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PERIODIC SOLUTIONS OF STRONGLY QUADRATIC NON-LINEAR OSCILLATORS BY THE ELLIPTIC PERTURBATION METHOD

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The elliptic perturbation method is applied to the study of the periodic solutions of strongly quadratic non-linear oscillators of the form $\ddot{x} + c_1 x + c_2 x^2 = \epsilon f(x, \dot{x})$, in which the Jacobian elliptic functions are employed. The generalized Van der Pol equation with $f(x, \dot{x}) = \mu_0 + \mu_1 x - \mu_2 x^2$ is studied in detail. Comparisons are made with the solutions obtained by using the Lindstedt–Poincaré method and Runge–Kutta method to show the efficiency of the present method.

1. INTRODUCTION

Jacobian elliptic functions can be adopted to exactly solve the following non-linear differential equation

$$\ddot{x} + c_1 x + c_3 x^3 = 0. \tag{1.1}$$

This is well documented in the work of McLachlan [1], Davis [2] and Nayfeh [3]. Barkham and Soudack [4–7] were the first to use Jacobian elliptic functions to construct an approximate solution for the equation

$$\ddot{x} + c_1 x + c_3 x^3 + \epsilon f(x, \dot{x}, t) = 0, \tag{1.2}$$

such that

$$x(t) = A(t)\,\mathrm{cn}\,(\omega t - \varphi(t), k). \tag{1.3}$$

Christopher [8] and Christopher and Brocklehurst [9] were also concerned with the same approximation technique but the amplitude A, frequency ω, phase φ and parameter k were dependent on time t. Their attention was restricted to the case $f(x, \dot{x}, t) = b\dot{x}$ and the oscillator was hard, i.e., $c_1 > 0$, $c_3 > 0$. Yuste and Bejarano [10] showed that the Christopher method could be extended to hard–soft oscillators, i.e., $c_1 > 0$, $c_3 < 0$ and to soft–hard ones, i.e., $c_1 < 0$, $c_3 > 0$. After that, Yuste and his colleagues presented methods to obtain approximate solutions in terms of Jacobian elliptic functions for the class of equation (1.2), such as the elliptic harmonic balance (EHB) method [11–17], elliptic Krylov–Bogoliubov (EKB) method [18–20], elliptic Galerkin method [21], elliptic Rayleigh method [22] and elliptic cubication technique [23, 24]. Many researchers also developed

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other techniques. For example, Cap [25], Coppola and Rand [26], Roy [27] presented different elliptic averaging methods to obtain the first order solution of equation (1.2). Recently, the authors have also presented two elliptic function methods: elliptic perturbation method (EPM) [28] and elliptic Lindstedt–Poincaré (ELP) method [29].

However, most of these methods are related to cubic non-linear oscillators, and very few of them have analyzed quadratic non-linear oscillators. In this paper, the elliptic perturbation method will be used to analyze the periodic solutions of quadratic non-linear oscillators of the form

$$\ddot{x} + c_1 x + c_2 x^2 = \epsilon f(x, \dot{x}), \tag{1.4}$$

which are associated with many physical systems such as betatron oscillations and vibrations of shells. It is therefore also an important area of non-linear vibration investigation.

2. THE ELLIPTIC PERTURBATION METHOD

Consider the equation having quadratic non-linearity

$$\ddot{x} + c_1 x + c_2 x^2 = \epsilon f(x, \dot{x}), \tag{2.1}$$

where ϵ is a small parameter. When $\epsilon = 0$, the solution of equation (2.1) can be written as

$$x_0 = a_0 \,\mathrm{cn}^2(\tau, k) + b_0. \tag{2.2}$$

Here

$$\tau = \omega_0 t, \qquad a_0 = 6\omega_0^2 k^2/c_2, \tag{2.3, 2.4}$$

$$b_0 = -4\omega_0^2(2k^2 - 1) + c_1/2c_2, \qquad \omega_0^4 = c_1^2/16(k^4 - k^2 + 1). \tag{2.5, 2.6}$$

