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VIBROACOUSTIC BEHAVIOUR OF A SIMPLIFIED MUSICAL WIND INSTRUMENT

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The influence of the wall vibrations of a musical wind instrument on tone quality remains an open question. In order to quantify the effects of these vibrations, a model of the vibroacoustic behaviour of a simplified instrument (clarinet-like instrument) is proposed. The reed, which is represented by mechanical and acoustical harmonic sources, excites a thin cylindrical shell, filled and surrounded with air. The sound radiation due to wall vibrations has two origins, which are decoupled in the model making use of artificial baffles. The first one corresponds to the direct radiation of the shell in the external fluid. The second one is created by the internal radiation of the shell, which is then radiated outside the tube, through its open end. Three kinds of vibroacoustic couplings are involved in this situation: structure/internal fluid, structure/external fluid and inter-modal acoustic coupling due to sound radiation at the open end of the duct. A modal formulation of the problem is proposed which takes into account these three couplings. Impedances describing the shifts of the internal acoustic resonance frequencies due to the effects of the three kinds of couplings are given and permit one to quantify the wall vibrations effect.

1. INTRODUCTION

The question of the effect of wall vibrations on the tone of a musical wind instrument remains open: we still do not know if this phenomenon is important for the emitted tone or not. The aim in this paper is to quantify this effect, by making use of a model of a simplified instrument, allowing one to understand the phenomena involved. Some papers in the literature deal with the role of the wall material on tone. First, experimental studies can be found. Coltman [1] showed, with blindfold tests, that flutists are unable to distinguish several instruments, having the same internal shape and made from various materials. Backus [2] presented some measurements of the sound pressure level, radiated outside by the wall vibrations of a clarinet. Angster *et al*. [3] determined experimentally the first structural modes of a flue organ pipe. Smith [4] gathered some results concerning brass instruments. Lawson and Lawson [5] presented an experimental comparison between several annealed French horn bell flares. Second, some theoretical studies have been proposed. Backus and Hundley [6] proposed a simplified model of the acoustic resonance frequency shifts of an air column, caused by wall vibrations. Making use of the finite elements method, Watkinson and Bowsher [7] studied the structural modes of a trombone and used them in order to estimate the vibratory response of the structure.

These experimental and theoretical studies encounter many difficulties: from an experimental point of view, constructing two instruments having strictly the same internal shape is difficult and prevents one's making convincing comparisons. Indeed, the wall vibrations effect, which seems to be particularly small, can be masked by a lot of parasitic

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phenomena. From a theoretical point of view, the complexity of the phenomena involved leads to approximate methods. The limitations of the simplified models used, which are rather difficult to estimate, do not allow one to draw firm conclusions. In this paper, in order to quantify the wall vibrations effect, a theoretical approach is proposed which takes into account three kinds of structure/fluid coupling: the internal and external radiation couplings and the inter-modal acoustic coupling due to sound radiation from the open end of the tube. The vibroacoustic behaviour of cylindrical shells has attracted a lot of attention, because of its great practical importance in a wide variety of applications such as muffler and industrial pipes acoustics. The problems, called internal problems, are dealing with the coupling between the vibrations of an elastic cavity and the acoustic behaviour of the contained fluid. For complex shapes, such a problem is usually solved by numerical methods [8]. However, some analytical solutions can be given for simple geometries by using integral methods [9–15]. First the vibratory response of the structure is expressed as an expression over the *in vacuo* structural modes. Second, the coupled acoustic field is written by making use of the Helmholtz–Huygens integral. Finally, the momentum equation of the structure, with account taken of the fluid-loading is projected over the structural eigenfunctions to give the governing equations of the problem. In the case of a cylindrical shell, the internal radiation impedances, characterizing the fluid/structure interaction, has been determined for various sets of acoustic boundary conditions at the ends of the duct. The external fluid-loading for simple structures has received a lot of attention [16–22] and has been calculated for a cylindrical shell with rigid extensions [23–25].

In this paper, in order to describe the internal acoustic field inside a cylindrical cavity with vibrating walls, two different modal expansions, involving both *in vacuo* structural modes and acoustic modes of the air column bounded by rigid walls, are used to express the coupled acoustic field, whereas the integral method would use only the second one. It has been shown [26, 27], that in the case of a cylindrical cavity, such a method leads to more simple expressions for the internal radiation impedances than those which are obtained by the classical integral method. The model used here, which takes into account the three types of couplings mentioned above, is presented in section 2. A theoretical formulation is given in section 3, leading to numerical estimations of the wall vibrations effect on tone (section 4).

2. MODEL OF THE MUSICAL WIND INSTRUMENT

2.1. THE MUSICAL WIND INSTRUMENT

A musical wind music instrument, such as a clarinet, can be considered as an acoustic waveguide (the instrument's body) interacting with an exciter (the energy source), and radiating in a fluid domain (the air). Each element is complex and has to be modelled, and for the purpose of this investigation an idealized instrument is assumed. The body of the instrument, characterized by the internal shape (assumed to be approximately cylindrical), the lateral holes and the keys is a very complex mechanical structure. Its vibratory behaviour depends on its shape, the material from which it is made, the boundary conditions and the kind of excitation. The different kinds of vibroacoustic coupling occurring between this complex structure and the fluid are schematically depicted in Figure 1.

The reed vibrations provide energy to the internal air column and create the internal acoustic pressure field (a). This acoustic field is coupled to the wall vibration (b). The vibratory field of the lateral wall produces sound radiation in the external fluid (c), called

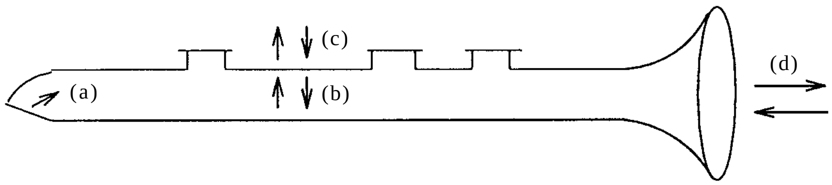

Figure 1. Vibroacoustic couplings occurring in a clarinet-like instrument.

lateral radiation. The open end of the tube produces sound radiation in front of the instrument, called frontal radiation (d).

2.2. THE WAVEGUIDE

The complexity of the instrument's body is not taken into account. The system studied is a thin cylindrical shell characterized by its density ρ_s, its ring natural frequency ω_a, its length ℓ, its mean radius a, and its thickness h (see Figure 2). The position vector is $\mathbf{r} = (r, \theta, z)$ and the surfaces S_0, S and S_ℓ correspond to the co-ordinate $z = 0$, the lateral surface of the cylinder ($r = a$) and the co-ordinate $z = \ell$, respectively. The internal domain, delimited by the surfaces S_0, S, S_ℓ, is denoted D_i and $\mathbf{n}$ is the unit vector normal to the cylinder in the outward direction. The shell motion is assumed to be described by the Flügge differential operator $\mathscr{L}$ (see reference [28]), which incorporates a model of structural damping. The shell motion is described by the displacement field $\mathbf{X}$, whose components u, v, w are the longitudinal, circumferential and radial displacements, respectively. The shell is assumed to be simply supported, leading to tractable analytical modal basis.

2.3. THE FLUID

The instrument produces sound in the air (density ρ and speed of sound c). The dissipative effects are taken into account only in the internal fluid, making use of the complex value $c = c_0(1 - \mathrm{j}\eta_f)$. The loss factor η_f can be obtained for each acoustic mode (plane modes and higher order modes) making use of the model described in Appendix A [29]. The acoustic behaviour of the fluid is characterized by the acoustic pressure $p(\mathbf{r})$ and the particle acoustic velocity $\mathbf{v}(\mathbf{r})$. The normal component of $\mathbf{v}(\mathbf{r})$, related to a surface S is denoted $v_S(\mathbf{r})$.

