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SELF-EXCITED VIBRATIONS OF THE VARIABLE MASS ROTOR/FLUID SYSTEM

L. Cveticanin

Faculty of Technical Sciences, Novi Sad, Yugoslavia

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In this paper the variable mass rotor/fluid system is considered. A rotor with variable mass is settled in hydrodynamic bearings. The dynamic properties of the rotor on which the fluid force and the impact force (due to mass variation) act are analyzed. The conditions of stable rotation are obtained applying the direct Lyapunov theorem. The self-excited vibrations are determined analytically. The Krylov–Bogolubov method is extended for solving the second order differential equation with a complex deflection function, small non-linearity and time variable parameters and a significant damping term. Analyzing the amplitude of self-excited vibrations, the conditions of unstable motion are defined. Special attention is given to the effect of interactive influence of the inertial fluid force and the impact force on the stability of rotation of the rotor. For the rotor on which the band winds up, the vibrations are obtained analytically. The results are compared with numerical ones.

1. INTRODUCTION

Dynamic phenomena induced by interaction between the rotor motion and bearing or seal fluid motion have been recognized for more than 70 years [1], before the first work about oil whip was published by Newirk and Taylor. After that, hundreds of papers present results of experimental investigations, discuss theoretical models of these dynamic phenomena and give specific recommendations of how to avoid vibrations that can seriously perturb the rotating machine normal operation. Many controversial results have appeared, and Hori [2] was the first to give a comprehensive theory that explained the facts consistently. The starting point for the theoretical consideration of a hydrodynamically generated oil film in a journal bearing is the well-known Reynolds equation for incompressible fluids in the laminar regime. The solution of this equation gives the pressure distribution in the journal bearing oil film. By integrating the pressures, the radial and transverse components of the force exerted by the oil film on the journal are obtained. Hori divided the vibrations of the rotor into small and large by comparing their amplitudes with the eccentricity of the bearing center. For large vibrations, a procedure is established for rapid estimation of the size and dispositions of amplitudes resulting from the combined action of rotating and static forces in the rotor assembly incorporating oil film bearings and damping [3]. The steady state journal centre orbit as a function of unbalance, gravity parameter and bearing parameter is analyzed for rigid [4] and flexible [5] rotors. Whirl orbit of a shaft in a finite journal bearing lubricated by micropolar fluid is predicted by numerical computation of the generalized Reynolds equation and the equations of the shaft motion [6]. The mentioned rotors are symmetric and the hydrodynamic bearings are equal. The results are extended to the vibrations of asymmetric flexible rotors supported by asymmetric bearings [7] and [8]. In the papers [9] and [10], the necessary conditions for backward whirling motion of a Jeffcott rotor supported on journal bearings are given.

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In the aforementioned papers and papers [11] and [12], the inertial effects of the fluid are omitted.

In the past ten years many theoretical and experimental investigations have been carried out at Bently Rotor Dynamics Research Corporation in the U.S.A., analyzing the stability of motion due to fluid/solid interaction. The results of the investigations show that some correction of the previous model is necessary. A new model of the fluid forces in seals and lightly loaded, fully lubricated bearings [13–15] is introduced. The inertia effects of the fluid film are included for lightly loaded high speed rotors. The fluid force model is based on the fluid circumferential average velocity of the flow. It proves to be an adequate way in which to represent the solid/fluid dynamic forces [16]. Using this model of fluid force the rotor/fluid model is formed and the instability conditions for rotors with constant mass are obtained [17–21].

The aim of this paper is to extend previous investigations and to analyze the self-excited vibrations of the rotor/fluid model where the mass of the rotor varies in time. The fluid force model based on the fluid circumferential average velocity of the flow is applied. Rotors with variable masses are very often the working elements of machines in the process industry. In those machines, mass is being added or separated from the rotor during its rotation.

In this paper, the threshold of the stable rotation of the rotor is defined. The direct Lyapunov method is applied for stability analysis of pure rotation motion of the rotor. The self-excited vibrations of the rotor are obtained analytically. The Krylov–Bogolubov method (the method of variable amplitude and phase) is extended and adopted for solving the second order differential equations with a complex function and slow time variable parameters where the damping terms are significant. The analytically obtained results are used to define the instability conditions of motion. Special attention is given to the case when the inertial fluid force acts. This component is usually omitted. The interaction between this component and the parameter of impact force on the stability of motion is analyzed in detail.

2. ROTOR/FLUID MODEL

The physical model of the system is a rigid rotor supported on journal bearings. The rotor is a shaft–disc system. The mass of the disc is varying over time. The shaft is supported on journal bearings. The connection between the shaft and the bearing is modelled as a system of two cylinders: one softly supported and rotating inside the other which is fixed. The clearance between them is relatively small and filled with fluid (Figure 1).

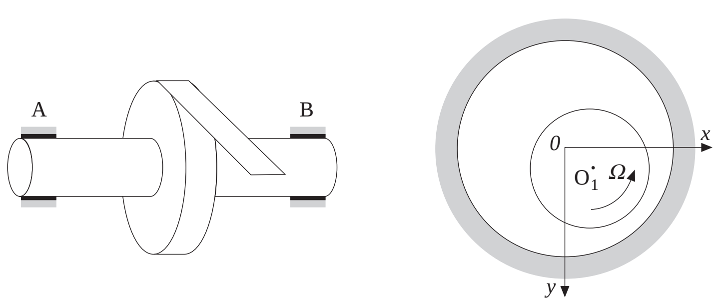

Figure 1. A model of the variable mass rotor/fluid system.

As mentioned in the introduction, various types of fluid force model have been developed. All of them can be divided into two groups: one based on the Reynolds equation for incompressible fluids in the laminar regime [2–12], and the other on the model in the form of a dashpot and a spring [13–21].

The starting point for the theoretical consideration of a hydrodynamically generated oil film in a journal bearing is the well known Reynolds equation for incompressible fluids in a laminar regime [2–12]. The usual Reynolds assumptions are: (a) the oil film is thin enough and the effect of its curvature may be neglected; (b) there is no pressure change across the film thickness; (c) the inertia force of the oil may be neglected; (d) the oil pressure in the unloaded region is zero; and (e) the pressure distribution is two-dimensional. The assumption for many journal bearings is that they are short in relation to their diameter. If only the first order movement of the oil film around the bearing circumference is considered, it produces only a first order change in pressure. For these small motions, the oil forces are (see [2–12])

$$F_1 = \mu R l \left(\frac{l}{c}\right)^2 [-\dot{\rho} g_2 + (\tfrac{1}{2}\Omega - \dot{\theta})\rho g_1],$$

$$F_2 = \mu R l \left(\frac{l}{c}\right)^2 [-\dot{\rho} g_1 + (\tfrac{1}{2}\Omega - \dot{\theta})\rho g_3],$$

where

$$g_1 = -\frac{2\pi}{(1-\rho^2)^2}, \qquad g_2 = \frac{1+2\rho^2}{(1-\rho^2)^{5/2}}\frac{\pi}{2}, \qquad g_3 = \frac{\pi}{2}\frac{1}{(1-\rho^2)^{3/2}},$$