$\mathrm{cn}(\tau, k)$ is the cosine Jacobian elliptic function, a_0, ω_0 and k are called the amplitude, the angular frequency and the modulus of the elliptic function respectively, and b_0 is called the bias. When $\epsilon \neq 0$, one assumes that the solution of equation (2.1) is still given by (2.2), i.e.,

$$x = a\,\mathrm{cn}^2(\tau, k) + b, \tag{2.7}$$

but with a, b and τ being dependent on the parameter ϵ. For the steady state solution, a and b should be independent of the time t so one can further expand them in powers of ϵ:

$$a = a_0 + \epsilon a_1 + \cdots = \sum_{n=0}^{\infty} \epsilon^n a_n, \qquad b = b_0 + \epsilon b_1 + \cdots = \sum_{n=0}^{\infty} \epsilon^n b_n \tag{2.8, 2.9}$$

and letting

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \omega(\tau) = \omega_0 + \epsilon\omega_1(\tau) + \cdots = \sum_{n=0}^{\infty} \epsilon^n \omega_n(\tau), \tag{2.10}$$

where ω_0 is a constant independent of τ. Then equation (2.7) can be rewritten as

$$x = \sum_{n=0}^{\infty} \epsilon^n x_n(\tau), \qquad x_n = a_n \,\mathrm{cn}^2(\tau, k) + b_n. \tag{2.11, 2.12}$$

It can be seen from equations (2.8) to (2.10) that when $\epsilon = 0$, a, b and ω become a_0, b_0 and ω_0 respectively. That is, $x = x_0$ as $\epsilon = 0$. Therefore the assumption of equation (2.7) is reasonable. Substituting equations (2.10) and (2.11) into equation (2.1) and equating the coefficients of like powers of ϵ yields the following equations:

$$\epsilon^0\colon\quad \omega_0^2 x_0'' + c_1 x_0 + c_2 x_0^2 = 0, \tag{2.13}$$

$$\epsilon^1\colon\quad \omega_0 \frac{\mathrm{d}}{\mathrm{d}\tau}(\omega_1 x_0') + \omega_1 \frac{\mathrm{d}}{\mathrm{d}\tau}(\omega_0 x_0') + \omega_0 \frac{\mathrm{d}}{\mathrm{d}\tau}(\omega_0 x_1') + (c_1 + 2c_2 x_0)x_1 = f(x_0, \omega_0 x_0') \tag{2.14}$$

$$\begin{aligned}\epsilon^2\colon\quad & \omega_0 \frac{\mathrm{d}}{\mathrm{d}\tau}(\omega_2 x_0') + \omega_2 \frac{\mathrm{d}}{\mathrm{d}\tau}(\omega_0 x_0') + \omega_1 \frac{\mathrm{d}}{\mathrm{d}\tau}(\omega_0 x_2') + \omega_1 \frac{\mathrm{d}}{\mathrm{d}\tau}(\omega_0 x_1') + \omega_0 \frac{\mathrm{d}}{\mathrm{d}\tau}(\omega_1 x_1') \\ & + \omega_1 \frac{\mathrm{d}}{\mathrm{d}\tau}(\omega_1 x_0') + (c_1 + 2c_2 x_0)x_2 + c_2 x_1^2 \\ & = f_x' (x_0, \omega_0 x_0')x_1 + f_{\dot{x}}' (x_0, \omega_0 x_0')(\omega_0 x_1' + \omega_1 x_0'),\end{aligned} \tag{2.15}$$

where $x' = \mathrm{d}x/\mathrm{d}\tau$, $f_x' = \partial f/\partial x$, $f_{\dot{x}}' = \partial f/\partial \dot{x}$.

It can be seen that equation (2.13) is obtained from the generating equation (2.1) by using the transformation (2.10). Therefore, when $c_1 > 0$, $c_2 > 0$, the solution of equation (2.13) is (2.2).