2.4. THE SOURCE

The reed, clamped on the mouthpiece of the clarinet, is a complex acoustic and mechanical source. The acoustic source is characterized by a given harmonic particle

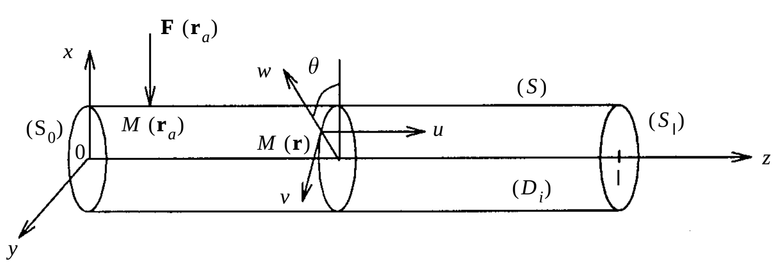

Figure 2. The vibrating cylinder.

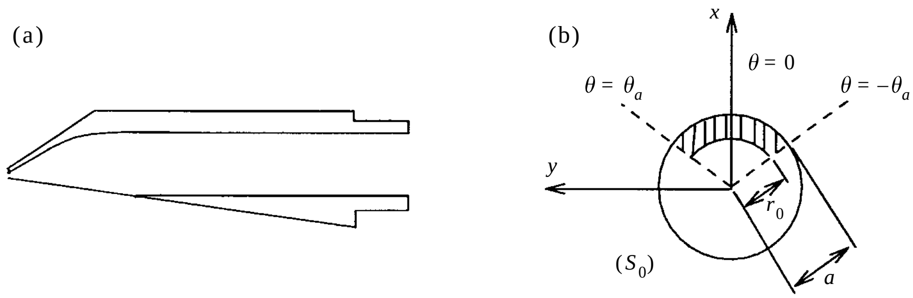

Figure 3. Mouthpiece of the real instrument (a) and the particle velocity distribution on the surface S_0 which represents it (b).

velocity distribution $v_{S_0}(\mathbf{r})$ at the surface S_0. Because of the complexity of the real mouthpiece's shape (see Figure 3(a)), the real velocity distribution, on the surface S_0 is not 0. The velocity distribution considered represents this asymmetry, which is at the origin of the excitation of the first structural modes (bending and ovalling modes). If a uniform velocity distribution is considered, only breathing modes are excited. Thus, the particle velocity distribution takes on a negligible value over the section S_0, except in the dashed region of S_0 (see Figure 3(b)) where the value is assumed to be equal to 1 ms^{-1} in this paper. The reed is also a mechanical source, because of the shocks it creates at the edge of the mouthpiece. The impulses generated by the reed at each period are represented by a mechanical forcing function $\mathbf{F}(\mathbf{r}, t)$, which is reduced in this study to a point force in the radial direction, and applied at $M(\mathbf{r}_a)$. Because of the Poisson summation formula, this impulse force can be written as an expansion of harmonic terms:

$$\mathbf{F}(\mathbf{r}, t) = F\delta(\mathbf{r} - \mathbf{r}_a) \sum_{N=-\infty}^{+\infty} \delta(2\pi N - \omega t)\mathbf{n} = F\delta(\mathbf{r} - \mathbf{r}_a) \frac{1}{2\pi} \sum_{N=-\infty}^{+\infty} \mathrm{e}^{-N\mathrm{j}\omega t}\mathbf{n}. \tag{1}$$

The system is assumed to be linear; thus the response to the impulse excitation can be obtained by summing the response corresponding to harmonic excitations. For this reason, in the following, only the harmonic excitation is considered

$$\mathbf{F}(\mathbf{r}, t) = \frac{F}{2\pi} \delta(\mathbf{r} - \mathbf{r}_a)\, \mathrm{e}^{-\mathrm{j}\omega t}\mathbf{n} = F_0\delta(\mathbf{r} - \mathbf{r}_a)\, \mathrm{e}^{-\mathrm{j}\omega t}\mathbf{n}, \tag{2}$$

where the magnitude F_0 is set to 1. The point $M(\mathbf{r}_a)$ is located at $\theta = 0$, $z = \ell/10$, $r = a$. Finally, the characteristics of the two kinds of sources, the particle velocity distribution $v_{S_0}(\mathbf{r})$ and the force $\mathbf{F}$, are expressed in the frequency domain, and the factor $\mathrm{e}^{-\mathrm{j}\omega t}$ is omitted.

2.5. THE BAFFLES

The coupling between the external fluid domain, D_e, and the shell vibration leads to the so-called lateral radiation, which is characterized by external radiation impedances. These impedances can be simply estimated if the shell is supposed to be connected with two semi-infinite rigid cylindrical baffles, denoted B_e, and corresponding to the surfaces ($r = a$, $z \leqslant 0$) and ($r = a$, $z \geqslant \ell$), as shown in Figure 4. Sandman [22] has shown that the baffles have little influence on the radiation impedances, allowing one to describe in a simple manner the external fluid-loading.

In a similar way, the sound radiation through the open end of the duct, called frontal radiation, can be taken into account if the tube is flanged (see Figure 5). The frontal

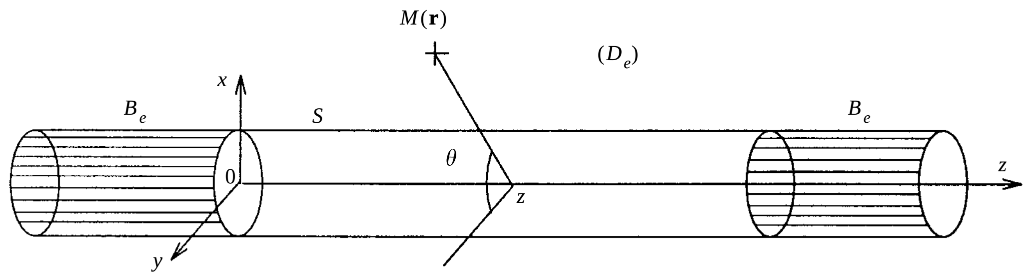

Figure 4. Lateral radiation.

radiation occurs in the half infinite domain D_f, limited by the rigid baffle B_f. With this configuration, the radiated sound field can easily be calculated in the domain D_f, without changing the kind of inter-modal coupling created by this radiation. The centre of the surface S_ℓ is denoted $\mathbf{r}_{S_\ell}$.

Two components of sound radiation have been distinguished: the lateral and frontal sound radiation. As already pointed out, in the real situation, the two acoustic fields are interfering with each other and the sources which create them are coupled. Considering artificial baffles leads to uncoupled sound sources, allowing an analytical solution to be expressed.

2.6. FORMULATION OF THE PROBLEM

The shell displacement field, $\mathbf{X}(\mathbf{r})$, and the acoustic pressure, $p(\mathbf{r})$, in the three fluid domains D_i, D_e and D_f, are the solutions of the following problem:

For D_i,

$$(\Delta + k^2)p(\mathbf{r}) = 0 \quad \text{for} \quad \mathbf{r} \in D_i,$$

$$v_{S_0}(\mathbf{r}) \text{ is assumed to be given for } \mathbf{r} \in S_0,$$

$$v_S(\mathbf{r}) = \dot{w}(\mathbf{r}) \quad \text{for} \quad \mathbf{r} \in S,$$

$$v_{S_\ell}(\mathbf{r} \to S_\ell^-) = v_{S_\ell}(\mathbf{r} \to S_\ell^+) \quad \text{and} \quad p(\mathbf{r} \to S_\ell^-) = p(\mathbf{r} \to S_\ell^+); \qquad \text{(3a–d)}$$

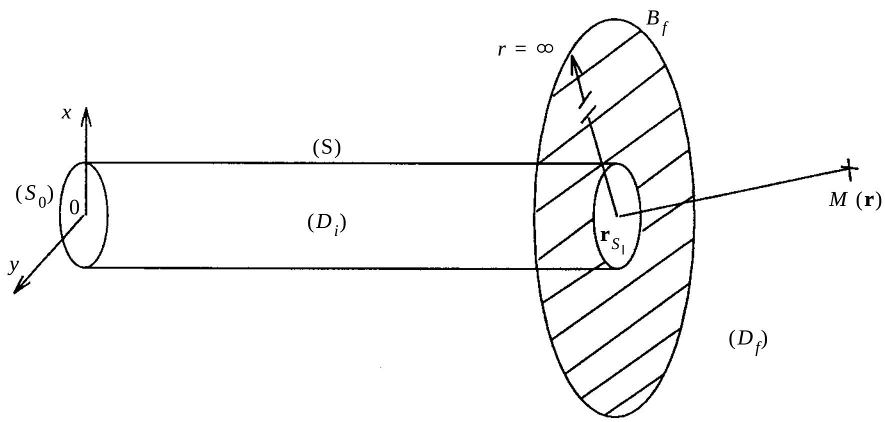

Figure 5. Frontal radiation.