$\rho = e/c$ is the eccentricity ratio, c is the radial clearance, e is the eccentricity of the journal in the bearing, l is the bearing or squeeze-film land length, R is the bearing radius, μ is the lubricant viscosity and Ω is the angular velocity of the rotor. In relation to a stationary co-ordinate system, the oil force is

$$F_z = F_1\,\mathrm{e}^{i\theta} + \mathrm{i}F_2\,\mathrm{e}^{i\theta}.$$

Assuming only the quasi-linear terms, it is

$$F_z = -[(K_0 + \psi(|z|)]z - [D + \psi_D(|z|)](\dot{z} - \lambda\Omega z i), \tag{1}$$

where $|z| = \rho$,

$$K_0 = \pi\mu R l \left(\frac{l}{c}\right)^2 \Omega, \qquad \psi(|z|) = 2\pi\mu R l \left(\frac{l}{c}\right)^2 \Omega |z|^2,$$

$$D = \tfrac{1}{2}\pi\mu R l \left(\frac{l}{c}\right)^2, \qquad \psi_D(|z|) = \tfrac{3}{4}\pi\mu R l \left(\frac{l}{c}\right)^2 |z|^2, \qquad \lambda = \tfrac{1}{2}.$$

The main deficiency of the model (1) is that the inertia force of the oil is dropped.

In this paper, the second model of fluid forces suggested by investigators from the Bently Rotor Dynamics Research Corporation in the U.S.A. is accepted. The advantage of the second model is its simplicity due to some assumptions that follow. The fluid dynamic forces in the clearance are modelled in the form of a mass which is connected with dashpot and a spring in the radial direction [13]. As stated in the paper of Muszynska and Bently [16], due to the rotation of the inner cylinder and the friction, the shaft rotating inside the clearance drags the fluid into rotative circumferential motion. It means that the fluid film related mass, dashpot and spring are not stationary, but rotate. In the fluid velocity profile inside the clearance, the angular velocity of the fluid boundary layer next to the shaft is the same as the shaft rotative speed Ω and the fluid layer next to the stator has zero velocity. It is assumed that when the journal is rotating, centered, fully developed fluid flow is established in the circumferential direction. To simplify the consideration, a function describing the fluid circumferential averge velocity ratio λ is introduced. Then the vital assumption is that the fluid force that results from averaging the circumferential flow, is rotating with angular velocity $\lambda\Omega$. It is supposed that shaft lateral vibrations are small enough to make modifications of this pattern negligible (see [14]). In the previous model it is assumed that the value of λ is 1/2. According to [13–21], the fluid force in stationary co-ordinates can, therefore, be expressed by

$$F = [K_0 + \psi(|z|)]z + [D + \psi_D(|z|)][\dot{z} - \mathrm{i}\lambda(|z|)\Omega z] + M_f[\ddot{z} - 2\mathrm{i}\lambda(|z|)\Omega\dot{z} - \lambda^2(|z|)\Omega^2 z], \tag{2}$$

where $z = x + \mathrm{i}y$ is the rotor lateral complex deflection, K_0 is the fluid film radial stiffness, $\psi(|z|)$ is the non-linear rigidity function, D is the fluid viscous damping, $\psi_D(|z|)$ is the non-linear damping function, M_f is the fluid inertial effect, Ω is the rotative speed, $\lambda(|z|)$ is the fluid circumferential average velocity ratio and $|z| = \sqrt{x^2 + y^2}$, in which x and y are horizontal and vertical components of the rotor lateral displacement in stationary co-ordinates. The fluid film inertia is included in this model. The inertia effects become important for lightly loaded high speed rotors. The properties of the suggested model are deeply investigated and experimentally proved in papers [13–21]. Comparing with the model (1), it is evident that the inertia force is taken into consideration and the fluid circumferential average velocity ratio is introduced.

Using the suggested model of the fluid force, the model of the rotor/fluid system is formed. The mathematical model of the rotor/fluid system is as follows:

$$\Phi = m(\tau)\ddot{z} + D_S\dot{z} + Kz + M_f[\ddot{z} - 2\mathrm{i}\lambda(|z|)\Omega\dot{z} - \lambda^2(|z|)\Omega^2 z] + [D + \psi_D(|z|)][\dot{z} - \mathrm{i}\lambda(|z|)\Omega z] + [K_0 + \psi(|z|)]z, \tag{3}$$

where $m(\tau)$, K and D_S are the mass, stiffness and damping parameters, respectively, $\tau = \epsilon t$ is the slow time, ϵ is a small parameter and Φ is the impact force. The impact force Φ appears due to the mass variation of the rotor. As defined by Meshcherski [22], it is the product of adding or separating mass and its relative velocity,

$$\Phi = -\epsilon \frac{\mathrm{d}m(\tau)}{\mathrm{d}\tau}(\dot{z} - v_z), \tag{4}$$

where v_z is the absolute velocity of adding or separating mass and $\dot{z}$ is the absolute velocity of the rotor. Usually, the relative velocity of adding mass is given as the fraction of the rotor velocity (Bessonov, [23]) and it is

$$\Phi = -\epsilon p \frac{\mathrm{d}m(\tau)}{\mathrm{d}\tau}\dot{z},$$

where $p = 0 - 1$. For $p = 0$, the relative velocity of mass separation or addition is zero. This means that the absolute velocity of the rotor and the absolute velocity of separated mass are the same. Then, the impact force is zero and the separation of the mass is without impact. For $p = 1$, the relative velocity of the mass variation is equal to the absolute velocity of the rotor. The absolute velocity of separated mass is zero. The impact force exists and it is known in the literature as the impact force of Levi–Chivita type.

The non-linear functions $\lambda(|z|)$, $\psi_D(|z|)$ and $\psi(|z|)$ are obtained experimentally. It is evident that the fluid circumferential average velocity ratio $\lambda(|z|)$ varies during a cycle. When the rotor is displaced from its concentric position inside the fixed cylinder, the average velocity decreases. As the variation of the velocity in the clearance between the fixed and inner cylinders is small, it is assumed that the fluid circumferential average velocity ratio has the constant value λ [16]. This value depends on the type of the fluid [15]. Assume that the non-linear function $\psi(|z|)$ is of the parabolic type

$$\psi(|z|) = k_3(|z|)^2. \tag{5}$$

The non-linear function $\psi_D(|z|)$ is omitted. Then, the differential equation of motion is

$$-\epsilon p \frac{\mathrm{d}m}{\mathrm{d}\tau}\dot{z} = m(\tau)\ddot{z} + D_S\dot{z} + Kz + M_f[\ddot{z} - 2\mathrm{i}\lambda\Omega\dot{z} - \lambda^2\Omega^2 z]$$

$$+ D[\dot{z} - \mathrm{i}\lambda\Omega z] + [K_0 + k_3(|z|)^2]z. \tag{6}$$

3. STABILITY ANALYSIS

To investigate the stability of the purely rotational motion of the shaft, the unbalance force is assumed to be equal to zero. The stability of the purely rotation motion of the shaft means the stability of the zero solution of the equation (6).

