of motion are normalized with respect to these origins. In this way, the frequency equation is simplified as much as possible.

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Numerical examples and comparisons end the paper, with use of some known results for classical boundary conditions.

2. EXACT ANALYSIS

Consider the beam in Figure 1, with span L, resting on a two-parameter elastic soil. Let x_1 and x_2 be two different reference frames with origins at the beam ends, and L_1 and L_2 the distance of the concentrated mass M from the origins of the two reference frames. If the Euler–Bernoulli slender beam theory is adopted, then the following equation of motion can easily be deduced by means of Hamilton's principle:

$$(EI)v_i''''(x_i, t) - k_1 v_i''(x_i, t) + k_0 v_i(x_i, t) + \rho A \ddot{v}_i(x_i, t) = 0. \tag{1}$$

Here, E is the Young modulus, I and A are the second moment of area and the area of the beam cross section, ρ is the mass density, k_0 is the Winkler modulus of the subgrade reaction, k_1 is the second foundation parameter, v_i is the vertical displacement, x_i is the abscissa, and t is the time.

The solution can be sought in the form

$$v_i(x_i, t) = V_i(x)\, e^{j\omega t}, \tag{2}$$

where ω is the circular frequency and $j = \sqrt{-1}$. Equation (1) then becomes

$$(EI)V_i''''(x_i) - k_1 V_i''(x_i) + (k_0 - \rho A \omega^2) V_i(x_i) = 0. \tag{3}$$

It is convenient to rewrite this equation in the more abstract form

$$V_i''''(x_i) - bV_i''(x_i) + cV_i(x_i) = 0, \tag{4}$$

with $b = k_1/EI$ and $c = (k_0 - \rho A \omega^2)/EI$. The characteristic polynomial of this equation is

$$r^4 - br^2 + cr = 0, \tag{5}$$

and its general solution is

$$V_i(x_i) = A_{i1}\, e^{r_1 x_i} + A_{i2}\, e^{r_2 x_i} + A_{i3}\, e^{r_3 x_i} + A_{i4}\, e^{r_4 x_i}, \tag{6}$$

where r_1, r_2, r_3 and r_4 are the roots of the polynomial equation (5).

In order to find the roots, it is important to take into account that (1a) k_1 is greater than zero, (1b) EI is greater than zero, (1c) $k_0 - \rho A \omega^2$ does not have a definite sign.

If one defines

$$p = r^2, \tag{7}$$

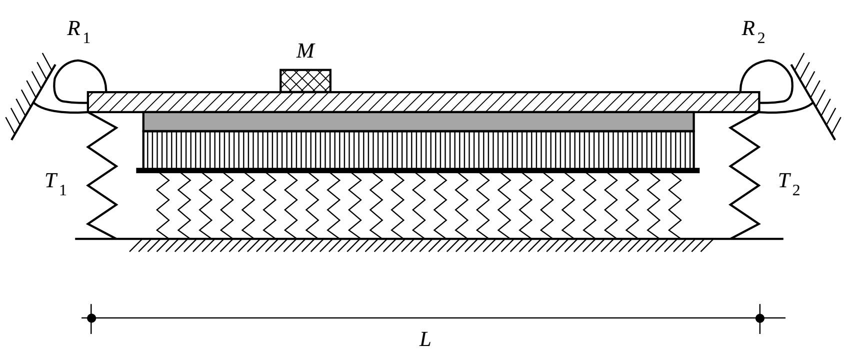


Figure 1. The structural scheme.

then equation (5) becomes a second order polynomial equation:

$$p^2 - bp + c = 0. \tag{8}$$

One can now define $\Delta = b^2 - 4c$, so that the following three cases can be distinguished:

$$\Delta > 0 \qquad \sqrt{\Delta} < b, \tag{9}$$

with roots

$$r_{1,2} = \pm\sqrt{p_1} = \pm\gamma, \qquad r_{3,4} = \pm\sqrt{p_2} = \pm\nu, \tag{10}$$

and solution given by

$$V_i\,(x_i) = C_{i1}\cosh\gamma x_i + C_{i2}\sinh\gamma x_i + C_{i3}\cos\nu x_i + C_{i4}\sin\nu x_i\,; \tag{11}$$