for D_e,

$$(\Delta + k^2)p(\mathbf{r}) = 0 \quad \text{for} \quad \mathbf{r} \in D_e,$$

$$v_S(\mathbf{r}) = \dot{w}(\mathbf{r}) \quad \text{for} \quad \mathbf{r} \in S,$$

$$v_{B_e}(\mathbf{r}) = 0 \quad \text{for} \quad \mathbf{r} \in B_e,$$

$$\text{Sommerfeld's conditions for } \|\mathbf{r}\| \to \infty; \tag{4a–d}$$

for D_f,

$$(\Delta + k^2)p(\mathbf{r}) = 0 \quad \text{for} \quad \mathbf{r} \in D_f,$$

$$v_{B_f}(\mathbf{r}) = 0 \quad \text{for} \quad \mathbf{r} \in B_f,$$

$$v_{S_\ell}(\mathbf{r} \to S_\ell^+) = v_{S_\ell}(\mathbf{r} \to S_\ell^-) \quad \text{and} \quad p(\mathbf{r} \to S_\ell^+) = p(\mathbf{r} \to S_\ell^-),$$

$$\text{Sommerfeld's condition for } \|\mathbf{r} \to \mathbf{r}_{s\ell}\| \to \infty; \tag{5a–d}$$

for the shell,

$$\rho_s h(\omega_a^2 \mathscr{L} + \omega^2)\mathbf{X}(\mathbf{r}) = [\, p^e(\mathbf{r}) - p^i(\mathbf{r})] \cdot \mathbf{n} - \mathbf{F}(\mathbf{r}) \quad \text{for} \quad \mathbf{r} \in S,$$

$$\text{simply supported boundary conditions for } z = 0, \ell. \tag{6a, b}$$

3. THEORY

3.1. THE SHELL MOTION

The harmonic motion of the shell, described by the displacement field $\mathbf{X}$, may be expanded over the *in vacuo* eigenmodes $\mathbf{\Phi}_\mu$ of the simply supported shell (see Appendix B for the description of these structural modes):

$$\mathbf{X} = \sum_\mu A_\mu \mathbf{\Phi}_\mu. \tag{7}$$

Inserting equation (7) into equation (6a), and making use of the orthogonality property of the structural modes $\mathbf{\Phi}_\mu$,one gets the generalized equation for the shell motion,

$$m_\mu A_\mu[-\omega^2 + \omega_\mu^2(1 - \mathrm{j}\eta)] = -P_\mu^e + P_\mu^i + F_\mu, \tag{8}$$

where η denotes the structural damping factor, m_μ denotes the modal mass of the shell mode $\mathbf{\Phi}_\mu$, whose natural angular frequency is ω_μ. The generalized force

$$F_\mu = \langle \mathbf{F}, \mathbf{\Phi}_\mu \rangle_S = \sum_{i=1}^{3} \langle F_i | \mathbf{\Phi}_{i\mu} \rangle_S = F_0 \mathbf{\Phi}_{3\mu}(\mathbf{r}_a), \tag{9}$$

and the generalized pressure

$$P_\mu^{i,e} = \langle p^{i,e}\mathbf{n}, \mathbf{\Phi}_\mu \rangle_S = \langle p^{i,e} | \mathbf{\Phi}_{3\mu} \rangle_S, \tag{10}$$

which appear in the right side of equation (8), are respectively the inner product of the given force $\mathbf{F}$ and the internal and external acoustic pressure over the shell mode $\mathbf{\Phi}_\mu = [\Phi_{1\mu} \;\; \Phi_{2\mu} \;\; \Phi_{3\mu}]^{\mathrm{T}}$. The inner product used for this projection is defined by $\langle f | g \rangle_{S_i} = \int_{S_i} f \cdot g^* \, \mathrm{d}s$. Solving equation (8) requires the internal and external acoustic pressure to be expressed in terms of the modal amplitudes A_μ.

3.2. THE COUPLING BETWEEN THE SHELL AND THE INTERNAL FLUID

By making use of an appropriate Green function, the classical integral method provides the expression of the acoustic field inside a cavity with vibrating walls [9]. However, it can be shown that a new method, called the Separate Modal Expansions method, provides coupling coefficients (internal radiation impedances), which are more simply expressed than with the classical integral method. A comparison between coupling terms coming from these different methods can be found in reference [26]. The Separate Modal Expansions method, which is used in this paper, leads to the following expression for the acoustic field inside a cylindrical cavity whose wall vibrations are described by modal amplitudes A_μ:

$$p(r, \theta, z) = p^r_S(r, \theta, z) + p^r_{S_{0\ell}}(r, \theta, z), \quad \text{for} \quad \mathbf{r} \in D_i, \tag{11}$$

with

$$p^r_S(r, \theta, z) = \rho c \sum_{\mu=(m,q,s,j)} -\mathrm{j}\omega A_\mu \left[\mathrm{j} \frac{k}{k_q} \frac{\mathrm{J}_m(k_q r)}{\mathrm{J}'_m(k_q a)} \right] \Phi_{3\mu}(\theta, z), \tag{12}$$

and

$$p^r_{S_{0\ell}}(r, \theta, z) = \sum_{\alpha=(m,n,s)} [B^+_\alpha \, \mathrm{e}^{\mathrm{j}k_{mn}z} + B^-_\alpha \, \mathrm{e}^{\mathrm{j}k_{mn}(\ell - z)}] \Psi_\alpha(r, \theta), \tag{13}$$

where the wavenumbers k_q and k_{mn} are defined by the relationships $k_q^2 = k^2 + (q\pi\ell)^2$, $k_{mn}^2 = k^2 - k^2_{W_{mn}}$ with $\mathrm{J}'_m(k_{W_{mn}}a) = 0$, all indices being described in Appendices B and C. The two modal expansions, corresponding to the terms $p^r_S(r, \theta, z)$ and $p^r_{S_{0\ell}}(r, \theta, z)$, use the radial component $\Phi_{3\mu}$ of the shell eigenmode $\mathbf{\Phi}_\mu$ (see Appendix B) and the well-known eigenfunctions of the two-dimensional transverse Neumann problem (see Appendix C). The term $p^r_S(r, \theta, z)$ can be interpreted as the internal radiation of the vibrating walls, whose motion is described by the modal amplitudes A_μ, when Dirichlet conditions are imposed on the surfaces S_0 and S_ℓ (see reference [26]). The complementary term $p^r_{S_{0\ell}}(r, \theta, z)$ corresponds to the radiation from the surfaces S_0 and S_ℓ, and the integration constants $B^\pm_\alpha$ are determined by making use of the acoustic boundary conditions (3b) and (3d). In order to express these conditions, the axial component of the acoustic velocity $\mathbf{v}$ has to be known. It can be determined, making use of the velocity potential Q which satisfies the relations $\mathbf{v} = -\nabla Q$ and $p = -\mathrm{j}\rho\omega Q$, p being the acoustic pressure given by equation (11). The inner product of the acoustic pressure and the axial velocity over the acoustic mode $\mathbf{\Psi}_\alpha$ can be written as follows:

$$\langle p, \mathbf{\Psi}_\alpha \rangle_{S_0} = B^+_\alpha + B^-_\alpha \, \mathrm{e}^{\mathrm{j}k_{mn}\ell}, \qquad \langle p, \mathbf{\Psi}_\alpha \rangle_{S_\ell} = B^+_\alpha \, \mathrm{e}^{\mathrm{j}k_{mn}\ell} + B^-_\alpha, \tag{14, 15}$$

$$\langle v, \Psi_\alpha \rangle_{S_0} = G_{0\alpha} + \frac{k_{mn}}{\rho c k} [B^+_\alpha - B^-_\alpha \, \mathrm{e}^{\mathrm{j}k_{mn}\ell}], \qquad \langle v, \Psi_\alpha \rangle_{S_\ell} = G_{\ell\alpha} + \frac{k_{mn}}{\rho c k} [B^+_\alpha \, \mathrm{e}^{\mathrm{j}k_{mn}\ell} - B^-_\alpha], \tag{16, 17}$$

where the generalized velocities $G_{i\alpha}$ ($i = 0$ or $i = \ell$), related to the wall vibrations effect in the axial generalized velocity $\langle v, \Psi_\alpha \rangle_{S_i}$, are given by

$$G_{i\alpha} = \sum_\mu -\mathrm{j}\omega A_\mu \frac{q\pi/\ell}{k_q \mathrm{J}'_m(k_q a)} \langle \mathrm{J}_m(k_q r) \sin(m\theta + s\pi/2) | \Psi_\alpha \rangle_S \begin{Bmatrix} 1 & \text{if } i = 0 \\ (-1)^q & \text{if } i = \ell \end{Bmatrix}. \tag{18}$$