Separating the real and imaginary parts in the equation (6), two second order differential equations are obtained:

$$-\epsilon p \frac{\mathrm{d}m}{\mathrm{d}\tau}\dot{x} = m(\tau)\ddot{x} + D_S\dot{x} + Kx + M_f[\ddot{x} + 2\lambda\Omega\dot{y} - \lambda^2\Omega^2 x]$$

$$+ D[\dot{x} + \lambda\Omega y] + [K_0 + k_3(x^2 + y^2)]x, \tag{7}$$

$$-\epsilon p \frac{\mathrm{d}m}{\mathrm{d}\tau}\dot{y} = m(\tau)\ddot{y} + D_S\dot{y} + Ky + M_f[\ddot{y} - 2\lambda\Omega\dot{x} - \lambda^2\Omega^2 y]$$

$$+ D[\dot{y} - \lambda\Omega x] + [K_0 + k_3(x^2 + y^2)]y. \tag{8}$$

For the stability analysis, the direct Lyapunov procedure will be applied, [24]. Forming the Lyapunov function

$$V = \tfrac{1}{2}[m(\tau) + M_f](\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}(x^2 + y^2)\left[K + K_0 - M_f\lambda^2\Omega^2 + \frac{D(D + D_S)}{2M_f}\right]$$

$$+ \frac{D}{2M_f}[m(\tau) + M_f](x\dot{x} + y\dot{y}) + \tfrac{1}{4}k_3(x^2 + y^2)^2, \tag{9}$$

which is positive definite for

$$m(\tau) + M_f \geqslant M_f > 0, \tag{10}$$

$$k_3 > 0, \tag{11}$$

$$K + K_0 - M_f\lambda^2\Omega^2 + \frac{D(D + D_S)}{2M_f} > 0, \tag{12}$$

$$\left(K + K_0 - M_f\lambda^2\Omega^2 + \frac{D(D+D_S)}{2M_f}\right) - \left(\frac{D}{2M_f}\right)^2[m(\tau) + M_f]$$

$$\geqslant \left(K + K_0 - M_f\lambda^2\Omega^2 + \frac{D(D+D_S)}{2M_f}\right) - \left(\frac{D}{2M_f}\right)^2[m_{max} + M_f] > 0, \quad (13)$$

where m_{max} is the maximal value of variable mass.

The first time derivative of V after substituting equations (7) and (8) is

$$\dot{V} = -\left\{\epsilon(p - \tfrac{1}{2})\frac{\mathrm{d}m}{\mathrm{d}\tau} + (D_S + D) - \frac{D}{2M_f}[m(\tau) + M_f]\right\}(\dot{x}^2 + \dot{y}^2)$$
$$- \frac{D}{2M_f}\epsilon\frac{\mathrm{d}m}{\mathrm{d}\tau}(p-1)(x\dot{x} + y\dot{y}) - \frac{D}{2M_f}[(K_0 + K) - M_f\lambda^2\Omega^2](x^2 + y^2)$$
$$- \frac{D}{2M_f}k_3(x^2 + y^2)^2. \quad (14)$$

The function (14) is negative definite for

$$K_0 + K - M_f\lambda^2\Omega^2 > 0, \quad (15)$$

$$(D_S + D) - \frac{D}{2M_f}[m_{max} + M_f] + (p - \tfrac{1}{2})\epsilon\left|\frac{\mathrm{d}m}{\mathrm{d}\tau}\right|_{max} > 0, \quad (16)$$

$$\frac{2D}{M_f}\left\{(p - \tfrac{1}{2})\epsilon\frac{\mathrm{d}m}{\mathrm{d}\tau} + D_S + D - \frac{D}{2M_f}[m(\tau) + M_f]\right\}[K_0 + K - M_f\lambda^2\Omega^2]$$
$$- \left(\frac{D}{2M_f}\right)^2\epsilon^2\left(\frac{\mathrm{d}m}{\mathrm{d}\tau}\right)^2(p-1)^2 \geqslant \frac{2D}{M_f}\left\{D_S + D - \frac{D}{2M_f}[m_{max} + M_f] - \epsilon\left|\frac{\mathrm{d}m}{\mathrm{d}\tau}\right|_{max}\right\}$$
$$\times [K_0 + K - M_f\lambda^2\Omega^2] - \left(\frac{D}{2M_f}\right)^2\epsilon^2\left(\frac{\mathrm{d}m}{\mathrm{d}\tau}\right)^2_{max} > 0, \quad (17)$$

where $|\mathrm{d}m/\mathrm{d}\tau|_{max}$ is the maximal value of the mass time derivative.

According to the direct Lyapunov theorem of asymptotic stability [24] (if, for differential equations (7) and (8) of perturbed motion, the positive definite function V exists, and if its first time derivative along the integrating line of the equations (7) and (8) is negative definite, the unperturbed motion is asymptotically stable) and the previous consideration, it can be concluded that the rotation of the rotor with zero deflection of the mass center is asymptotically stable for equation (11),

$$(D_S + D) - \frac{D}{2M_f}[m_{max} + M_f] + (p - \tfrac{1}{2})\epsilon\left|\frac{\mathrm{d}m}{\mathrm{d}\tau}\right|_{max} > 0, \quad (18)$$

and

$$K + K_0 - M_f\lambda^2\Omega^2 - \left(\frac{D}{2M_f}\right)^2[m_{max} + M_f] > 0, \quad (19)$$

$$\frac{2D}{M_f}\left\{D_S + D - \frac{D}{2M_f}[m_{max} + M_f] - \epsilon\left|\frac{dm}{d\tau}\right|_{max}\right\}$$

$$\times [K_0 + K - M_f\lambda^2\Omega^2] - \left(\frac{D}{2M_f}\right)^2\epsilon^2\left(\frac{dm}{d\tau}\right)^2_{max} > 0, \qquad (20)$$

i.e.,

$$\Omega < \frac{1}{\lambda}\sqrt{\frac{K + K_0}{M_f} - \frac{1}{M_f}\left(\frac{D}{2M_f}\right)^2[m_{max} + M_f]}, \qquad (21)$$

or

$$\Omega < \frac{1}{\lambda}\sqrt{\frac{K + K_0}{M_f} - \frac{D}{8M_f^2}\frac{\epsilon^2(dm/d\tau)^2_{max}}{[D_S + D - (D/2M_f)[m_{max} + M_f] - \epsilon|dm/d\tau|_{max}]}}. \qquad (22)$$

The stronger criteria of equations (21) and (22) are the threshold of the asymptotically stable rotation of the rotor with zero deflection of the mass center. It depends not only on the properties of the bearing and the rotor, but also on the mass variation. In Figure 2, the criteria of asymptotically stable motion are plotted. For $M_f = 0{\cdot}1$, $D_S = 1{\cdot}92$, $D = 0{\cdot}08$, $\lambda = 0{\cdot}5$ and various values of the maximal mass of the rotor and the velocity of the mass increase ($dm/d\tau$), the angular velocity of the rotor for which the motion is asymptotically stable is obtained according to the equations (21) and (22).