(1b),

$$\Delta > 0, \qquad \sqrt{\Delta} > b, \tag{12}$$

with roots

$$r_{1,2} = \pm\sqrt{p_1} = \pm\gamma, \qquad r_{3,4} = \pm\sqrt{p_3} = \pm\mu, \tag{13}$$

and solution given by

$$V_i\,(x_i) = C_{i1}\cosh\gamma x_i + C_{i2}\sinh\gamma x_i + C_{i3}\cos\mu x_i + C_{i4}\sin\mu x_i \tag{14}$$

with

$$p_1 = \frac{b+\sqrt{\Delta}}{2}, \qquad p_2 = \frac{b-\sqrt{\Delta}}{2}, \qquad p_3 = \frac{-b+\sqrt{\Delta}}{2}\,; \tag{15–17}$$

and finally (1c)

$$\Delta < 0, \tag{18}$$

with roots [15]

$$r_{1,2,3,4} = \pm(\alpha \pm i\beta), \tag{19}$$

where

$$\alpha = \sqrt{\sqrt{\frac{c}{4}} + \frac{b}{4}}, \qquad \beta = \sqrt{\sqrt{\frac{c}{4}} - \frac{b}{4}}, \tag{20, 21}$$

and solution given by

$$\begin{aligned} V_i\,(x_i) = {} & C_{i1}\cos\beta x_i\cosh\alpha x_i + C_{i2}\cos\beta x_i\sinh\alpha x_i. \\ & + C_{i3}\sin\beta x_i\cosh\alpha x_i + C_{i4}\sin\beta x_i\sinh\alpha x_i. \end{aligned} \tag{22}$$

This solution can be normalized with respect to the origin of the reference frame, by imposing

$$\begin{vmatrix} V_{i1}\,(0) & V'_{i1}\,(0) & V''_{i1}\,(0) & V'''_{i1}\,(0) \\ V_{i2}\,(0) & V'_{i2}\,(0) & V''_{i2}\,(0) & V'''_{i2}\,(0) \\ V_{i3}\,(0) & V'_{i3}\,(0) & V''_{i3}\,(0) & V'''_{i3}\,(0) \\ V_{i4}\,(0) & V'_{i4}\,(0) & V''_{i4}\,(0) & V'''_{i4}\,(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}, \tag{23}$$

and

$$V_i\,(x_i) = C_{i1}\,V_{i1} + C_{i2}\,V_{i2} + C_{i3}\,V_{i3} + C_{i4}\,V_{i4}. \tag{24}$$

Henceforth, if $\Delta > 0$ one obtains the following: (1a) $\sqrt{\Delta} < b$,

$$V_{i1} = \frac{1}{\nu^2 - \gamma^2}(\nu^2 \cosh \gamma x_i - \gamma^2 \cosh \nu x_i), \qquad V_{i2} = \frac{1}{\nu^2 - \gamma^2}\left(\frac{\nu^2 \sinh \gamma x_i}{\gamma} - \frac{\gamma^2 \sinh \nu x_i}{\nu}\right), \tag{25, 26}$$

$$V_{i3} = \frac{1}{\nu^2 - \gamma^2}(-\cosh \gamma x_i + \cosh \nu x_i), \qquad V_{i4} = \frac{1}{\nu^2 - \gamma^2}\left(\frac{-\sinh \gamma x_i}{\gamma} + \frac{\sinh \nu x_i}{\nu}\right); \tag{27, 28}$$

(1b), $\sqrt{\Delta} > b$,

$$V_{i1} = \frac{1}{\mu^2 + \gamma^2}(\mu^2 \cosh \gamma x_i + \gamma^2 \cos \mu x_i), \qquad V_{i2} = \frac{1}{\mu^2 + \gamma^2}\left(\frac{\mu^2 \sinh \gamma x_i}{\gamma} + \frac{\gamma^2 \sin \mu x_i}{\mu}\right), \tag{29, 30}$$

$$V_{i3} = \frac{1}{\mu^2 + \gamma^2}(\cosh \gamma x_i - \cos \mu x_i), \qquad V_{i4} = \frac{1}{\mu^2 + \gamma^2}\left(\frac{\sinh \gamma x_i}{\gamma} - \frac{\sin \mu x_i}{\mu}\right); \tag{31, 32}$$

(2), $\Delta < 0$,

$$V_{i1} = = \cos \beta x \cosh \alpha x_i - (\alpha^2 - \beta^2) \sin \beta x_i \sinh \alpha x_i\,/2\alpha\beta, \tag{33}$$

$$V_{i2} = \left[\frac{3\beta^2 - \alpha^2}{\beta} \cosh \alpha x_i \sin \beta x_i + \frac{3\alpha^2 - \beta^2}{\alpha} \cos \beta x_i \sinh \alpha x_i\right] \frac{1}{2(\alpha^2 + \beta^2)}, \tag{34}$$

$$V_{i3} = \sin \beta x_i \sinh \alpha x_i\,/2\alpha\beta \tag{35}$$

$$V_{i4} = \left[\frac{\cosh \alpha x_i \sin \beta x_i}{\beta} - \frac{\cos \beta x_i \sinh \alpha x_i}{\alpha}\right] \frac{1}{2(\alpha^2 + \beta^2)}. \tag{36}$$