The inner product $\langle v, \Psi_\alpha \rangle_{S_0} = V_{S_{0\alpha}}$ is a given quantity because the particle velocity distribution $v_{S_0}(\mathbf{r})$ is supposed to be known. Consequently, the relationship (16) represents the boundary condition on the surface S_0. The unknown acoustic pressure $p(\mathbf{r})$ on S_ℓ will be fixed in the next section, by making use of the continuity relation between domains D_i and D_f (see equations (3d) and (5c)). Thus, by using the projections $P_\alpha = \langle p, \Psi_\alpha \rangle_{S_\ell}$, the boundary condition on S_ℓ is given by equation (15). Finally, the amplitudes $B_\alpha^\pm$ are the solutions of the linear system:

$$\begin{bmatrix} e^{jk_{mn}\ell} & 1 \\ 1 & -e^{jk_{mn}\ell} \end{bmatrix} \begin{bmatrix} B_\alpha^+ \\ B_\alpha^- \end{bmatrix} = \begin{bmatrix} P_\alpha \\ \rho ck/k_{mn}[V_{S_{0\alpha}} - G_{0\alpha}} \end{bmatrix}. \tag{19}$$

The integration constants $B_\alpha^\pm$ can be split into three terms, which can be interpreted by making use of other integration constants corresponding to more simple problems. These problems are labelled by using three letters. The first, second and third letters represent the condition on the surfaces S_0, S and S_ℓ, respectively, and are chosen among N (for Neumann), D (for Dirichlet), V (for imposed velocity), and P (for imposed pressure). For example, with this notation, the problem associated to Neumann boundary conditions on S_0, velocity condition on S and Dirichlet boundary conditions on S_ℓ is denoted (NVD). Each integration constant $(B_\alpha^{-NVD}, B_\alpha^{+NVD})$, $(B_\alpha^{+VND}, B_\alpha^{-VND})$, $(B_\alpha^{+NNP}, B_\alpha^{-NNP})$ correspond to the solution of the linear system (19) when the right side terms $(V_{S_{0\alpha}}, P_\alpha)$, $(G_{0\alpha}, P_\alpha)$, and $(V_{S_{0\alpha}}, G_{0\alpha})$ are respectively forced to zero. Thus, the constants $B_\alpha^\pm$ can be written as the sums

$$B_\alpha^+ = B_\alpha^{+VVP} = B_\alpha^{+NVD} + B_\alpha^{+VND} + B_\alpha^{+NNP},$$
$$B_\alpha^- = B_\alpha^{-VVP} = B_\alpha^{-NVD} + B_\alpha^{-VND} + B_\alpha^{-NNP}. \tag{20}$$

Inserting equations (20) into equation (11), one gets the acoustic pressure $p(\mathbf{r})$, leading to the following expression for the internal generalized pressure, equation (10):

$$P_\mu^i = P_\mu^{NVD} + P_\mu^{VND} + P_\mu^{NNP}. \tag{21}$$

By making use of the internal radiation impedance $Z_{\mu\mu'}^i$, which characterized wall radiation into the cylindrical cavity for the (NVD) problem, the generalized pressure P_μ^{NVD} can be written as (see Appendix D)

$$P_\mu^{NVD} = +j\omega \sum_{\mu'} Z_{\mu\mu'}^i A_{\mu'}. \tag{22}$$

The other generalized pressures P_μ^{VND} and P_μ^{NNP} depend only on the given coefficients $V_{S_{0\alpha}}$ and on the unknown modal amplitude P_α, and are given by

$$P_\mu^{VND} = \rho c \sum_\mu V_{S_{0\alpha}} T_{\mu\alpha}, \qquad P_\mu^{NNP} = \sum_\mu P_\alpha H_{\mu\alpha}, \tag{23, 24}$$

where the expressions for the impedance $H_{\mu\alpha}$ and the 0 function $T_{\mu\alpha}$ are given in Appendix D.

3.3. INTER-MODAL ACOUSTIC COUPLING DUE TO SOUND RADIATION FROM THE OPEN END OF THE DUCT

Upon making use of the Green function $G_{Df}(\mathbf{r}, \mathbf{r}_0)$, which satisfies Neumann boundary conditions on the surface $(S_\ell + B_f)$ [30],

$$G_{Df}(\mathbf{r}, \mathbf{r}_0) = e^{jk\|\mathbf{r} - \mathbf{r}_{S\ell}\|}/2\pi\|\mathbf{r} - \mathbf{r}_{S\ell}\|, \tag{25}$$

the acoustic pressure in the frontal domain (D_f) can be expressed as

$$p(\mathbf{r}) = -\mathrm{j}\rho\omega \int_{S_\ell} G_{Df}(\mathbf{r}, \mathbf{r}_0) v_{S_\ell}(\mathbf{r}_0)\, \mathrm{d}s_0, \qquad \mathbf{r} \in D_f. \tag{26}$$

Using the continuity relations equations (3d) and (5c) leads to the following impedance-like relationship [31]:

$$P_\alpha = \langle p_{S_\ell} | \Psi_\alpha \rangle_{S_\ell} = \sum_{\alpha'} Z^r_{\alpha\alpha'} \langle v_{S_\ell} | \Psi_{\alpha'} \rangle_{S_\ell}, \tag{27}$$

where the frontal radiation impedances $Z^r_{\alpha\alpha'}$ describe the inter-modal coupling which is induced by sound radiation from the open end of the duct. The expressions for $Z^r_{\alpha\alpha'}$ are given in Appendix E. Inserting equation (27) into equation (17), one gets

$$P_\alpha = \sum_{\alpha'} Z^r_{\alpha\alpha'} \left(G_{\ell\alpha'} + \frac{k_{mn}}{\rho c k} [B^+_{\alpha'}\, \mathrm{e}^{\mathrm{j}k_{mn}\ell} - B^-_{\alpha'}] \right) = \sum_{\alpha'} Z^r_{\alpha\alpha'} (V^{NVD}_{\alpha'} + V^{VND}_{\alpha'} + V^{NNP}_{\alpha'}), \tag{28, 29}$$

where the axial velocities $V^{NVD}_{\alpha'}$, $V^{VND}_{\alpha'}$, $V^{NNP}_{\alpha'}$ correspond to the (NVD), (VND) and (NND) problems. The axial velocity V^{NVD}_α depends only on the modal amplitudes A_μ:

$$V^{NVD}_\alpha = \mathrm{j}\omega \sum_{q,j} A_\mu H_{\mu\alpha}, \qquad \text{with} \quad \mu = (m, q, s, j), \tag{30}$$

where the coupling term $H_{\mu\alpha}$ is given in Appendix D. The other generalized velocities V^{VND}_α and V^{NNP}_α can be expressed as

$$V^{VND}_\alpha = J_\alpha V_{S_{0\alpha}}, \qquad V^{NNP}_\alpha = (K_\alpha / \rho c) P_\alpha, \tag{31, 32}$$

the transfer functions J_α and K_α being given in Appendix E.