Analyzing the criteria of stability equations (11) and (18–20) it can be seen that the motion may also be asymptotically stable for the case in which damping is neglected, but the impact parameter has the value

$$p \geqslant 1/2, \qquad (23)$$

and

$$\Omega < \frac{1}{\lambda}\sqrt{\frac{K + K_0}{M_f}}. \qquad (24)$$

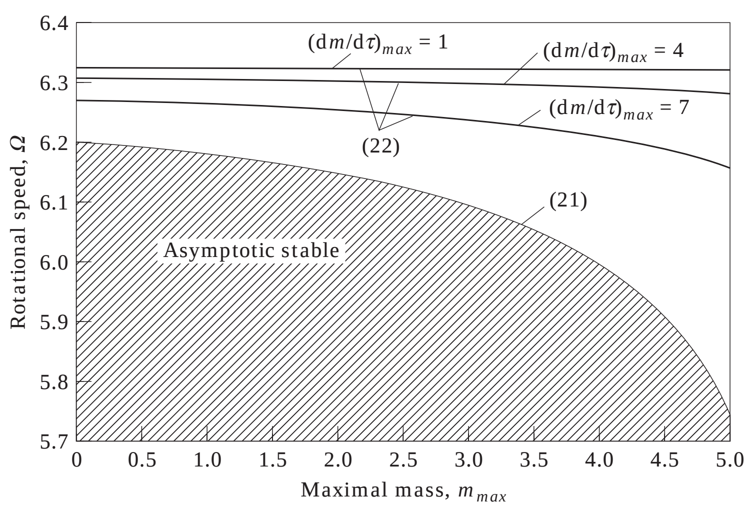

Figure 2. The threshold of asymptotic stable rotation.

Then, the impact force acts as motion stabilizer.

Consider the case in which $p = 0$; i.e., the impact force is zero. Due to the definition of the simple stability given by Lyapunov [24] (if, for differential equations (7) and (8) of perturbed motion, the positive definite function V exists if its first time derivative along the integrating line of the equations (7) and (8) is a negative function or zero, the unperturbed motion is stable) it can be concluded that the rotation of the rotor with zero deflection of the mass center, when the mass variation is without impact, is simple stable if the condition (11) and

$$K + K_0 - M_f \lambda^2 \Omega^2 \geqslant 0, \tag{25}$$

$$(D_S + D) - \frac{D}{2M_f}[m_{max} + M_f] - \frac{\epsilon}{2}\left|\frac{\mathrm{d}m}{\mathrm{d}\tau}\right|_{max} \geqslant 0, \tag{26}$$

are satisfied. From the last inequality it can be seen that the simple stable motion is realized only for the case when damping exist. Otherwise, the conditions of stability are not fulfilled.

4. SELF-EXCITED VIBRATION

Consider the case in which the non-linearity is small and the unbalance is omitted. The mathematical model of motion is

$$m(\tau)\ddot{z} + D_S\dot{z} + Kz + M_f[\ddot{z} - 2\mathrm{i}\lambda\Omega\dot{z} - \lambda^2\Omega^2 z] + D[\dot{z} - \mathrm{i}\lambda\Omega z] + K_0 z = \epsilon\psi(|z|)z - \epsilon p\,\frac{\mathrm{d}m(\tau)}{\mathrm{d}\tau}\,\dot{z}, \tag{27}$$

where $\epsilon \ll 1$ is a small parameter. The equation is a second order differential equation with a complex function, small non-linearity and slow time variable parameter. The damping terms in the equation are significant. The approximate solutions of the equation (27) are to be determined. In reference [25] the procedure of variable amplitude and phase (Krylov–Bogolubov method) is extended for the systems with a complex function. Using the principles of the aforementioned procedure, an extension of the method is used for the differential equation (27) with a complex deflection function and small non-linearities, and also slow time variable parameters. The solution of equation (27) has the same form as the solution of the equation with constant parameters and without non-linearity (when $\epsilon = 0$)

$$m_0\ddot{z} + D_S\dot{z} + Kz + M_f[\ddot{z} - 2\mathrm{i}\lambda\Omega\dot{z} - \lambda^2\Omega^2 z] + D[\dot{z} - \mathrm{i}\lambda\Omega z] + K_0 z = 0, \tag{28}$$

where m_0 is the mass for $\tau = 0$. The solution of equation (28) is assumed as

$$z = A\,\mathrm{e}^{\delta t}\,\mathrm{e}^{\mathrm{i}(\omega t + \theta)}. \tag{29}$$

Substituting (29) and its time derivatives into equation (28) and separating the real and imaginary parts, two algebraic equations are obtained. The unknown values are

$$\delta = -\frac{D + D_S}{2(m_0 + M_f)} \pm \frac{\sqrt{-R_1 + \sqrt{(R_1)^2 + (R_2)^2}}}{2\sqrt{2}(m_0 + M_f)}, \tag{30}$$

$$\omega = \frac{M_f\lambda\Omega}{m_0 + M_f} \pm \frac{\sqrt{R_1 + \sqrt{(R_1)^2 + (R_2)^2}}}{2\sqrt{2}(m_0 + M_f)}, \tag{31}$$

where

$$R_1 = 4(m_0 + M_f)(K + K_0) - 4m_0 M_f \lambda^2 \Omega^2 - (D + D_S)^2, \tag{32}$$

$$R_2 = 4\lambda\Omega(m_0 D - M_f D_S). \tag{33}$$

The values of A and θ are obtained for the arbitrary initial conditions. According to equation (29), the solution of equation (27) is assumed in the form

$$z = A(t)\,\mathrm{e}^{\Delta(t)}\,\mathrm{e}^{\mathrm{i}\varphi(t)}, \tag{34}$$

where

$$\Delta = \int_0^t \delta(\tau)\,\mathrm{d}t = \int_0^t \left\{-\frac{D + D_S}{2[m(\tau) + M_f]} \pm \frac{\sqrt{-R_1(\tau) + \sqrt{[R_1(\tau)]^2 + [R_2(\tau)]^2}}}{2\sqrt{2}[m(\tau) + M_f]}\right\}\mathrm{d}t, \tag{35}$$

$$\varphi(t) = \int_0^t \omega(\tau)\,\mathrm{d}t + \theta(t)$$

$$= \int_0^t \left\{\frac{M_f \lambda \Omega}{m(\tau) + M_f} \pm \frac{\sqrt{R_1(\tau) + \sqrt{[R_1(\tau)]^2 + [R_2(\tau)]^2}}}{2\sqrt{2}[m(\tau) + M_f]}\right\}\mathrm{d}t + \theta(t), \tag{36}$$

$$R_1(\tau) = 4[m(\tau) + M_f](K + K_0) - 4m(\tau) M_f \lambda^2 \Omega^2 - (D + D_S)^2, \tag{37}$$

$$R_2(\tau) = 4\lambda\Omega[m(\tau) D - M_f D_S]. \tag{38}$$

Substituting equation (31) into equation (24) and separating the real and imaginary parts and after a transformation, two first order differential equations are obtained:

$$\dot{A} = \left[\frac{\epsilon\delta(\tau)}{m(\tau) + M_f}\,\mathrm{Re} + \frac{\epsilon\omega(\tau)}{m(\tau) + M_f}\,\mathrm{Im}\right]\frac{1}{[\delta(\tau)]^2 + [\omega(\tau)]^2}$$

