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A VARIANT OF THE METHOD OF ORTHOGONAL POLYNOMIALS

R. O. Grossi and C. M. Albarracín

Facultad de Ingenieria, Universidad Nacional de Salta, Buenos Aires 177, 4400-*Salta*, *Argentina*

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1. INTRODUCTION

In a very interesting paper Bhat [1] proposed a set of characteristic orthogonal polynomials for use in the Rayleigh–Ritz method for the study of the vibration of rectangular plates. He showed that the procedure for obtaining the set of orthogonal polynomials is simple, and that the method yields results superior to other methods for lower modes, particularly when the plate has one or more free edges. The method has also been applied among others in the following problems: the determination of natural frequencies and mode shapes of a rotating uniform cantilever beam with a tip mass [2]; the determination of natural frequencies of transverse vibration of thin, rectangular plates of non-uniform thickness [3]; the determination of natural frequencies of edge restrained tapered rectangular plates [4].

It is the purpose of the present paper to present a variant of Bhat's method, based on use of the Rayleigh–Schmidt method of undetermined powers, [5]. This procedure allows the use of a lower number of orthogonal polynomials. This is an important aspect in various situations which require the use of a large number of polynomials in the Bhat's method and generate some numerical instability. This situation can be generally avoided with the present variant.

2. GENERATION OF ORTHOGONAL POLYNOMIALS WITH UNDETERMINED POWERS

In the case of beams the shape function is assumed to be a linear combination of orthogonal polynomials;

$$u(x) = \sum_{i=0}^{N} c_i p_i(x). \tag{1}$$

The set of orthogonal polynomial is generated by using the classical Gram–Schmidt procedure as follows:

The first member $p_0(x)$ is chosen as $p_0(x) = \sum_{i=0}^{5} a_i x^{n_i}$, where the coefficients a_i are determined from the boundary conditions and the n_i are adjustable parameters. The exponents n_i are determined in such a way as to minimize or nearly minimize the approximate eigenvalues.

The other members of the orthogonal set of polynomials in the interval $[a, b]$ are generated as follows:

$$p_1(x) = (x - A_1)p_0(x), \qquad p_i(x) = (x - A_i)p_{i-1}(x) - B_i p_{i-2}(x), \qquad i > 1.$$

$$A_i = \int_a^b x w(x) p_{i-1}^2(x)\,\mathrm{d}x \Big/ \int_a^b w(x) p_{i-1}^2(x)\,\mathrm{d}x,$$

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$$B_i = \int_a^b x w(x) p_{i-1}(x) p_{i-2}(x)\,\mathrm{d}x \Big/ \int_a^b w(x) p_{i-2}^2(x)\,\mathrm{d}x,$$

where $w(x)$ is the weighting function. For a uniform beam, $w(x)$ is unity, and in the case of a tapered beam, $w(x)$ can be conveniently chosen as the non-uniformity of the beam. In the present approach $p_0(x)$ satisfies all the boundary conditions, both geometric and natural; the other members of the orthogonal set satisfy only the geometric boundary conditions.

In the case of plates the shape function is assumed to be

$$u(x, y) = \sum_{i=0}^{N} \sum_{j=0}^{M} c_{ij} p_i(x) q_j(y), \tag{2}$$

where p_i and q_j are obtained by means of the procedure described above and both $p_0(x)$ and $q_0(x)$ have adjustable exponents. Application of the Rayleigh–Ritz method yields an eigenvalue problem of the type $[\mathbf{K}] - \omega^2[\mathbf{M}] = 0$. Solution of this eigenvalue equation yields the natural frequencies.

3. NUMERICAL RESULTS

Tables 1 and 2 show values of $\sqrt{\lambda_1}$, where $\lambda_1 = \sqrt{(\rho A/EI)}\omega_1 L^2$ is the fundamental frequency coefficient, for a uniform beam of length L, elastically restrained against rotation and translation. The values obtained with the present method using only one adjustable exponent in $p_0(x)$, are compared with the exact values reported by Maurizi *et al.* [6] and Rao and Mirza [7], respectively. Also, the values obtained by means of Bhat's method of characteristic orthogonal polynomials are included. Excellent agreement was obtained between the present values and the exact results.

Table 3 depicts a comparison of values of the fundamental frequency coefficient $\Omega_1 = \sqrt{(\rho h_0/D_0)}\omega_1 a^2$ for a rectangular tapered plate elastically restrained against rotation and translation. The plate thickness is described by $h(x) = h_0(1 + \alpha(x/a))$. The values were obtained by means of the classical Bhat's method with eight terms in each

TABLE 1

Values of $\sqrt{\lambda_1}$ for a uniform beam subject to a rotational restraint at $x = 0$ and a translational restraint at $x = 1$. ($T_1 = \infty$, $R_2 = 0$, $R_1 = r_1 L/EI$, $T_2 = t_2 L^3/EI$).