3.4. THE COUPLING BETWEEN THE SHELL AND THE EXTERNAL FLUID

By making use of the Green function $G_{De}(\mathbf{r}, \mathbf{r}_0)$ which is the solution for the exterior Neumann boundary value problem [24, 25, 30],

$$G_{De}(\mathbf{r}, \mathbf{r}_0) = \frac{-1}{4\pi^2} \sum_{m=0}^{\infty} \varepsilon_m \cos(m\theta - \theta_0) \int_{-\infty}^{+\infty} \frac{\mathrm{H}^{(1)}_m(k_r a)}{k_r \mathrm{H}^{(1)\prime}_m(k_r a)}\, \mathrm{e}^{\mathrm{j}k_z(z - z_0)}\, \mathrm{d}k_z, \tag{33}$$

where $k_r = (k^2 - k_z^2)^{1/2}$, ε_m is the Neumann factor, $\mathrm{H}^{(1)}_m$ and $\mathrm{H}^{(1)\prime}_m$ denote the mth order Hankel function and its derivative respectively, the radiated pressure into the external domain D_e can be expressed as

$$p(\mathbf{r}) = -\mathrm{j}\rho\omega \int_S \dot{w}(\mathbf{r}_0) G_{De}(\mathbf{r}, \mathbf{r}_0)\, \mathrm{d}S_0. \tag{34}$$

Using the modal expansion (7), one can write the generalized external pressure, equation (10), as

$$P^e_\mu = -\mathrm{j}\omega \sum_{\mu'} Z^e_{\mu\mu'} A_\mu, \tag{35}$$

where the external radiation impedances $Z^e_{\mu\mu'}$ are given in Appendix F.

3.5. THE COUPLING EQUATIONS: SYNTHESIS

The unknown modal amplitudes A_μ and P_α satisfy the two equations governing the problem, equations (8) and (29). The first one, associated to equations (9) and (21)–(24) describes the shell motion under fluid and mechanical loadings. The second one, associated to equations (30)–(32), describes the modal coupling due to the acoustic radiation from the open end of the duct. By making use of the matricial notations defined in Appendix G, the governing equations can be written as follows:

$$\begin{bmatrix} \mathbf{M}_m - \mathrm{j}\omega(\mathbf{Z}^i_m + \mathbf{Z}^e_m) & -\mathbf{H}_m \\ -\mathrm{j}\omega\mathbf{Z}^r_m \cdot \mathbf{H}^t_m & 1-(\rho c)^{-1}\mathbf{Z}^r_m \cdot \mathbf{K}_m \end{bmatrix} \cdot \begin{bmatrix} \mathbf{A}_m \\ \mathbf{P}_m \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \rho c \mathbf{T}_m \\ \mathbf{0} & \mathbf{Z}^r_m \cdot \mathbf{J}_m \end{bmatrix} \cdot \begin{bmatrix} \mathbf{F}_m \\ \mathbf{V}_m \end{bmatrix}. \tag{36}$$

4. NUMERICAL RESULTS

Solving equation (36) requires the computation of three kinds of impedance matrices $\mathbf{Z}^i_m$, $\mathbf{Z}^e_m$ and $\mathbf{Z}^r_m$ (see Appendix G) which characterize the three couplings: internal and external radiation coupling, and inter-modal coupling due to sound radiation through the open end of the duct. The calculation of the internal radiation impedances requires the estimation of the series given by equation (D2). Numerical tests have shown that satisfactory results can be obtained if the truncation adopted takes into account the 15 first terms of the series. The two other kinds of impedances require intensive calculations. Upon making use of impedance matrices $\mathbf{Z}^i_m$, $\mathbf{Z}^e_m$ and $\mathbf{Z}^r_m$, the direct inversion of the coupled system (36) provides the unknown vectors $\mathbf{P}_m$ and $\mathbf{A}_m$.

4.1. VIBROACOUSTIC RESPONSE

In this section, the sound power radiated from the lateral surface into the external domain,

$$\Pi_S = \frac{1}{2}\int_S \mathrm{Re}\,[\,p^e(\mathbf{r})\dot{w}^*(\mathbf{r})]\,\mathrm{d}S = \frac{\omega^2}{2}\sum_{\mu\mu'} \mathrm{Re}\,(A_\mu A^*_{\mu'}\ Z^e_{\mu\mu'}), \tag{37}$$

is used as a global indicator of the sound level produced by wall vibrations. This vibroacoustic indicator is investigated separately for the two kinds of excitation and is

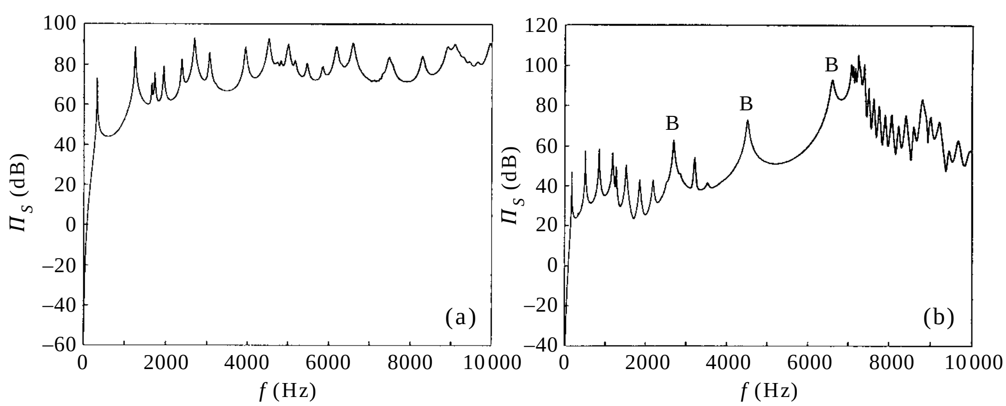

Figure 6. Sound power level, radiated from the lateral surface S for a mechanical (a) and an acoustical (b) excitation (referenced to 10^{-12} W).

shown in Figures 6(a) and (b). For the mechanical excitation, the magnitude of the force is set to 1 Newton,

$$F_0 = 1\ \mathrm{N} \quad \text{and} \quad v_{S_0}(\mathbf{r}) \equiv 0, \quad \mathbf{r} \in S_0, \tag{38}$$

and an unitary acoustic source defined by (see Figure 3(b))

$$F_0 = 0\ \mathrm{N}, \quad \text{and} \quad \left\{ \begin{array}{ll} v_{S_0}(\mathbf{r}) = 1\ \mathrm{ms}^{-1} & \text{if } -\theta_a < \theta < \theta_a \text{ and } r_0 < r < a \\ v_{S_0}(\mathbf{r}) = 0 & \text{otherwise} \end{array} \right\}, \tag{39}$$

the geometrical parameters (defining the dashed part of S_ℓ) being defined by $r_0 = 0{\cdot}9a$ and $\theta_a = \pi/3$. For the steel shell which is studied (length $\ell = 0{\cdot}5$ m, radius $a = 0{\cdot}01425$ m, thickness $h = 0{\cdot}05$ m), the structural modes whose natural *in vacuo* frequencies are in the range 0–10 kHz are taken into account. This corresponds to the first 43 mechanical modes, and involves the circumferential indexes $0 \leqslant m \leqslant 4$. The structural damping factor is set to $\eta = 10^{-2}$ as given in reference [18] for a steel shell.

For a mechanical excitation, Π_S indicates a strong acoustic power radiated for each shell resonance. The acoustic resonances of the internal column of fluid do not lead to particularly significant values of Π_S. For acoustic excitation, significant peaks in Π_S appear for the first acoustic internal resonances (plane modes resonances), for the first higher order acoustic resonances (up to the first cut-off frequency (equal to 7060 Hz)), and for bending modes of the shell (denoted B on Figure 6(b)). The resolution of the coupled system also provides the coefficients of the modal expansion of the acoustic pressure on the section S_ℓ, allowing one to calculate the sound power radiated across this surface:

$$\Pi_S = \frac{1}{2} \int_{S_\ell} \mathrm{Re}\,[\, p(\mathbf{r}) v^*_{S_\ell}(\mathbf{r})]\, \mathrm{d}S = \frac{1}{2} \sum_{\alpha\alpha'} \mathrm{Re}\,(V_\alpha V^*_{\alpha} Z^r_{\alpha\alpha'}). \tag{40}$$

As expected, Π_S takes on high values for frequencies which are near to odd harmonics of the fundamental frequency $f = c/4\ell$ (plane modes resonances). Up to the first cut-off frequency (7060 Hz), the resonances of the higher order mode $m = 1$, $n = 0$ are added to those of plane modes. The indicator Π_{S_ℓ} can be calculated for a vibrating or a rigid tube,