$$- \frac{\epsilon A}{[\delta(\tau)]^2 + [\omega(\tau)]^2}\left[\delta(\tau)\frac{\mathrm{d}\delta(\tau)}{\mathrm{d}\tau} + \omega(\tau)\frac{\mathrm{d}\omega(\tau)}{\mathrm{d}\tau}\right], \tag{39}$$

$$A\dot{\theta} = \left[\frac{\epsilon\delta(\tau)}{m(\tau) + M_f}\,\mathrm{Im} - \frac{\epsilon\omega(\tau)}{m(\tau) + M_f}\,\mathrm{Re}\right]\frac{1}{[\delta(\tau)]^2 + [\omega(\tau)]^2}$$

$$+ \frac{\epsilon A}{[\delta(\tau)]^2 + [\omega(\tau)]^2}\left[\omega(\tau)\frac{\mathrm{d}\delta(\tau)}{\mathrm{d}\tau} - \delta(\tau)\frac{\mathrm{d}\omega(\tau)}{\mathrm{d}\tau}\right]. \tag{40}$$

where

$$\delta(\tau) = -\frac{D + D_S}{2[m(\tau) + M_f]} \pm \frac{\sqrt{-R_1(\tau) + \sqrt{[R_1(\tau)]^2 + [R_2(\tau)]^2}}}{2\sqrt{2}[m(\tau) + M_f]}, \tag{41}$$

$$\omega(\tau) = \frac{M_f \lambda \Omega}{m(\tau) + M_f} \pm \frac{\sqrt{R_1(\tau) + \sqrt{[R_1(\tau)]^2 + [R_2(\tau)]^2}}}{2\sqrt{2}[m(\tau) + M_f]}, \tag{42}$$

$$\mathrm{Re} \equiv A\psi(A\,\mathrm{e}^{\Delta}) - pA\frac{\mathrm{d}m(\tau)}{\mathrm{d}\tau}\delta(\tau), \tag{43}$$

$$\text{Im} \equiv -pA \frac{\mathrm{d}m(\tau)}{\mathrm{d}\tau} \omega(\tau). \tag{44}$$

If the non-linearity is omitted, equation (39) transforms into the form

$$\dot{A} = -\frac{\mathrm{d}m(\tau)}{\mathrm{d}\tau} \frac{\epsilon A p}{m(\tau) + M_f} - \frac{\epsilon A}{2} \frac{1}{[\delta(\tau)]^2 + [\omega(\tau)]^2} \frac{\mathrm{d}[\delta(\tau)]^2 + [\omega(\tau)]^2}{\mathrm{d}\tau}. \tag{45}$$

Integrating equation (45) for the initial conditions $A(0) = A_0$, $m(0) = m_0$, $\delta(0) = \delta_0$, $\omega(0) = \omega_0$, it is

$$A = A_0 \left(\frac{m_0 + M_f}{m(\tau) + M_f} \right)^p \left(\frac{\delta_0^2 + \omega_0^2}{[\delta(\tau)]^2 + [\omega(\tau)]^2} \right)^{1/2}. \tag{46}$$

The value of A depends on the impact parameter p. For the case in which it is zero, the $A - t$ function is

$$A = A_0 \sqrt{\frac{\delta^2 + \omega^2}{[\delta(\tau)]^2 + [\omega(\tau)]^2}}. \tag{47}$$

Next, we analyze equations (41), (42), (46) and (47) for the case in which the damping of the system is extremely strong. As the damping parameter is constant and the mass of the system varies, it causes variation of the damping function. For the case in which the mass increases, the damping function (41) and the frequency function (42) decrease. Then, the properties of the function A depend on the velocity of the added mass. If the mass is added with a velocity that is smaller than the velocity of the rotor (the parameter $p < 1$) and when the adding of mass is with the velocity equal to the absolute velocity of the rotor ($p = 0$) (see equation (47)), the function A increases. For the case in which the absolute velocity of the adding mass is zero ($p = 1$), the function A (see equation (46)) is constant. According to equation (34), it can be seen that the amplitude of self-excited vibrations is the product of the function A and the function e^{Δ}. For the case of intensive damping, it can be seen that e^{Δ} decreases exponentially (see equation (35)) and the amplitude of vibration disappears.

The more complex situation is when the damping in the system is not the dominant value. Then, the amplitude of self-excited vibrations has a tendency to increase if

$$\frac{\mathrm{d}}{\mathrm{d}t} (A\, \mathrm{e}^{\Delta}) \geqslant 0,$$

i.e., $[\dot{A} + A\delta(\tau)]\, \mathrm{e}^{\Delta} \geqslant 0$. If the vibrations have a tendency to increase, the motion of rotor is unstable. As $\mathrm{e}^{\Delta} > 0$, the condition for unstable motion is

$$\dot{A} + A\delta(\tau) \geqslant 0. \tag{48}$$

For the case in which the non-linearity is neglected, the condition of unstable motion of the rotor is

$$-\frac{\mathrm{d}m(\tau)}{\mathrm{d}\tau} \frac{\epsilon A p}{m(\tau) + M_f} - \frac{\epsilon A}{2} \frac{1}{[\delta(\tau)]^2 + [\omega(\tau)]^2} \frac{\mathrm{d}[\delta(\tau)]^2 + [\omega(\tau)]^2}{\mathrm{d}\tau} + A\delta(\tau) \geqslant 0, \tag{49}$$

where A is given as (46). It is evident that the unstable motion is strongly correlated with the mass variation and the parameter of impact force.

4.1. FLUID INERTIAL EFFECT NEGLECTED

As suggested in reference [13], the fluid dynamic forces in the clearance between the stationary and rotating cylinders are very often modelled in the form of a dashpot and a spring. Then, the inertial effects of the fluid are neglected. This assumption is assumed due to the fact that the mass M_f is quite small.

For the case in which the fluid inertial effect is neglected, the simplified system of the first order differential equations that describe the rotor vibrations is

$$\dot{A} = \left[\frac{\epsilon\delta(\tau)}{m(\tau)}\mathrm{Re} + \frac{\epsilon\omega(\tau)}{m(\tau)}\mathrm{Im}\right]\frac{1}{[\delta(\tau)]^2 + [\omega(\tau)]^2} - \frac{\epsilon A}{[\delta(\tau)]^2 + [\omega(\tau)]^2} \times \left[\delta(\tau)\frac{\mathrm{d}\delta(\tau)}{\mathrm{d}\tau} + \omega(\tau)\frac{\mathrm{d}\omega(\tau)}{\mathrm{d}\tau}\right], \tag{50}$$

$$A\dot{\theta} = \left[\frac{\epsilon\delta(\tau)}{m(\tau)}\mathrm{Im} - \frac{\epsilon\omega(\tau)}{m(\tau)}\mathrm{Re}\right]\frac{1}{[\delta(\tau)]^2 + [\omega(\tau)]^2} + \frac{\epsilon A}{[\delta(\tau)]^2 + [\omega(\tau)]^2} \times \left[\omega(\tau)\frac{\mathrm{d}\delta(\tau)}{\mathrm{d}\tau} - \delta(\tau)\frac{\mathrm{d}\omega(\tau)}{\mathrm{d}\tau}\right], \tag{51}$$