R_1	T_2	(I)	(II)	(III)	N in eq. (1)
0·00	0·1	0·739730	0·739730	0·739730	4
0·00	1000·0	3·126081	3·126681	3·126083	9
0·01	0·1	0·757696	0·757696	0·757696	2
0·01	1000·0	3·127658	3·128258	3·127659	10
0·10	0·1	0·878208	0·878208	0·878208	2
0·10	1000·0	3·141553	3·142162	3·141554	11
1·00	0·1	1·287038	1·287038	1·287038	5
1·00	1000·0	3·256645	3·257324	3·256646	10
1000·00	0·1	1·888235	1·888235	1·888235	5
1000·00	1000·0	3·894008	3·895111	3·894011	9

(I) reference [6]; (II) classical Bhat's method with $N = 12$ in equation (1); (III) present approach.

TABLE 2

Values of $\sqrt{\lambda_1}$ *for a generally restrained Bernoulli—Euler beam.* ($R_1 = r_1L/EI$, $T_1 = t_1L^3/EI$, $R_2 = r_2L/EI$, $T_2 = t_2L^3/EI$, $T_1 = T_2/100 = T$, $R_1 = R_2/100 = R$).

T	R	(I)	(II)	(III)	N in eq. (1)
0·10	0·01	1·238122	1·238123	1·238123	11
0·10	1000·00	1·691790	1·691790	1·691790	5
1·00	0·01	1·531633	1·533096	1·531634	6
1·00	1000·00	2·291046	2·291046	2·291046	11
10·00	0·01	2·324034	2·324640	2·324034	8
10·00	1000·00	2·720894	2·721326	2·720899	11
100·00	0·01	3·107827	3·107912	3·107828	12
100·00	1000·00	3·844697	3·845066	3·844699	10

(I) reference [7]; (II) classical Bhat's method with $N = 13$ in equation (1); (III) present approach.

direction, and the present approach. In almost all cases a very good agreement is obtained with the use of only 3 terms in each direction and only one adjustable exponent in $p_0(x)$ and $q_0(x)$.

TABLE 3

Values of Ω_1 *of a rectangular plate with edge restraints* ($R_1 = R_2 = R_4 = S_1 = S_2 = S_4 = \infty$, $R_3 = r_3a/D_0$, $S_3 = t_3a^3/D_0$, $R_3 = S_3$, $\lambda = a/b$, *where the notations follow those of reference* [4]).

α	λ	R_3	(I)	(II)	$N = M$ in eq. (2)
0·0	0·5	0	22·641	22·676	4
		1	22·706	22·743	3
		10	22·995	23·010	3
		100	23·951	23·944	3
		1000	24·513	24·505	3
		∞	24·578	24·581	3
0·2	0·5	0	24·861	24·907	3
		1	24·916	24·961	3
		10	25·182	25·200	3
		100	26·172	26·142	3
		1000	26·907	26·881	3
		∞	26·985	26·988	4
0·0	1·0	0	23·927	23·980	3
		1	24·111	24·162	3
		10	24·893	24·920	3
		100	28·223	28·230	3
		1000	34·445	34·465	3
		∞	35·985	35·990	3
		0	26·278	26·337	3
		1	26·435	26·495	3
0·2	1·0	10	27·163	27·198	3
		100	30·206	30·216	3
		1000	37·379	37·339	3
		∞	39·510	39·514	4

(I) classical Bhat's method with N = M = 8 in equation (2); (II) present approach.

4. CONCLUSIONS

A variant of Bhat's method of orthogonal polynomials, based on the use of undetermined parameters has been developed. In this approach the first member, in the orthogonal set of polynomials, is constructed with adjustable exponents so as to satisfy all the boundary conditions, both geometrical and natural. The other members of the orthogonal set satisfy only the geometric boundary conditions. This procedure refines the shape function $p_0(x)$ by optimizing the exponents n_i and consequently allows the use of a lower number of terms in the approximating function.

The method has been applied to the determination of natural frequencies of generally restrained beams and plates. The analysis of Tables 1–3 show that the present approach implies the use of a lower number of polynomials and yields excellent results, even if only one adjustable exponent is used. This situation is particularly relevant in those cases that require the use of a large number of orthogonal polynomials in the classical method proposed by Bhat.

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