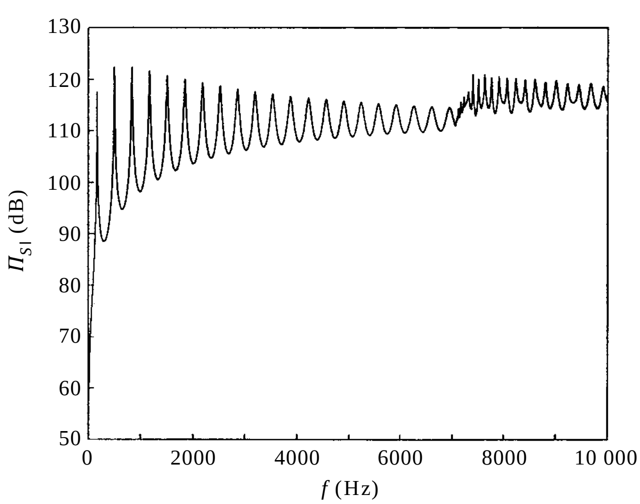

Figure 7. Sound power level, radiated from the surface S_ℓ for acoustic excitation.

leading to the same curves on Figure 7, according to its scale. The small discrepancies between the two results is exhibited by making use of the indicator $\Delta\Pi_{S_\ell}$ defined in the next section. Note that the acoustic power emitted from the vibrations of the lateral walls is much lower than the acoustic power radiated at the output of the duct (see Figures 6(b) and 7).

4.2. IMPORTANCE OF SHELL VIBRATIONS ON THE SOUND POWER RADIATED FROM THE OPEN END OF THE DUCT

Because the sound power Π_{S_ℓ} radiated from the open end of the duct is a good indicator of what could be heard by the musician, interest in this section is in comparing this indicator when wall vibrations are taken into account (sound power level $\Pi_{S_\ell 1}$) and when the tube is rigid (sound power level $\Pi_{S_\ell 2}$). For the computation of the sound power level $\Pi_{S_\ell 2}$, the impedance matrices $\mathbf{Z}^i_m$ and $\mathbf{Z}^e_m$ in equation (36) are forced to zero. The difference between the two results,

$$\Delta\Pi_{S_\ell} = |\Pi_{S_\ell 1} - \Pi_{S_\ell 2}|, \tag{41}$$

is shown in Figure 8.

Each acoustic resonance frequency is shifted and damped because of the interaction with the shell. The algebraic difference $\Delta\Pi_{S_\ell}$, which quantifies the discrepancies between the sound power level radiated by a vibrating and a rigid tube, leads to a local maximum at each resonance frequency. Thus, each peak of the indicator $\Delta\Pi_{S_\ell}$, shows a frequency shift. This phenomena is found to be particularly significant near the first cut-off frequency, thus concerning the higher order modes associated to $m = 1$.

4.3. CORRECTED FRONTAL RADIATION IMPEDANCES

In order to quantify the frequency shifts mentioned in the previous section, corrected values of the frontal radiation impedance, taking into account the wall vibrations effects, are derived. Upon neglecting the inter-modal couplings, due to the internal and external radiation of the shell, and considering only the interaction between one structural mode $\mathbf{\Phi}_\mu$ and one acoustic mode Ψ_α, the linear system (36) reduces to a 2×2 system which is singular when

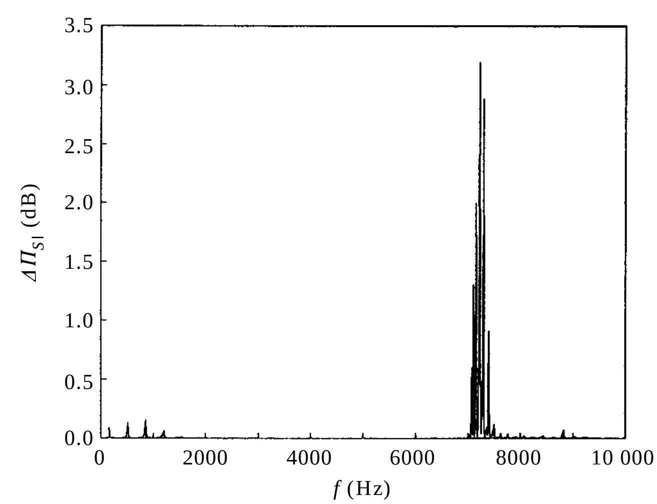

Figure 8. Difference between the sound power level, radiated from the open end of the tube, calculated with and without wall vibrations.

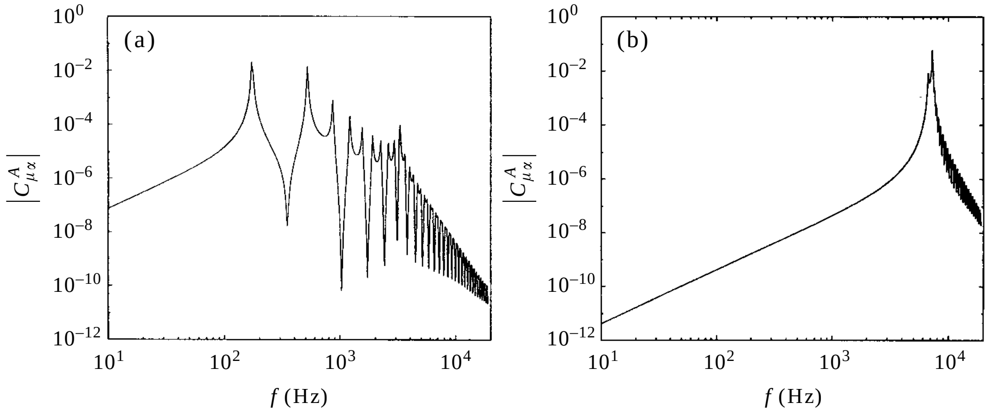

Figure 9. Modulus of $C^A_{\mu\alpha}$, for $\alpha = (0, 0, 1)$ $\mu = (0, 1, 1, 1)$ (a), and for $\alpha = (1, 0, 1)$, $\mu = (1, 5,1,1)$ (b).

$$-m_\mu\omega^2 - \mathrm{j}\omega(Z^i_{\mu\mu} + Z^e_{\mu\mu}) + m_\mu\omega^2_\mu = \mathrm{j}\omega C^M_{\mu\alpha}, \tag{42}$$

or when

$$1 - (Z^r_{\alpha\alpha}/\rho c)(k_{mn}/k)\mathrm{j}\tan(k_{mn}\ell) = C^A_{\mu\alpha}, \tag{43}$$

where the terms $C^M_{\mu\alpha}$ and $C^A_{\mu\alpha}$ are defined by

$$C^M_{\mu\alpha} = H^2_{\mu\alpha}Z^r_{\alpha\alpha}/[1 - (Z^r_{\alpha\alpha}/\rho c)(k_{mn}/k)\mathrm{j}\tan(k_{mn}\ell)], \tag{44}$$

$$C^A_{\mu\alpha} = \mathrm{j}\omega H^2_{\mu\alpha}Z^r_{\alpha\alpha}/[-m_\mu\omega^2 - \mathrm{j}\omega(Z^i_{\mu\mu} + Z^e_{\mu\mu}) + m_\mu\omega^2_\mu], \tag{45}$$

the coupling coefficient $H_{\mu\alpha}$ being given by equation (D5). The solutions of equations (42) and (43) are the coupled resonance frequencies. When the terms $C^M_{\mu\alpha}$ and $C^A_{\mu\alpha}$ are forced to zero, solutions of equations (42) and (43) correspond to the resonance frequencies of the fluid-loaded shell (which do not takes into account the frontal radiation) and to the acoustic resonance frequencies of the internal column of fluid (bounded by a rigid wall) taking into account the end impedance $Z^r_{\alpha\alpha}$, respectively. The term $C^A_{\mu\alpha}$ allows one to take into account the effect of the three kinds of coupling upon the acoustic resonance frequencies. Equation (43) shows that the wall vibrations effect can be taken into account if the frontal radiation impedance $Z^r_{\alpha\alpha}$ takes on the corrected value

$$\bar{Z}^r_{\alpha\alpha} = Z^r_{\alpha\alpha} \cdot C^A_{\mu\alpha}/(1 - C^A_{\mu\alpha}) \simeq Z^r_{\alpha\alpha} \cdot (1 + C^A_{\mu\alpha}), \tag{46}$$

the coupled acoustic resonance frequencies being the solutions of

$$1 - (\bar{Z}^r_{\alpha\alpha}/\rho c)(k_{mn}/k)\mathrm{j}\tan(k_{mn}\ell) = 0. \tag{47}$$

The variation of $C^A_{\mu\alpha}$'s modulus versus frequency is shown in Figures 9(a) and (b). The first one (Figure 9(a)) concerns the interaction of the plane wave modes ($m = 0$, $n = 0$) with the first breathing shell mode ($m = 0$, $q = 1$). The second one (Figure 9(b)) is related to the interaction between the first higher order acoustical mode ($m = 1$, $n = 0$) and the bending shell mode of axial index $q = 5$.