Multiplying both sides of equation (2.14) by x_0' and integrating, one obtains

$$\omega_0 \omega_1 x_0'^2 |_0^\tau = \int_0^\tau f(x_0, \omega_0 x_0')x_0' \,\mathrm{d}\tau - \frac{a_1}{a_0}(\omega_0 x_0')^2|_0^\tau - x_1 (c_1 x_0 + c_2 x_0^2)|_0^\tau. \tag{2.16}$$

Note that $x_0(0) = x_0(2K)$, $x_1(0) = x_1(2K)$, $x_0'(0) = x_0'(2K)$, where K is the complete elliptic integral of the first kind. One assumes that $\omega_1(\tau)$ is periodic with period equal to $2K$. Thus letting $\tau = 2K$ and $\tau = K$ in equation (2.16), one has respectively

$$\int_0^{2K} f(x_0, \omega_0 x_0')x_0' \,\mathrm{d}\tau = 0, \tag{2.17}$$

$$\int_0^{K} f(x_0, \omega_0 x_0')x_0' \,\mathrm{d}\tau = (a_1 + b_1)[c_1(a_0 + b_0) + c_2(a_0 + b_0)^2] - b_1(c_1 b_0 + c_2 b_0^2). \tag{2.18}$$

Similarly, multiplying both sides of equation (2.15) by x_0' and then integrating, one obtains

$$\begin{aligned}\omega_0 \omega_2 x_0'^2 |_0^\tau = & \int_0^\tau f_x' (x_0, \omega_0 x_0')x_1 x_0' \,\mathrm{d}\tau + \int_0^\tau f_{\dot{x}}' (x_0, \omega_0 x_0')(\omega_0 \mathrm{x}_1' + \omega_1 x_0')x_0' \,\mathrm{d}\tau \\ & - (1/2a_0^2)(2a_0 a_2 \omega_0^2 + 2a_0 a_1 \omega_0 \omega_1 + a_0^2 \omega_1^2 - a_1^2 \omega_0^2)x_0'^2 |_0^\tau \\ & - (1/a_0)(a_0 x_2 - a_1 x_1)(c_1 x_0 + c_2 x_0^2)|_0^\tau - \tfrac{1}{2} x_1^2 (c_1 + 2c_2 x_0)|_0^\tau.\end{aligned} \tag{2.19}$$

Let $\tau = 2K$ in equation (2.19), then

$$\int_0^{2K} f_x' (x_0, \omega_0 x_0')x_1 x_0' \,\mathrm{d}\tau + \int_0^{2K} f_{\dot{x}}' (x_0, \omega_0 x_0')(\omega_0 \mathrm{x}_1' + \omega_1 \mathrm{x}_0')\mathrm{x}_0' \,\mathrm{d}\tau = 0 \tag{2.20}$$

So a_0, b_0, ω_0 and k^2 can be determined from equations (2.4)–(2.6) and (2.17) while a_1, b_1 and $\omega_1(\tau)$ can be obtained from equations (2.16), (2.18) and (2.20). They can be expressed as follows for convenience.

$$a_1 = -I_n/I_d, \qquad b_1 = a_1 B_1 + B_0, \qquad \omega_1(\tau) = a_1 W_1(\tau) + W_0(\tau), \tag{2.21–2.23}$$

where

$$I_n = B_0 \int_0^{2K} f'_x(x_0, \omega_0 x'_0) x'_0 \, \mathrm{d}\tau + \int_0^{2K} f'_{\dot{x}}(x_0, \omega_0 x'_0) W_0(\tau) x_0'^2 \, \mathrm{d}\tau = 0, \tag{2.24}$$

$$I_d = \frac{1}{a_0}\left\{\int_0^{2K} f'_x(x_0, \omega_0 x'_0) x_0 x'_0 \, \mathrm{d}\tau + \int_0^{2K} f'_{\dot{x}}(x_0, \omega_0 x'_0)\,[\omega_0 + a_0 \mathrm{W}_1(\tau)] \mathrm{x}_0'^2 \, \mathrm{d}\tau \right.$$
$$\left. + \int_0^{2K} (a_0 B_1 - b_0) f'_x(x_0, \omega_0 x'_0) x'_0 \, \mathrm{d}\tau \right\}, \tag{2.25}$$