TABLE 1

Numerical comparisons with reference [9] *for a simply supported beam*; *the second row gives the exact result*

K_0 $\bar{K}_1$	0	0·5	1	2·5
0	3·1415	3·4767	3·7306	4·2970
	3·14159	3·4767	3·7360	4·2970
1	3·1496	3·4826	3·7407	4·3001
	3·1496	3·48267	3·74078	4·30016
100	3·7483	3·9608	4·1437	4·5824
	3·74836	3·9608	4·1437	4·58239
10 000	10·024	10·036	10·048	10·084
	10·024	10·036	10·048	10·084
1 000 000	31·623	31·623	31·624	31·625
	31·6235	31·6239	31·624	31·625

TABLE 2

Numerical comparisons with reference [7] *for a clamped–clamped beam*

K_0 $\bar{K}_1$	0	0	0·5	0·5	1	1	2·5	2·5
	Reference [7]	Exact	Reference [7]	Exact	Reference [7]	Exact	Reference [7]	Exact
0	4·73	4·73	4·87	4·869	5·32	4·994	5·32	5·32
	7·85	7·854	7·97	7·968	8·38	8·078	8·38	8·381
	11·0	10·996	11·09	11·086	11·43	11·174	11·43	11·43
100	4·95	4·95	5·23	5·071	5·54	5·182	5·48	5·477
	7·90	7·904	8·16	8·017	8·39	8·124	8·42	8·423
	11·01	11·014	11·24	11·104	11·43	11·192	11·44	11·444
10 000	10·12	10·123	10·16	10·137	10·21	10·152	10·41	10·194
	10·84	10·839	10·94	10·883	11·04	10·927	11·38	11·055
	12·53	12·526	12·68	12·588	12·81	12·648	13·21	12·825
1 000 000	31·64	31·626	31·65	31·627	31·65	31·628	31·67	31·629
	31·67	31·653	31·67	31·654	31·68	31·666	31·71	31·662
	31·75	31·738	31·76	31·741	31·77	31·745	31·81	31·757

The boundary conditions are by no means intuitive, and it is necessary to use an energetic approach, in order to be sure of not missing some term. One thus has the following:

at $x_1 = 0$

$$EIV_1''(0) = k_{R1}\, V_1'(0), \qquad EIV_1'''(0) + k_{T1}\, V_1(0) = k_1\, V_1'(0); \tag{37}$$

at $x_2 = 0$,

$$EIV_2''(0) = k_{R2}\, V_2'(0), \qquad EIV_2'''(0) + k_{T2}\, V_2(0) = k_1\, V_2'(0); \tag{38}$$

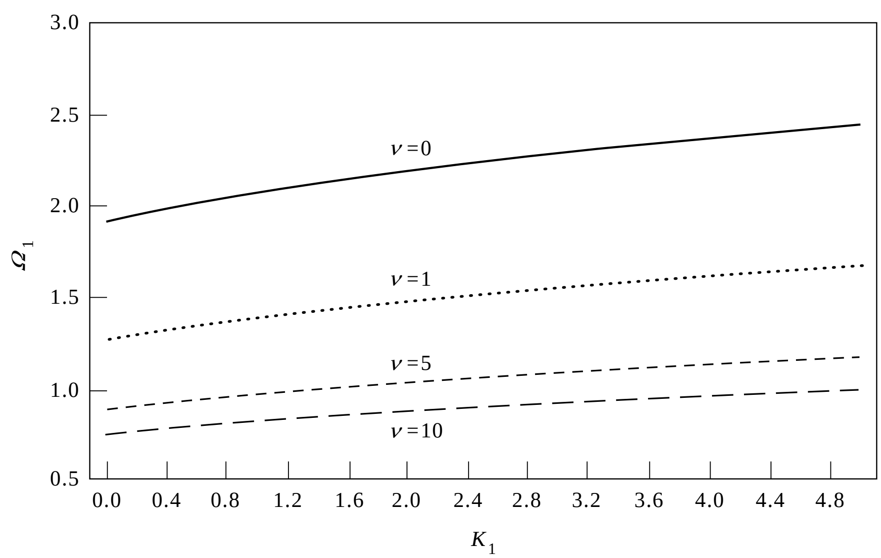


Figure 2. Cantilever beam with tip mass. First non-dimensional frequency coefficient versus second foundation parameter for $k_0 = 1$ and for various values of the concentrated mass.