5. CONCLUSION

In order to quantify the wall vibrations effect of the body of a wind music instrument on tone, a model for the vibroacoustic behaviour of an ersatz clarinet has been presented. Three kinds of coupling have been identified and included in the model: the internal and external radiation couplings and the inter-modal acoustic coupling created by the radiation from the open end of the instrument. The radiated sound power from the lateral wall is calculated for mechanical and acoustical excitations and is found to be much lower than the sound power radiated from the open end. The difference between the sound power radiated from the open end for a vibrating tube and a rigid one are exhibited and interpreted in terms of acoustic resonance frequency shifts due to wall vibrations. This effect has been shown to be significant near the first cut-off frequency of the cylindrical tube. Corrected values for the frontal radiation impedance of the tube, taking into account the wall vibrations effect, are given and quantify these frequency shifts, related to small tone changes.

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APPENDIX A: COMPLEX SPEED OF SOUND TAKING INTO ACCOUNT THE INTERNAL FLUID LOSSES

The dissipative phenomena (viscous and thermal losses) inside the waveguide can be taken into account by considering the complex speed of sound $c = c_0(1 - \mathrm{j}\eta_f)$, which can be obtained for each acoustic mode as follows [29]. The Helmholtz equation $(\Delta + k^2)p(\mathbf{r}) = 0$, with $k = \omega/c_0$ in an infinite rigid circular tube, associated to the mixed condition on the wall

$$\frac{\partial p}{\partial r}(r = a) = -\mathrm{j}k\beta p, \tag{A1}$$

where the wall admittance describing the loss effects, have the form

$$\beta(\omega) = (\mathrm{j}k)^{1/2}\left[\left(1 - \frac{k^2_{W_{mn}} - m^2/a^2}{k^2}\right)\sqrt{l'_v} + (\gamma - 1)\sqrt{l_h}\right] \tag{A2}$$

(l'_v and l_h being viscous and thermal characteristic length, respectively, and γ being the specific heat ratio), leads to the following expression for the axial acoustic wavenumber, associated to the circumferential index m and the radial index n,

$$k^2_{mn} = k^2 - k^2_{W_{mn}} + X, \tag{A3}$$

with

$$X = \frac{1 + \mathrm{j}}{\sqrt{2}}\frac{2}{a}k^{3/2}\left[\left(1 - \frac{k^2_{W_{mn}} - m^2/a^2}{k^2}\right)\sqrt{l'_v} + (\gamma - 1)\sqrt{l_h}\right]\frac{1}{1 - \gamma^2_{mn}}, \tag{A4}$$

where $k_{W_{mn}}$ and γ_{mn} are given in Appendix B. Note that the term X is equal to zero when the fluid is lossless. The comparison of equation (A3), in which $k = \omega/c_0$, with the relation $k_{mn} = (\omega/c)^2 - k_{W_{mn}}$, allows one to characterize the complex speed c, taking into account the loss phenomena.

APPENDIX B: STRUCTURAL MODES

The modes of a simply supported shell can be written as [28]

$$\mathbf{\Phi}_\mu = \begin{bmatrix} \Phi_{1\mu} \\ \Phi_{2\mu} \\ \Phi_{3\mu} \end{bmatrix} = \begin{bmatrix} U_\mu \cos(q\pi z/\ell)\sin(m\theta + s\pi/2) \\ V_\mu \sin(q\pi z/\ell)\cos(m\theta + s\pi/2) \\ \sin(q\pi z/\ell)\sin(m\theta + s\pi/2) \end{bmatrix}, \tag{B1}$$

where the subscript μ denotes the four indexes:

$$\mu = (m, q, s, j), \tag{B2}$$

in which m denotes the circumferential index ($m \geqslant 0$), q denotes the axial index ($q \geqslant 1$), s is the symmetry index ($s = 0, 1$), and j is the mode type index ($j = 1$ for bending modes, $j = 2$ for extension/compression modes, $j = 3$ for twisting modes). The modal coefficients U_μ, V_μ and the natural angular frequency ω_μ can be found by solving a 3×3 linear system [28], obtained by inserting equation (B1) into the homogeneous equation of the shell motion.

APPENDIX C: ACOUSTICAL MODES

The solutions of the Neumann two-dimensional transverse problem are written as

$$\Psi_\alpha = \mathrm{J}_m(k_{W_{mn}} r)\sin(m\theta + s\pi/2)/\Lambda_\alpha, \tag{C1}$$

with

$$\Lambda_\alpha^2 = (\pi a^2/\varepsilon_m)(1 - \gamma_{mn}^2)\mathrm{J}_m^2(k_{W_{mn}} a) \tag{C2}$$

and

$$\gamma_{mn}^2 = \begin{Bmatrix} 0 & m = 0 \\ m^2/(k_{W_{mn}} a)^2 & m > 0 \end{Bmatrix}, \tag{C3}$$

and are associated to the eigenwavenumber $k_{mn} = (k^2 - k_{W_{mn}}^2)^{1/2}$, $(k_{W_{mn}} a)$ being the successive zeros of J'_m. The parameter α denotes a triplet of integers,

$$\alpha = (m, n, s), \tag{C4}$$

where m is the circumferential index ($m \geqslant 0$), n the radial index ($n \geqslant 0$) and s the symmetry index ($s = 0, 1$).

APPENDIX D: INTERNAL RADIATION IMPEDANCES

The internal radiation impedances $Z_{\mu\mu'}^i$, equation (22), describe the interactions between the shell and the internal fluid when Neumann and Dirichlet boundary conditions are applied on S_0 and S_ℓ, respectively. They can be written as [16]

$$Z_{\mu\mu'}^i = \delta_{mm'}\delta_{ss'}(Z_{\mu\mu'}^c + \delta_{\mu\mu'} Z_{\mu\mu}^d), \tag{D1}$$

where $\mu = (m, q, s, j)$ and $\mu' = (m', q', s', j')$, and where $\delta_{\mu\mu'}$ denotes the Kronecker symbol. The cross impedance, $Z_{\mu\mu'}^c$, and the complementary term, $Z_{\mu\mu}^d$, that one needs to obtain the direct impedance $Z_{\mu\mu}^i = Z_{\mu\mu}^c + Z_{\mu\mu}^d$, are given by

$$\frac{Z_{\mu\mu'}^c}{\rho c 2\pi a \ell} = -\mathrm{j}\frac{2}{\varepsilon_m}\frac{a}{\ell} ka \frac{q\pi a}{\ell}\frac{q'\pi a}{\ell}\sum_n \frac{\tan(k_{mn}\ell)}{k_{mn}a[1 - \gamma_{mn}^2][(k_{mn}a)^2 - (q\pi a/\ell)^2][(k_{mn}a)^2 - (q'\pi a/\ell)^2]}, \tag{D2}$$

$$\frac{Z^{d}_{\mu\mu}}{\rho c 2\pi a \ell} = \frac{-\mathrm{j}}{2\varepsilon_m} \frac{k}{k_q} \frac{\mathrm{J}_m(k_q a)}{\mathrm{J}'_m(k_q a)}. \tag{D3}$$