where

$$\delta(\tau) = -\frac{D + D_S}{2m(\tau)} \pm \frac{\sqrt{-R_1(\tau) + \sqrt{[R_1(\tau)]^2 + [R_2(\tau)]^2}}}{2\sqrt{2}m(\tau)}, \tag{52}$$

$$\omega(\tau) = \pm\frac{\sqrt{R_1(\tau) + \sqrt{[R_1(\tau)]^2 + [R_2(\tau)]^2}}}{2\sqrt{2}m(\tau)}, \tag{53}$$

$$R_1(\tau) = 4m(\tau)(K + K_0) - (D + D_S)^2, \tag{54}$$

$$R_2(\tau) = 4\lambda\Omega m(\tau)D. \tag{55}$$

Substituting equations (43) and (44) into the differential equation (50), it becomes

$$\dot{A} = \frac{\epsilon A\psi(A\,\mathrm{e}^{\Delta})\delta(\tau)}{m(\tau)}\frac{1}{[\delta(\tau)]^2 + [\omega(\tau)]^2} - \frac{\mathrm{d}m(\tau)}{\mathrm{d}\tau}\frac{\epsilon Ap}{m(\tau)} - \frac{\epsilon A}{2}\frac{1}{[\delta(\tau)]^2 + [\omega(\tau)]^2}\frac{\mathrm{d}[\delta(\tau)]^2 + [\omega(\tau)]^2}{\mathrm{d}\tau}. \tag{56}$$

According to equation (48), the motion is unstable if

$$\frac{\epsilon A\psi(A\,\mathrm{e}^{\Delta})\delta(\tau)}{m(\tau)}\frac{1}{[\delta(\tau)]^2 + [\omega(\tau)]^2} - \frac{\mathrm{d}m(\tau)}{\mathrm{d}\tau}\frac{\epsilon Ap}{m(\tau)} - \frac{\epsilon A}{2}\frac{1}{[\delta(\tau)]^2 + [\omega(\tau)]^2}\frac{\mathrm{d}[\delta(\tau)]^2 + [\omega(\tau)]^2}{\mathrm{d}\tau} + A\delta(\tau) \geqslant 0. \tag{57}$$

Analyzing equation (57), it can be concluded that the stability of the vibrations depends not only on the mass variation properties of the system but also on the average circumferential angular velocity of the fluid force.

4.2. DAMPING NEGLECTED

As discussed in section 4, if the damping is extremely strong in the system, the influence of slow mass variation can be neglected. The self-excited vibrations decrease and the motion is stable. If the damping in the system is quite small, the effects of the mass variation and of the impact force have very important roles. The character of the motion depends directly on the mass variation and the impact force. To give the correct conclusions about the motion, assume that the damping is omitted. Then, the vibrations of the rotor are described by two first order differential equations:

$$\dot{A} = \frac{\epsilon}{[m(\tau) + M_f]\omega(\tau)} \mathrm{Im} - \frac{\epsilon A}{\omega(\tau)} \frac{\mathrm{d}\omega(\tau)}{\mathrm{d}\tau}, \tag{58}$$

$$A\dot{\theta} = -\frac{\epsilon}{[m(\tau) + M_f]\omega(\tau)} \mathrm{Re}, \tag{59}$$

where

$$\delta(\tau) = 0, \tag{60}$$

$$\omega(\tau) = \frac{M_f \lambda \Omega}{m(\tau) + M_f} \pm \frac{\sqrt{[m(\tau) + M_f](K + K_0) - m(\tau) M_f \lambda^2 \Omega^2}}{m(\tau) + M_f}, \tag{61}$$

i.e.,

$$\dot{A} = -\frac{\epsilon p A[\mathrm{d}m(\tau)/\mathrm{d}\tau]}{m(\tau) + M_f} - \frac{\epsilon A}{\omega(\tau)} \frac{\mathrm{d}\omega(\tau)}{\mathrm{d}\tau}, \tag{62}$$

$$\dot{\theta} = -\frac{\epsilon \psi(A)}{[m(\tau) + M_f]\omega(\tau)}. \tag{63}$$

If the non-linearity is given as a parabolic function (5), the differential equation of motion is

$$m(\tau)\ddot{z} + (K + K_0)z + M_f[\ddot{z} - 2\mathrm{i}\lambda\Omega\dot{z} - \lambda^2\Omega^2 z] = -\epsilon p \frac{\mathrm{d}m}{\mathrm{d}\tau}\dot{z} - \epsilon k_3(|z|)^2 z. \tag{64}$$

For the initial conditions $A(0) = A_0$, $\theta(0) = \theta_0$ and $\omega(0) = \omega_0$, the solutions of equations (62) and (63) are

$$A = A_0 \left[\frac{m_0 + M_f}{m(\tau) + M_f}\right]^p \left[\frac{\omega_0}{\omega(\tau)}\right], \tag{65}$$

$$\theta = -\epsilon k_3 A_0^2 \omega_0^2 (m_0 + M_f)^{2p} \int \frac{\mathrm{d}t}{[m(\tau) + M_f]^{2p+1}[\omega(\tau)]^3} + \theta_0, \tag{66}$$

or after substituting equation (61), it is

$$A = A_0 \left[\frac{m_0 + M_f}{m(\tau) + M_f}\right]^{p-1} \left[\frac{M_f\lambda\Omega \pm \sqrt{[m_0 + M_f](K + K_0) - m_0 M_f \lambda^2\Omega^2}}{M_f\lambda\Omega \pm \sqrt{[m(\tau) + M_f](K + K_0) - m(\tau) M_f \lambda^2\Omega^2}}\right], \tag{67}$$

$$\theta = -\epsilon k_3 A_0^2 \omega_0^2 (m_0 + M_f)^{2p}$$

$$\times \int \frac{\mathrm{d}t}{[m(\tau) + M_f]^{2(p-1)}[M_f\lambda\Omega + \sqrt{[m(\tau) + M_f](K + K_0) - m(\tau) M_f \lambda^2\Omega^2}]^3} + \theta_0. \tag{68}$$

The solution (67) exists only for

$$\Omega \leqslant \frac{1}{\lambda}\sqrt{\frac{K+K_0}{M_f}\left(1+\frac{M_f}{m_0}\right)}. \tag{69}$$

The amplitude of vibration is not a function of non-linear rigidity. In spite of that, the phase angle is a function of non-linearity (5).

For the case in which the damping is neglected, the condition of instability transforms to the relation $\dot{A} \geqslant 0$; i.e.,

$$\frac{pA[\mathrm{d}m(\tau)/\mathrm{d}\tau]}{m(\tau)+M_f} + \frac{A}{\omega(\tau)}\frac{\mathrm{d}\omega(\tau)}{\mathrm{d}\tau} \leqslant 0. \tag{70}$$

It is evident from equation (70) that the non-linear rigidity force has no influence on instability conditions.