$$B_0 = -\int_0^K f(x_0, \omega_0 x'_0) x'_0 \, \mathrm{d}\tau / (c_1 a_0 + c_2 a_0^2 + 2c_2 a_0 b_0),$$

$$B_1 = -[c_1(a_0 + b_0) + c_2(a_0 + b_0)^2]/(c_1 a_0 + c_2 a_0^2 + 2c_2 a_0 b_0), \tag{2.26, 2.27}$$

$$W_0(\tau) = \frac{1}{\omega_0 x_0'^2}\left\{\int_0^{\tau} f(x_0, \omega_0 x'_0) x'_0 \, \mathrm{d}\tau + B_0 [c_1(a_0 + b_0) + c_2(a_0 + b_0)^2 - (c_1 x_0 + c_2 x_0^2)]\right\}, \tag{2.28}$$

$$W_1(\tau) = \frac{1}{\omega_0 x_0'^2}\left\{-\frac{1}{a_0}(\omega_0 x'_0)^2 + (1 + B_1)[c_1(a_0 + b_0) + c_2(a_0 + b_0)^2] \right.$$
$$\left. + (\mathrm{cn}^2 \tau + B_1)(c_1 x_0 + c_2 x_0^2)\right\}. \tag{2.29}$$

It is worth pointing out that when $c_1 > 0$, $c_2 < 0$, the solution of equation (2.13) can be expressed by

$$x_0(\tau) = \bar{a}_0 \, \mathrm{sn}^2 \tau + \bar{b}_0, \tag{2.30}$$

where

$$\bar{a}_0 = -a_0, \qquad \bar{b}_0 = a_0 + b_0. \tag{2.31, 2.32}$$

It can be shown that equation (2.30) is indeed identical to equation (2.2), because

$$a_0 \, \mathrm{cn}^2 \tau + b_0 = a_0(1 - \mathrm{sn}^2 \tau) + b_0 = \bar{a}_0 \, \mathrm{sn}^2 \tau + \bar{b}_0.$$

Similarly, when $c_1 < 0$, $c_2 > 0$, the solution of equation (2.13) can be expressed by

$$x_0(\tau) = \bar{\bar{a}}_0 \, \mathrm{dn}^2 \tau + \bar{\bar{b}}_0. \tag{2.33}$$

Here

$$\bar{\bar{a}}_0 = a_0/k^2, \qquad \bar{\bar{b}}_0 = b_0 - a_0(1-k^2)/k^2. \tag{2.34, 2.35}$$

It can also be proved that equation (2.33) is equivalent to equation (2.2). Therefore, one can use equations (2.2) as a unified solution of equation (2.13) later.

3. THE GENERALIZED VAN DER POL OSCILLATOR

As an application of the elliptic perturbation method, one considers the generalized Van der Pol oscillator

$$\ddot{x} + c_1 x + c_2 x^2 = \epsilon(\mu_0 + \mu_1 x - \mu_2 x^2)\dot{x}. \tag{3.1}$$

Here $f(x, \dot{x}) = (\mu_0 + \mu_1 x - \mu_2 x^2)\dot{x}$. Let

$$I_1(\tau) = \int f(x_0, \omega_0 x_0')x_0' \, d\tau. \tag{3.2}$$

Substituting equation (2.2) into (3.2), one obtains

$$I_1(\tau) = 4\omega_0 a_0^2 [(C_a I_{11}^k + C_b I_{12}^k + C_c I_{13}^k)\tau + C_{11} Z(\tau) + C_{12} \operatorname{sn}\tau \operatorname{cn}\tau \operatorname{dn}\tau + C_{13} \operatorname{sn}\tau \operatorname{cn}^3\tau \operatorname{dn}\tau + C_{14} \operatorname{sn}\tau \operatorname{cn}^5\tau \operatorname{dn}\tau + C_{15} \operatorname{sn}\tau \operatorname{cn}^7\tau \operatorname{dn}\tau. \tag{3.3}$$