TABLE 3

First three non-dimensional frequencies for beams with classical boundary conditions and concentrated mass $v = 2$ *at mid-span, the first column is a comparison with reference* [5]

	(K_0, K_1)						
	$(2\pi^4, 0)$	(1, 2·5)	(10, 2·5)	(100, 2·5)	(1, 10)	(10, 10)	(100, 10)
C.F.	2·9141	1·8340	1·9759	2·6348	2·1475	2·2283	2·7214
	4·2193	3·7534	3·7838	4·0771	4·2238	4·2469	4·4699
	7·9522	7·9532	7·9570	8·0012	8·2194	8·2247	8·2644
C.C.	3·2489	3·0463	3·0588	3·1755	3·1720	3·1831	3·2875
	7·9519	7·9124	7·9170	7·9619	8·0808	8·0851	8·1273
	9·6916	9·6785	9·6808	9·7039	9·7868	9·7890	9·8113
C.P.	2·9910	2·6622	2·6838	2·8737	2·6423	2·8646	3·0225
	6·9835	6·9202	6·9268	6·9920	6·8793	7·1507	7·2101
	9·1404	9·1245	9·1273	9·1550	9·101	9·2561	9·2827
P.P.	2·7460	2·2210	2·2601	2·5683	2·4949	2·5224	2·7563
	6·6710	6·3813	6·3900	6·4745	6·6489	6·6565	6·7315
	8·1559	8·1320	8·1358	8·1735	8·3001	8·3037	8·3392
C.S.	2·9229	1·9573	2·0737	2·6361	2·1875	2·2583	2·7219
	4·7936	4·4648	4·4845	4·6761	4·7106	4·7278	4·8951
	8·5129	8·4908	8·4945	8·5304	8·6526	8·6560	8·6899
P.S.	2·7398	1·4451	1·6415	2·4128	1·7927	1·9048	2·5097
	4·6418	4·2364	4·2618	4·4992	4·5302	4·5511	4·7493
	7·3109	7·2649	7·2704	7·3251	7·4746	7·4797	7·5301

at $x_1 = L_1$ and $x_2 = L_2$,

$$V_1(L_1) = V_2(L_2), \qquad V_1'(L_1) = -V_2'(L_2), \qquad V_1''(L_1) = V_2''(L_2),$$
$$EIV_1'''(L_1) + EIV_2'''(L_2) = -M\omega^2 V_1(L_1). \tag{39}$$

Here k_{R1}, k_{R2} are the rotational stiffnesses of the beam ends, and k_{T1}, k_{T2} are the axial stiffnesses at the same ends.

This linear homogeneous system has non-trivial solutions if the determinant of the coefficients is equal to zero (see the Appendix).

TABLE 4

First three non-dimensional frequencies for a beam with $v = 10$, $T_1 = 5$, $K_0 = 10$ *and* $K_1 = 1$, *and for various mass abscissae*

	μ		
Ω	0·25	0·5	0·75
I	1·1041	1·4168	2·0216
II	4·7549	3·4138	2·9370
III	6·7918	7·8992	6·0397

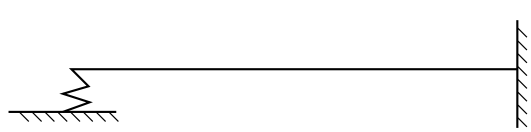
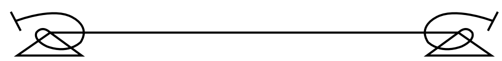


TABLE 5

First three non-dimensional frequencies for a beam with $v = 10$, $R_1 = R_2 = 0{\cdot}5$, $K_0 = 10$ and $K_1 = 1$, and for various mass abscissae

	μ		
Ω	0·25	0·5	0·75
I	1·9490	1·6933	1·9490
II	5·1866	6·5903	5·1866
III	9·0228	8·1560	9·0228



3. NUMERICAL EXAMPLES

It is convenient to define non-dimensional coefficients of the end flexibilities,

$$R_1 = EI/k_{R1}\,L, \qquad R_2 = EI/k_{R2}\,L, \qquad T_1 = EI/k_{T1}\,L^3, \qquad T_2 = EI/k_{T2}\,L^3, \tag{40}$$

and non-dimensional soil parameter coefficients,

$$K_0 = k_0\,L^4/EI, \qquad K_1 = k_1\,L^2/EI, \qquad v = M/\rho AL, \qquad \mu = L_1/L. \tag{41}$$