Consider two special cases: the first when the relative velocity of mass adding is zero, i.e., the absolute velocity of adding mass is equal to the velocity of the rotor ($p = 0$), and the second when the absolute velocity of adding mass is zero ($p = 1$).

4.2.1. *Impact force neglected* ($p = 0$)

For the first case when the impact force is neglected the amplitude–time function is

$$A = A_0\left[\frac{m(\tau)+M_f}{m_0+M_f}\right]\left[\frac{M_f\lambda\Omega \pm \sqrt{[m_0+M_f](K+K_0) - m_0 M_f\lambda^2\Omega^2}}{M_f\lambda\Omega \pm \sqrt{[m(\tau)+M_f](K+K_0) - m(\tau)M_f\lambda^2\Omega^2}}\right]. \tag{71}$$

The amplitude of self-excited vibrations is the function of angular velocity of the rotor. It is necessary to define the area in which the motion of the rotor is unstable. For the case when the impact force is zero ($p = 0$), the condition of unstable motion (70) is simplified to

$$\mathrm{d}\omega/\mathrm{d}\tau \leqslant 0, \tag{72}$$

i.e.,

$$\frac{\mathrm{d}m}{\mathrm{d}\tau}\frac{1}{m(\tau)+M_f}\left\{\frac{1}{m(\tau)+M_f}\left[M_f\lambda\Omega \pm \sqrt{[m(\tau)+M_f](K+K_0) - m(\tau)M_f\lambda^2\Omega^2}\right]\right.$$

$$\left.\mp \frac{(K+K_0) - M_f\lambda^2\Omega^2}{2\sqrt{[m(\tau)+M_f](K+K_0) - m(\tau)M_f\lambda^2\Omega^2}}\right\} \geqslant 0. \tag{73}$$

If it is assumed that the mass of the rotor increases it is $\mathrm{d}m(\tau)/\mathrm{d}\tau > 0$. Hence, the inequality (73) exists for equation (69).

4.2.2. *Impact force exists* ($p = 1$)

If the impact force exists, the amplitude–time function is, according to equation (67),

$$A = A_0\left[\frac{M_f\lambda\Omega \pm \sqrt{[m_0+M_f](K+K_0) - m_0 M_f\lambda^2\Omega^2}}{M_f\lambda\Omega \pm \sqrt{[m(\tau)+M_f](K+K_0) - m(\tau)M_f\lambda^2\Omega^2}}\right]. \tag{74}$$

The motion is unstable for

$$\frac{[\mathrm{d}m(\tau)/\mathrm{d}\tau]}{m(\tau)+M_f} + \frac{1}{\omega(\tau)}\frac{\mathrm{d}\omega(\tau)}{\mathrm{d}\tau} \leqslant 0, \tag{75}$$

i.e.,

$$\frac{\mathrm{d}m}{\mathrm{d}\tau}\frac{K+K_0-M_f\lambda^2\Omega^2}{\{M_f\lambda\Omega+\sqrt{[m(\tau)+M_f](K+K_0)-m(\tau)M_f\lambda^2\Omega^2}\}}$$
$$\times\frac{1}{\sqrt{[m(\tau)+M_f](K+K_0)-m(\tau)M_f\lambda^2\Omega^2}}\leqslant 0. \tag{76}$$

For $\mathrm{d}m/\mathrm{d}t > 0$ and equation (69), the condition (76) cannot be satisfied. This means that for adding mass with zero absolute velocity and the rotor velocity (69), the rotation of the rotor is never unstable. The amplitude of self-excited vibrations decreases and the rotation of the rotor is stable, as mentioned in the previous section.

The results of section 4.2 will be discussed in an example.

4.3. DAMPING AND FLUID INERTIAL FORCE NEGLECTED

For this special case, the amplitude of vibration is

$$A=A_0\left[\frac{m_0}{m(\tau)}\right]^{p-1/2}, \tag{77}$$

$$\theta=-\epsilon k_3A_0^2\omega_0^2(m_0)^{2p}\int\frac{\mathrm{d}t}{[m(\tau)]^{2(p-1)}[\sqrt{[m(\tau)(K+K_0)}]^3}+\theta_0. \tag{78}$$

It is interesting to conclude that the amplitude of vibration is not dependent on rigidity coefficients of the rotor and fluid layer. Besides, the amplitude of vibration does not depend on the angular velocity of the rotor. For a rotor with increasing mass, the amplitude of vibration decreases for $p \geqslant 1/2$, and increases for $p < 1/2$. This means that if the adding of the mass is with the velocity which is the halfed or minor than the absolute velocity of the rotor, the motion of the rotor is stable. For the mass adding with the velocity which is higher or equal to the velocity of the rotor, the motion of the rotor is unstable.

5. EXAMPLE

Consider the motion of the rotor on which the band is winding up. The rotor is the symmetrical shaft–disc system. The mass of the disc varies in time due to winding of the band. As shown in reference [23], the mass variation is a linear function of time,

$$m=m_0+\epsilon t, \tag{79}$$

where m_0 is the mass of the empty rotor and $\epsilon=\rho l v h$, in which ρ is the density of the band material, l is the length of the disc, v is the velocity of band winding and h is the thickness of the band. Let us assume that ϵ is a small value and the mass variation is slow. For a real rotor in the textile industry, it is [26]

$$m=0{\cdot}9+0{\cdot}1t.$$

The rotor is supported on journal bearings. For the case in which the damping properties of the bearing can be omitted, the differential equations of motion are

$$m(\tau)\ddot{x}+Kx+M_f[\ddot{x}+2\lambda\Omega\dot{y}-\lambda^2\Omega^2x]+[K_0+\epsilon k_3(x^2+y^2)]x=-\epsilon p\frac{\mathrm{d}m}{\mathrm{d}\tau}\dot{x}, \tag{80}$$

$$m(\tau)\ddot{y}+Ky+M_f[\ddot{y}-2\lambda\Omega\dot{x}-\lambda^2\Omega^2y]+[K_0+\epsilon k_3(x^2+y^2)]y=-\epsilon p\frac{\mathrm{d}m}{\mathrm{d}\tau}\dot{y}. \tag{81}$$

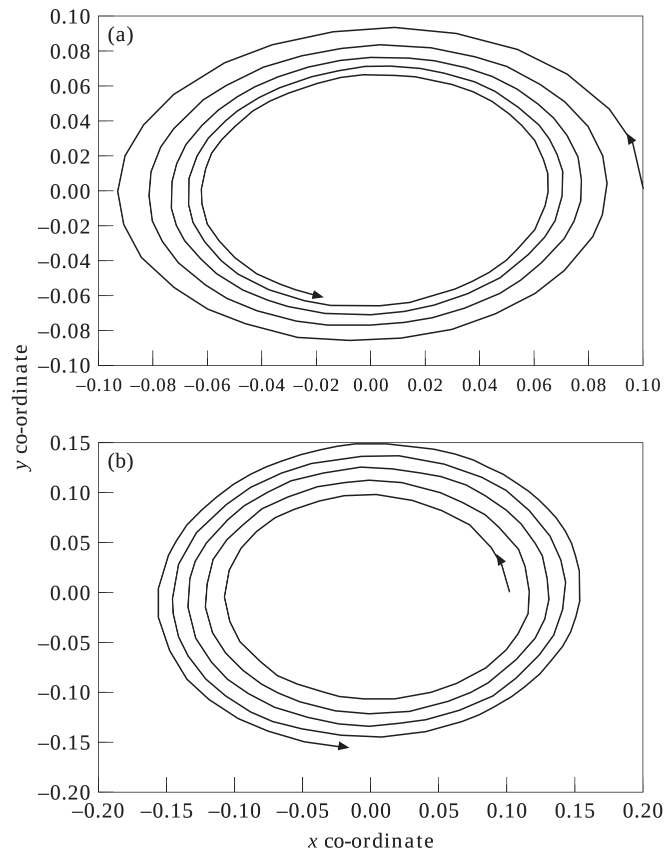

Figure 3. Motion of rotor center in the x–y plane: (a) $p = 1$; (b) $p = 0$.