The other coefficients, $T_{\mu\alpha}$ and $H_{\mu\alpha}$, describing the internal radiation of the shell and defined by equations (23) and (24), are given by

$$T_{\mu\alpha} = 2\mathrm{j}\left[\frac{\pi}{\varepsilon_m(1-\gamma^2_{mn})}\right]^{1/2} \frac{kq\pi/\ell}{k_{mn}} \frac{\tan(k_{mn}\ell)}{[k^2_{mn} - (q\pi/\ell)^2]}, \tag{D4}$$

$$H_{\mu\alpha} = -4\left[\frac{\pi}{\varepsilon_m(1-\gamma^2_{mn})}\right]^{1/2} \frac{q\pi/\ell}{[k^2_{mn} - (q\pi/\ell)^2]} \frac{1}{\cos k_{mn}\ell} \begin{Bmatrix} \sin^2(k_{mn}\ell/2) & q \text{ even} \\ \cos^2(k_{mn}\ell/2) & q \text{ odd} \end{Bmatrix}. \tag{D5}$$

APPENDIX E: FRONTAL RADIATION IMPEDANCES

The frontal radiation impedances, equation (27), characterize the inter-modal coupling caused by sound radiation from the open end S_ℓ of the duct, and are given by [31]

$$\frac{Z^{r}_{\alpha\alpha'}}{\rho c} = -\delta_{ss'}\delta_{mm'}\mathrm{j}\int_0^{\infty} \tau(\tau^2-1)^{(-1/2)}\mathscr{D}_{mn}(\tau)\mathscr{D}_{mn'}(\tau)\,\mathrm{d}\tau, \tag{E1}$$

where

$$\mathscr{D}_{mn}(\tau) = k^2\int_0^{a} \mathrm{J}_m(\tau k r)\mathrm{J}_m(k_{mn} r) r\,\mathrm{d}r, \tag{E2}$$

$$\alpha = (m, n, s), \qquad \alpha' = (m', n', s'). \tag{E3}$$

The other coupling coefficients, J_α and K_α involved in the frontal radial equations (31) and (32) are written as:

$$J_\alpha = 1/\cos(k_{mn}\ell), \qquad K_\alpha = (k_{mn}/k)\mathrm{j}\tan(k_{mn}\ell). \tag{E4, E5}$$

APPENDIX F: EXTERNAL RADIATION IMPEDANCES

The external radiation impedances, $Z^{e}_{\mu\mu'}$, describe the interaction between the shell and the external fluid and can be written as [23–25]

$$Z^{e}_{\mu\mu'} = \mathrm{j}\rho c \frac{a}{\varepsilon_m}\int_{-\infty}^{+\infty} \frac{k}{k_r} \frac{\mathrm{H}^1_m(k_r a)}{\mathrm{H}^{1\prime}_m(k_r a)} \hat{\Phi}_{3\mu}(k_z)\hat{\Phi}^{*}_{3\mu'}(k_z)\,\mathrm{d}k_z, \tag{F1}$$

with

$$k_r = (k^2 - k_z^2)^{1/2}, \qquad \text{with} \quad 0 \leqslant \arg(k_r) \leqslant \pi/2, \tag{F2}$$

and

$$\hat{\Phi}_{3\mu}(k_z) = \int_0^{\ell} \sin(q\pi z/\ell)\,\mathrm{e}^{-\mathrm{j}k_z z}\,\mathrm{d}z. \tag{F3}$$

After some calculations, the external radiation impedance is expressed in the numerically tractable form

$$\frac{Z^e_{\mu\mu'}}{\rho c 2\pi a\ell} = \frac{\mathrm{j}4qq'\pi}{\varepsilon_m (k\ell)^3}\int_0^{+\infty} \frac{\mathrm{H}_m(ka(1-u^2)^{1/2})}{\mathrm{H}'_m(ka(1-u^2)^{1/2})}\frac{F_{qq'}(u)}{[u^2-(q\pi/k\ell)^2][u^2-(q'\pi/k\ell)^2][1-u^2]^{1/2}}\,\mathrm{d}u, \tag{F4}$$

where

$$F_{qq'}(u) = \left\{\begin{matrix} \cos^2(ku\ell/2) & \text{if } q, & q' \text{ odd,} \\ \sin^2(ku\ell/2) & \text{if } q, & q' \text{ even} \\ 0 & \text{if } q(q'), & q'(q), \text{ even, odd} \end{matrix}\right\}. \tag{F5}$$

APPENDIX G: MATRIX NOTATIONS

For the shell studied (see section 4.1), the resonance frequencies of the extension/compression and twisting modes ($j = 2, 3$) have high values compared to those of bending modes ($j = 1$). Thus, only bending modes are taken into account. The mechanical and acoustic excitation sources are symetrical with respect to the plane ($x0z$). Thus, the vibroacoustic responses are also symmetrical, implying $s = 1$. In the following, the indexes s and j are set to 1. The vectors

$$\mathbf{A}_m = \begin{bmatrix} A_{(m,1,s,j)} \\ \vdots \\ A_{(m,Q,s,j)} \end{bmatrix}, \qquad \mathbf{P}_m = \begin{bmatrix} P_{(m,1,s)} \\ \vdots \\ P_{(m,N,s)} \end{bmatrix}, \tag{G1}$$

describe the unknown modal amplitudes $A_\mu = A_{(m,q,s,j)}$ and $P_\alpha = P_{(m,n,s)}$ for a given circumferential index m and axial mechanical expansion and for the radial acoustic expansion truncated to Q and N, respectively. The inner products

$$\mathbf{F}_m = \begin{bmatrix} F_{(m,1,s,j)} \\ \vdots \\ F_{(m,Q,s,j)} \end{bmatrix}, \qquad \mathbf{V}_m = \begin{bmatrix} V_{S0(m,1,s)} \\ \vdots \\ V_{S0(m,N,s)} \end{bmatrix} \tag{G2}$$

describe the excitation sources. The three kinds of radiation impedances $Z^i_{\mu\mu'}$, $Z^e_{\mu\mu}$ and $Z^r_{\alpha\alpha'}$ do not depend on the s and j indexes. Thus, one defines

$$\mathbf{Z}^{i,e}_m = \begin{bmatrix} Z^{i,e}_{(m,1,s,j),(m,1,s,j)} & \cdots & Z^{i,e}_{(m1,s,,j),(m,Q,s,j)} \\ \vdots & & \vdots \\ Z^{i,e}_{(m,Q,s,j),(m,1,s,j)} & \cdots & Z^{i,e}_{(m,Q,s,j),(m,Q,s,j)} \end{bmatrix}, \tag{G3}$$

$$\mathbf{Z}^v_m = \begin{bmatrix} Z^r_{(m,1,s),(m,1,s)} & \cdots & Z^r_{(m,1,s),(m,N,s)} \\ \vdots & & \vdots \\ Z^r_{(m,N,s),(m,1,s)} & \cdots & Z^r_{(m,N,s),(m,N,s)} \end{bmatrix}. \tag{G4}$$

Upon making use of the relations (8) and (29) the other coupling matrices are defined by:

$$\mathbf{H}_m = \begin{bmatrix} H_{(m,1,s,j),(m,1,s)} & \cdots & H_{(m,1,s,j),(m,N,s)} \\ \vdots & & \vdots \\ H_{(m,Q,s,j),(m,1,s)} & \cdots & H_{(m,Q,s,j),(m,N,s)} \end{bmatrix}, \qquad \mathbf{J}_m = \begin{bmatrix} \mathrm{J}_{(m,1,s,j)} & & (0) \\ & \ddots & \\ (0) & & \mathrm{J}_{(m,N,s,j)} \end{bmatrix}, \tag{G5}$$

$$\mathbf{T}_m = \begin{bmatrix} T_{(m,1,s,j),(m,1,s)} & \cdots & T_{(m,1,s,j),(m,N,s)} \\ \vdots & & \vdots \\ T_{(m,Q,s,j),(m,1,s)} & \cdots & T_{(m,Q,s,j),(m,N,s)} \end{bmatrix}, \qquad \mathbf{K}_m = \begin{bmatrix} K_{(m,1,s,j)} & & (0) \\ & \ddots & \\ (0) & & K_{(m,N,s,j)} \end{bmatrix}, \quad \text{(G6)}$$

$$\mathbf{M}_m = \begin{bmatrix} m_{(m,1,s,j)}(-\omega^2 + \omega^2_{(m,1,s,j)}(1 - \mathrm{j}\eta)) & & (0) \\ & \ddots & \\ (0) & & m_{(m,Q,s,j)}(-\omega^2 + \omega^2_{(m,Q,s,j)}(1 - \mathrm{j}\eta)) \end{bmatrix}. \quad \text{(G7)}$$