Finally, it is convenient to express the results in terms of the non-dimensional frequency parameter

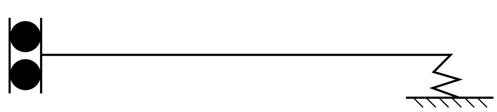
$$\Omega_i = \sqrt{\sqrt{\rho A\omega_i^2\,L^4/EI}}. \tag{42}$$

First of all, a comparison with the results given in references [7, 9] is shown in Tables 1 and 2, where the non-dimensional free frequency coefficients have been given as functions of the two soil parameters, for both simply supported and clamped–clamped beams. It should be noted that, for the sake of comparison, the definition $\overline{K}_1 = K_1\,\pi^2$ is used here. Another comparison is shown in Table 3, where a concentrated mass $v = 2$ at the mid-span has been introduced, and the first three non-dimensional frequencies have been reported as functions of the soil parameters for different classical boundary conditions. The Winkler case has already been given in reference [4].

TABLE 6

First three non-dimensional frequencies for a beam with $v = 10$, $T_2 = 5$, $K_0 = 10$ and $K_1 = 1$, and for various mass abscissae

	μ		
Ω	0·25	0·5	0·75
I	0·9457	0·9754	0·9557
II	2·4232	2·6980	2·4474
III	5·5227	4·4751	5·3641



In Figure 2 the first non-dimensional frequency is given as a function of the second foundation parameter, for different values of the mass: $v = 0$, 1, 5, 10. The frequency increases with the second foundation parameter, and decreases for increasing values of the mass.

In Table 4 the influence of the constraint flexibility is taken into account by calculating the first three non-dimensional free frequencies for a beam with elastic support at the right end and with a clamped right end, in the presence of a concentrated mass $v = 10$ placed at $\mu = 0{\cdot}25$, 0·5, 0·75. The non-dimensional elastic flexibility of the support is equal to 5, and the soil is defined by $K_0 = 10$, $K_1 = 1$.

The same structure is examined in Table 5, in the presence of rotationally flexible ends with flexibilities $R_1 = R_2 = 0{\cdot}5$, and in Table 6 for a beam with sliding at left and an axially flexible end at right with $T_2 = 5$.

Finally, it is worth noting that the position of the mass strongly influences the values of the frequencies.

4. CONCLUSIONS

The exact dynamic analysis of Euler beams resting on Pasternak soil has been performed in the presence of rotationally and axially flexible ends and concentrated masses placed at arbitrary abscissae.

Some numerical examples show good agreement between exact and approximate results.

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APPENDIX

$$a_{11} = R_1 L V_{12} + V_{13} + K_1 T_1 R_1 L^2 V_{11}, \qquad a_{12} = -T_1 L^3 V_{11} + V_{14},$$

$$a_{13} = -(R_2 L V_{22} + V_{23} + K_1 T_2 R_2 L^2 V_{21}), \qquad a_{14} = -(-T_2 L^3 V_{21} + V_{24}),$$

$$a_{21} = R_1 L V'_{12} + V'_{13} + K_1 T_1 R_1 L^2 V'_{11}, \qquad a_{22} = -T_1 L^3 V'_{11} + V'_{14},$$

$$a_{23} = R_2 L V'_{22} + V'_{23} + K_1 T_2 R_2 L^2 V'_{21}, \qquad a_{24} = -T_2 L^3 V'_{21} + V'_{24},$$

$$a_{31} = R_1 L V''_{12} + V''_{13} + K_1 T_1 R_1 L^2 V''_{11}, \qquad a_{32} = -T_1 L^3 V''_{11} + V''_{14},$$

$$a_{33} = -(R_2 L V''_{22} + V''_{23} + K_1 T_2 R_2 L^2 V''_{21}), \qquad a_{34} = -(-T_2 L^3 V''_{21} + V''_{24}),$$

$$a_{41} = R_1 L V'''_{12} + V'''_{13} + K_1 T_1 R_1 L^2 V'''_{11} + a_{11} M\omega^2/EI,$$

$$a_{42} = -T_1 L^3 V'''_{11} + V'''_{14} + a_{12} M\omega^2/EI$$

$$a_{43} = R_2 L V'''_{22} + V'''_{23} + K_1 T_2 R_2 L^2 V'''_{21}, \qquad a_{44} = -T_2 L^3 V'''_{21} + V'''_{24}.$$