As the parameters are $D = D_S = 0$, $K + K_0 = 1$, $M_f = 0{\cdot}1$, $\lambda = 0{\cdot}5$, $k_3 = 0{\cdot}1$ and $\epsilon = 0{\cdot}1$, and the initial conditions are $x(0) = 0{\cdot}1$, $\dot{x} = 0$, $y(0) = 0$ and $\dot{y}(0) = 0{\cdot}1$, equations (80) and (81) can be solved numerically by applying the Runge–Kutta method. In Figure 3, the motion of the rotor center in the x–y plane is plotted for $\Omega = 1$ and $p = 1$ (Figure 3(a)) and $p = 0$ (Figure 3(b)). The first group of parameters satisfies the conditions of asymptotic stable motion (23) and (24) and the motion of the rotor center is bounded (see Figure 3(a)). The second group of parameters does not satisfy the relation (23) and the conditions of the stable motion are not fulfilled. As can be seen in Figure 3(b), the motion of the rotor center is not bounded.

To prove the correctness of the analytically solved procedures for obtaining self-excited vibrations, the analytically obtained solutions are compared with the numerical ones. The analytical approximate solutions of the differential equations (67) and (68) are compared with numerical solutions of the differential equations (80) and (81). In Figure 4(a) and 4(b), the amplitude–time diagrams are plotted for $p = 0$ and $p = 1$, respectively. For the first case, when $\Omega = 4{\cdot}15$ the amplitude has a tendency to increase. In the second case, the amplitude has a tendency to decrease for $\Omega = 2{\cdot}5$. The analytically and numerically obtained solutions show very good agreement. The analytical solution is the averaged value of the numerical one.

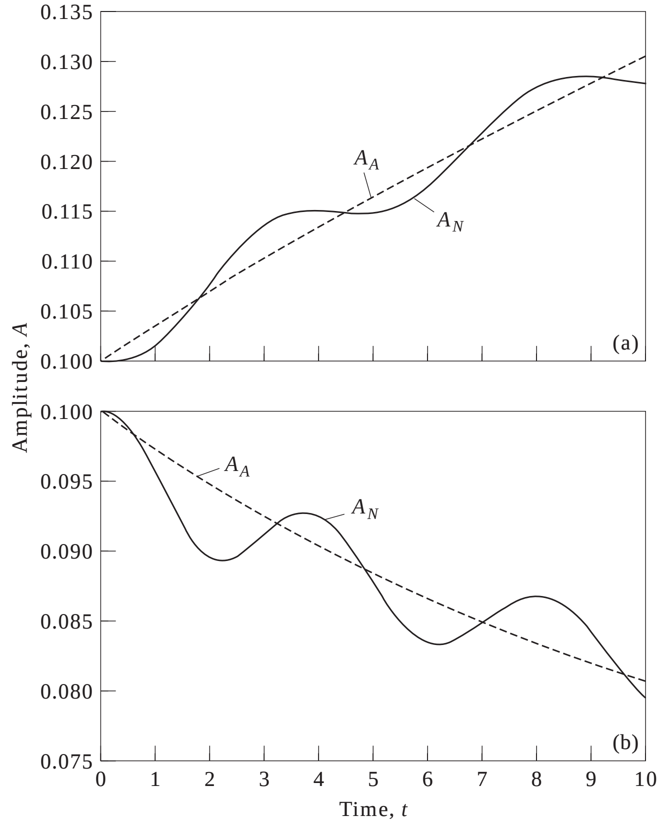

Figure 4. Amplitude–time diagrams obtained analytically A_A and numerically A_N: (a) $p = 0$; (b) $p = 1$.

6. CONCLUSIONS

The following conclusions can be drawn.

(1) The rotation of the rotor with the variable mass and zero deflection of the mass center which is settled on journal bearings is asymptotically stable if the conditions (11), (18), (21) and (22) are satisfied. The threshold of stability is a function not only of the physical properties of the rotor and the fluid film but also of the mass increase: the higher the value of adding mass on the rotor the lower the limit of the angular velocity of the rotor. This means that the stability of rotation of the rotor with angular velocity Ω will not be disturbed only if the added mass is quite small. On the contrary, to stabilize the motion the angular velocity of motion of the rotor must be decreased.

(2) If the damping in the system is so small that it can be omitted, the motion of the rotor with mass adding is stable only if the impact force exists and the parameter is $p > 1/2$.

(3) The self-excited vibrations of the rotor are obtained analytically by applying the asymptotic method of Krylov–Bogolubov. The solutions of the differential equations obtained analytically are compared with the numerical ones. They are in good agreement for a mass increase up to 50% of the initial mass.

(4) Using the asymptotic analytical solutions, the criteria of unstable motion are obtained. The threshold of unstable motion depends not only on the physical properties of the rotor and journal but also on the mass variation parameters.

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APPENDIX: NOTATION

A	amplitude of vibration
A_0	initial amplitude
c	radial clearance
D	fluid damping parameter
D_S	damping parameter of the rotor
e	eccentricity of journal in bearing
F	fluid force
i	imaginary unit
K	stiffness coefficient
k_3	coefficient of non-linearity
K_0	fluid film radial stiffness
l	bearing or squeeze film land length
m_0	initial mass
$m(\tau)$	variable mass
m_{max}	maximal mass value
M_f	fluid mass
p	constant parameter
R	bearing radius
$r(\tau)$	position of center of unbalanced rotor
t	time
v_z	velocity of adding or separating mass
V	Lyapunov function
x, y	horizontal and vertical components of rotor lateral displacement in stationary co-ordinates
z	complex deflection function
$\|z\|$	$= (x^2 + y^2)^{1/2}$
$\dot{z}$	absolute velocity of the rotor
δ	coefficient of damping
δ_0	initial coefficient of damping
Δ	variable damping function
ϵ	small parameter
θ	phase of vibration
θ_0	initial phase of vibration
$\lambda(\|z\|)$	fluid circumferential average velocity
μ	lubricant viscosity
ρ	eccentricity ratio
τ	slow variable time
$\varphi(t)$	variable phase function
Φ	impact force
$\psi(\|z\|)$	non-linear rigidity function
$\psi_D(\|z\|)$	non-linear damping force
ω	frequency of vibration
ω_0	initial frequency of vibration
Ω	rotational speed