Using the condition of equation (2.17) and the periodic property of elliptic functions, one has

$$C_a I_{11}^k + C_b I_{12}^k + C_c I_{13}^k = 0. \tag{3.4}$$

Therefore

$$I_1(\tau) = 4\omega_0 a_0^2 [C_{11} Z(\tau) + C_{12} \operatorname{sn}\tau \operatorname{cn}\tau \operatorname{dn}\tau + C_{13} \operatorname{sn}\tau \operatorname{cn}^3\tau \operatorname{dn}\tau + C_{14} \operatorname{sn}\tau \operatorname{cn}^5\tau \operatorname{dn}\tau + C_{15} \operatorname{sn}\tau \operatorname{cn}^7\tau \operatorname{dn}\tau]. \tag{3.5}$$

where

$$C_a = \mu_0 + \mu_1 b_0 - \mu_2 b_0^2, \qquad C_b = \mu_1 a_0 - 2\mu_2 a_0 b_0, \qquad C_c = -\mu_2 a_0^2, \tag{3.6–3.8}$$

$$I_{11}^k = (1/15k^4)[(1-k^2)(k^2-2) + 2(k^4-k^2+1)E/K]. \tag{3.9}$$

$$I_{12}^k = (1/105k^6)[(1-k^2)(3k^4-15k^2+8) + (2k^2-1)(3k^4-3k^2+8)E/K]. \tag{3.10}$$

$$I_{13}^k = (1/315k^8)[(1-k^2)(5k^6-45k^4+48k^2-16) + (10k^8-20k^6+66k^4-56k^2+16)E/K]. \tag{3.11}$$

The coefficients C_{11} to C_{15} are listed in Appendix 1. $Z(\tau)$ is called the Jacobi zeta function with period $2K$, and is defined by

$$Z(\tau) = E(\tau) - (E/K)\tau, \tag{3.12}$$

where $E(\tau)$ is the elliptic integral of the second kind [30] and E is the complete integral of $E(\tau)$. According to equation (3.5), one has $I_1(0) = I_1(K) = 0$, so

$$\int_0^K f(x_0, \omega_0 x_0')x_0' \, d\tau = 0. \tag{3.13}$$

From equation (2.26)

$$B_0 = 0. \tag{3.14}$$

and from equation (2.28)

$$W_0(\tau) = I_1(\tau)/\omega_0 x_0'^2. \tag{3.15}$$

Let

$$\begin{aligned} g(\tau) &= f'_{\dot{x}}(x_0, \omega_0 x_0') W_0(\tau) x_0'^2 \\ &= [\mu_0 + \mu_1 (a_0 \,\mathrm{cn}^2 \tau + b_0) - \mu_2 (a_0 \,\mathrm{cn}^2 \tau + b_0)^2] I_1(\tau)/\omega_0, \end{aligned} \tag{3.16}$$

$$\because g(-\tau) = -g(\tau) \quad \text{and} \quad g(\tau + 2K) = g(\tau), \qquad \therefore \int_0^{2K} g(\tau)\,\mathrm{d}\tau = 0.$$

So $I_n = 0$ from equation (2.24) and $a_1 = 0$, $b_1 = 0$ from equations (2.21) and (2.22). Finally one gets

$$x_1(\tau) = 0, \qquad \omega_1(\tau) = W_0(\tau) = I_1(\tau)/\omega_0 x_0'^2. \tag{3.17, 3.18}$$

Therefore a final approximate solution of equation (3.1) to $O(\epsilon^2)$ can be expressed by

$$x = a_0 \,\mathrm{cn}^2\tau + b_0 + O(\epsilon^2), \qquad \dot{x} = -2a_0 [\omega_0 + \epsilon\omega_1(\tau)] \mathrm{sn}\,\tau \,\mathrm{cn}\tau \,\mathrm{dn}\,\tau + O(\epsilon^2). \tag{3.19, 3.20}$$

This means that the elliptic perturbation method is suitable for the generalized Van der Pol oscillator because $x_1 = 0$ in this case. Certainly, x_1 is not always zero for the general function of $f(x, \dot{x})$ in equation (2.1).

4. EXAMPLES

Example 1. *Consider the equation*

$$\ddot{x} + 2x - x^2 = \epsilon(0{\cdot}1 + x - x^2)\dot{x}. \tag{4.1}$$

This is the second type of oscillator ($c_1 > 0$, $c_2 < 0$) with $c_1 = 2$, $c_2 = -1$, $\mu_0 = 0{\cdot}1$, $\mu_1 = 1$ and $\mu_2 = 1$. One gets $k^2 = 0{\cdot}5061$ from equation (3.4) and then obtains $a_0 = -1{\cdot}7532$, $b_0 = 1{\cdot}0142$ and $\omega_0 = 0{\cdot}7598$ from equations (2.4)–(2.6). The limit cycle phase portraits for the cases $\epsilon = 0{\cdot}1$ and $\epsilon = 1{\cdot}5$ are shown in Figure 1. Comparisons are also made with the results of the Runge–Kutta (RK) numerical method and classical Lindstedt–Poincaré (LP) method (see Appendix 2).

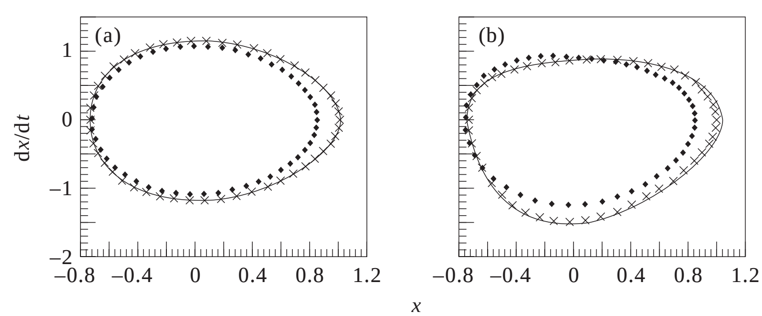

Figure 1. Limit cycles of equation (4.1). (a) $\epsilon = 0{\cdot}1$; (b) $\epsilon = 1{\cdot}5$; ——, RK method; +, present method; ◆, LP method.

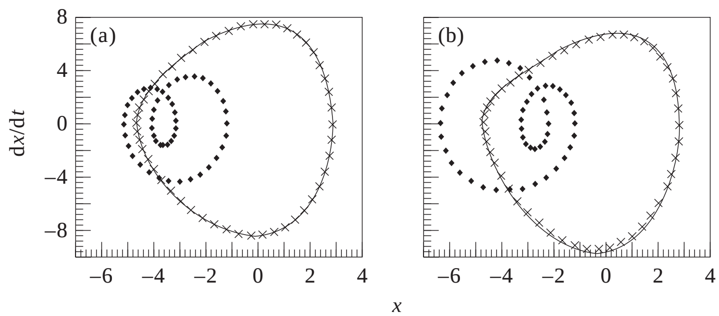

Figure 2. Limit cycles of equation (4.2). (a) $\epsilon = 0{\cdot}1$; (b) $\epsilon = 0{\cdot}3$; key as for Figure 1.

Example 2. *Consider the equation*

$$\ddot{x} + 6x + x^2 = \epsilon(1 + x - 0{\cdot}1x^2)\dot{x}. \tag{4.2}$$

This is the first type of oscillator ($c_1 > 0$, $c_2 > 0$) with $c_1 = 6$, $c_2 = 1$, $\mu_0 = 1$, $\mu_1 = 1$ and $\mu_2 = 0{\cdot}1$. One gets $k^2 = 0{\cdot}74736$, $a_0 = 7{\cdot}4682$, $b_0 = -4{\cdot}6479$ and $\omega_0 = 1{\cdot}29053$ from equations (2.4)–(2.6) and (3.4). The limit cycle phase portraits for the cases $\epsilon = 0{\cdot}1$ and $\epsilon = 0{\cdot}3$ are shown in Figure 2. Comparisons are also made with the results of the RK method and classical LP method.

Example 3. *Consider the equation*

$$\ddot{x} - 4x + x^2 = \epsilon(-15 + 8x - x^2)\dot{x}. \tag{4.3}$$

This is the third type of oscillator ($c_1 < 0$, $c_2 > 0$) with $c_1 = -4$, $c_2 = 1$, $\mu_0 = -15$, $\mu_1 = 8$ and $\mu_2 = 1$. In this example, $k^2 = 0{\cdot}56484$, $a_0 = 3{\cdot}9024$, $b_0 = 1{\cdot}7013$, $\omega_0 = 1{\cdot}0731$. The limit cycle phase portraits for the cases $\epsilon = 0{\cdot}1$ and $\epsilon = 0{\cdot}5$ are shown in Figure 3. It should be emphasized that one cannot use the LP method to solve equation (4.3) directly because $c_1 < 0$ in this case. To overcome this difficulty, it is convenient to introduce a change of the dependent variable from x to u, $u = x - x_0$. Where x_0 is the location of the so-called center point, $x_0 = -c_1/c_2 = 4$. Thus equation (4.3) becomes

$$\ddot{u} + 4u + u^2 = \epsilon(1 - u^2)\dot{u}. \tag{4.4}$$

This equation now belongs to the first type oscillator and therefore it can be solved by using the LP method now. The solutions of the LP method are also shown in Figure 3 for comparison.

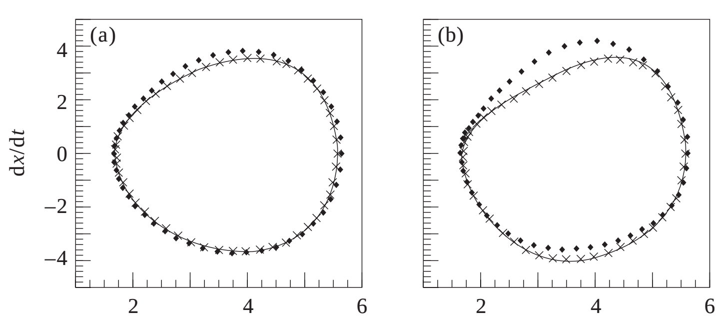

Figure 3. Limit cycles of equation (4.3). (a) $\epsilon = 0{\cdot}1$; (b) $\epsilon = 0{\cdot}5$; key as for Figure 1.

It can be seen from Figures 1–3 that:

(1) The solutions obtained by the present method are nearly identical with those given by the RK method for all examples all the cases of $\epsilon = 0{\cdot}1$, $\epsilon = 0{\cdot}3$, $\epsilon = 0{\cdot}5$ even $\epsilon = 1{\cdot}5$.

(2) The solutions of the classical LP method have obvious errors even when $\epsilon = 0{\cdot}1$, because the quadratic non-linear term is not weak.

(3) The departure of the solutions of the LP method in example 2 is very large. It can be concluded that it is not acceptable. It is therefore necessary to analyze this interesting phenomenon. One can see from Appendix 2 that

$$x(0) = M_0 + \epsilon(C_{c0} + M_1 + C_{c2}). \tag{4.5}$$

If $c_2 > 0$, $\mu_0 > 0$ and $\mu_1 > 0$, then $M_1 < 0$, $C_{c0} < 0$, $C_{c2} > 0$. In the case of $\epsilon = 0{\cdot}1$, $M_0 = 6{\cdot}3246$, $C_{c0} = -33{\cdot}3333$, $C_{c2} = 11{\cdot}1111$, $M_1 = -52{\cdot}7046$, so $x(0) = -1{\cdot}1681$, while the solution of the elliptic perturbation method is $x(0) = a_0 + b_0 = 2{\cdot}8203$. Obviously, the value of $x(0)$ is greatly reduced by C_{c0} and M_1, and therefore the departure is very large. It is mainly due to $c_1 > 0$, $c_2 > 0$ and $\mu_1 > 0$. In Example 3, although $c_2 > 0$ and $\mu_1 > 0$ in equation (4.3), $c_1 < 0$. After using transformation, equation (4.3) becomes equation (4.4). Applying the LP method to equation (4.4), one obtains $\mu_1 = 0$ then $M_1 = 0$. In the case of $\epsilon = 0{\cdot}1$, $C_{c0} = -5$, $C_{c2} = 1{\cdot}6666$, so $x(0) = 1{\cdot}6666$. This value is near to that of the starting point $x(0) = b_0 = 1{\cdot}7013$. Therefore it seems to point to the conclusion that the classical LP method cannot be applied to equation (3.1) with c_1, c_2 and μ_1 being positive simultaneously.

CONCLUSION

The elliptic perturbation method is an efficient method for calculating periodic solutions of strongly quadratic nonlinear oscillators. All the numerical results are in excellent agreement with those obtained by the RK method even for moderately large values of the parameter ϵ.

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Appendix 1 and 2 overleaf.

APPENDIX 1

The coefficients C_{11} to C_{15} occurring in equation (3.4) are

$$C_{11} = C_a\,(k^4 - k^2 + 1)/(15k^4) + C_b\,(2k^2 - 1)\,(3k^4 - 3k^2 + 8)/(105k^6) + C_c\,(10k^8 - 20k^6 + 66k^4 - 56k^2 + 16)/(315k^8),$$

$$C_{12} = C_a\,(2k^2 - 1)/(15k^2) + C_b\,[4(2k^2 - 1)^2 + 10k^2(1 - k^2)]/(105k^4) + C_c\,(2k^2 - 1)\,[8(2k^2 - 1)^2 + 27k^2(1 - k^2)]/(315k^6),$$

$$C_{13} = -C_a/5 + C_b\,(2k^2 - 1)/(35k^2) + C_c\,[6(2k^2 - 1)^2 + 14k^2(1 - k^2)]/(315k^4),$$

$$C_{14} = -C_b/7 + C_c\,(2k^2 - 1)/(63k^2), \qquad C_{15} = -C_c/9.$$

APPENDIX 2

The solution of equation (3.1) by using the Lindstedt–Poincaré method: let $\tau = \omega t$ and $\bar{c}_2 = c_2/\epsilon$, then equation (3.1) becomes

$$\omega^2 x'' + c_1 x + \epsilon \bar{c}_2 x^2 = \epsilon(\mu_0 + \mu_1 x - \mu_2 x^2)\omega x'$$

Using the classical LP procedure, one finally obtains

$$\omega = \omega_0 + \epsilon^2 \omega_2 + O(\epsilon^3), \qquad x = x_0 + \epsilon x_1 + O(\epsilon^2)$$

where

$$\omega_0 = \sqrt{c_1},$$

$$\omega_2 = (64\mu_0\,\mu_1\,M_0 - 16\mu_1^2\,M_0^2 - 18\mu_0\,\mu_2\,M_0^2 - 16\mu_1\,\mu_2\,M_0^3 + 3\mu_2^2\,M_0^4)/(384\omega_0) - 5\bar{c}_2^3\,M_0^2/(12\omega_0^3),$$

$$x_0 = M_0 \cos\tau, \qquad x_1 = C_{c0} + M_1 \cos\tau + N_1 \sin\tau + C_{c2}\cos 2\tau + C_{s2}\sin 2\tau + C_{s3}\sin 3\tau,$$

$$M_0 = 2\sqrt{\mu_0/\mu_2}, \qquad M_1 = -\mu_1\,\bar{c}_2\,M_0^3/(8\omega_0^2\,\mu_0), \qquad N_1 = 3\mu_2\,M_0^3/(32\omega_0) - \mu_1\,M_0^2/(3\omega_0),$$

$$C_{c0} = -\bar{c}_2\,M_0^2/(2\omega_0^2), \qquad C_{c2} = \bar{c}_2\,M_0^2/(6\omega_0^2), \qquad C_{s2} = \mu_1\,M_0^2/(6\omega_0),$$

$$C_{s3} = -\mu_2\,M_0^3/(32\omega_0).$$