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DETECTION OF FATIGUE CRACKS IN FLEXIBLE GEOMETRICALLY NON-LINEAR BARS BY VIBRATION MONITORING

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The results of mathematical simulation of flexural vibrations of a geometrically non-linear cracked bar under external harmonic excitation are presented. It is shown, that thanks to the influence of elastic non-linearity of the crack, new non-linear properties which are impossible in the initial undamaged structure, appear in the system (self-excitation of subharmonic regimes, appearance of even-numbered harmonic components in frequency spectrum of 3/1 order superharmonic regime). By the utilization of these non-linear effects a new method for detection of cracks is developed. Unlike traditional procedures, the proposed method permits recognition of the legitimated response to non-linearity of the crack through interfering signals from the geometrical non-linearity of a bar.

1. INTRODUCTION

There is a great number of beam-type mechanical structural elements (aerials, long pipelines, wings of some types of aeroplanes, blades of helicopters, etc.), the dynamic behaviour of which under external harmonic excitation is strongly dependent on their geometrical non-linearity [1, 2]. Dynamic loads acting on such structures in operating conditions encourage the origin of fatigue cracks in cross-sections of bars. Immediate visual detection of fatigue cracks in many cases is a very difficult problem due to hindered access to the testing object (equipping of an object with thin-walled skin or protective enclosure, presence of heat insulation or anticorrosive coating on the surface of testing object, etc.).

Therefore non-destructive testing of such structures is often carried out by the use of traditional linear vibration techniques, based on mathematical presentation of a crack through the local decrease of the flexural rigidity of the damaged cross-section of a bar (constantly open crack) [3, 4]. By this linear approach the presence of cracks in a testing object is detected through the monitoring of changes in resonant frequencies [4, 5] or in damping factor [6]. But these linear vibration procedures do not always come up to practical requirements because of low sensitivity to defects. For example, the origin of a crack, which makes up about 15–20% of the undamaged cross-section area, reduces the natural frequencies of a bar only on 1–1·5% [5, 7]. Besides, natural frequencies and damping factors of real structures can change their values not only due to the cracking process, but also in consequence of relaxation, wear, etc., and this latter change of parameters may be highly significant (up to 10–15%). Therefore for the practical realization of linear vibration methods of diagnostics it is necessary to determine initially

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the natural frequencies and damping characteristics of a structure in an undamaged condition and then to monitor the change of these parameters while it is in use. The procedure is very laborious and requires a great many working hours of failure shooting. Besides, due to low sensitivity of linear vibration methods to defects high-precision and expensive instrumentation has to be used.

The above mentioned disadvantages are not inherent to non-linear methods of vibration monitoring [8–12], based on the utilization of non-linear vibration effects as a collection of diagnostic factors, which point to the appearance of a fault. Under such an approach the non-linear model of an open and closed crack is used. At the present time various non-linear models of a crack are known, which are distinguished from each other by their mathematical complexity and physical nature [7–16]. The most adequate and detailed description of stress distribution near the crack can be given with the use of finite element models [7, 13, 14]. But these models are too intricate and require for their realization high-speed computer facilities with specialized software and a large memory.

In the present paper the mathematically more simple piece-wise linear crack model is used [9, 11, 17]. This model considers the opening and the closing of crack edges as a momentary process of changing the flexural rigidity of the damaged cross-section of the bar via the sign of the bend angle. According to [17], a piecewise linear crack model (in spite of its mathematical simplicity) ensures a sufficiently high accuracy in the dynamic analysis of structures with relatively small transverse cracks, whose value is not more than 20–25% of the cross-sectional area of the undamaged bar (such cracks are considered in this paper).

The necessity of taking account of geometrical non-linearity in testing a bar introduces an additional complication into the mathematical formulation of the problem to be analyzed. In this case the application of familiar non-linear vibration procedures [8–10] can result in quantitative and qualitative errors, because of the interfering influence of geometrical non-linearity on legitimate responses to the non-linearity of a crack. Therefore this paper deals with the development of more perfect methods of vibration monitoring, which will make it possible to recognize the response to non-linearity of a crack through signals from the geometrical non-linearity of a bar.

2. DYNAMIC MODEL

The dynamic model to be analyzed is a uniform viscoelastic and flexible cantilever bar, performing bending vibrations excited by the test harmonic force $P \sin \omega t$ (Figure 1).

A differential equation of vibration has been formed by the use of the well known differential relationships in bending:

$$d^2M/dx^2 = dQ/dx = -q, \tag{1}$$

where M and Q are the bending moment and transverse shear force in the beam cross-section with the co-ordinate x; q is intensity of a distributed load.

According to the equation of the elastic line of a flexible beam [18], the bending moment M can be represented as

$$M = -EI\, \partial^2 y/\partial x^2/[1 + (\partial y/\partial x)^2]^{3/2}, \tag{2}$$

where EI is rigidity in bending of a bar.

Only the variant of so called small geometric non-linearity [1], when longitudinal displacements of a bar are negligible in comparison with the flexural ones (such variant takes place in majority of applied problems of non-destructive testing), is analyzed. In this

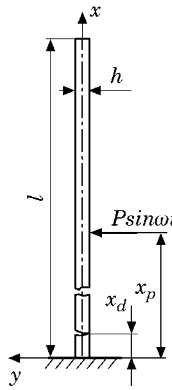

Figure 1. Model considered in dynamic analysis.

case the expression (2) with the aid of the binomial theorem can be approximately reduced to the following form:

$$M = -EI\,\partial^2 y/\partial x^2[1 - \tfrac{3}{2}(\partial y/\partial x)^2 + \tfrac{15}{8}(\partial y/\partial x)^4 - \tfrac{105}{48}(\partial y/\partial x)^6 + \cdots]$$
$$= -EI\,\partial^2 y/\partial x^2[1 + f_g(\partial y/\partial x)], \quad (3)$$

where

$$f_g(\partial y/\partial x) = -\tfrac{3}{2}(\partial y/\partial x)^2 + \tfrac{15}{8}(\partial y/\partial x)^4 - \tfrac{105}{48}(\partial y/\partial x)^6 + \cdots$$

is a functional describing the geometrical non-linearity of a bar.

Taking account of internal friction in the material of a bar [18], the expression (3) can be rearranged to the form:

$$M = -EI(1 + b\,\partial/\partial t)\,\partial^2 y/\partial x^2[1 + f_g(\partial y/\partial x)], \quad (4)$$

where b is a coefficient of internal friction.

On the other hand, the intensity of the distributed load q is determined by external harmonic excitation and distributed inertial forces of the beam:

$$q = P\sin\omega t\delta_1(x - x_p) - \mu\,\partial^2 y/\partial t^2, \quad (5)$$

where P and ω are the amplitude and frequency of the external harmonic excitation; $\delta_1(x - x_p)$ is a Dirac delta function; x_p is the coordinate of the cross-section with applied external harmonic force; μ is distributed mass of the bar.

By substituting the mathematical expressions (4) and (5), the initial differential equation (1) of bending vibrations of the bar can be transformed into the following form:

$$EI\left[1 + f_g\left(\frac{\partial y}{\partial x}\right)\right]\left(1 + b\frac{\partial}{\partial t}\right)\frac{\partial^4 y}{\partial x^4} + 2EI\frac{\partial}{\partial x}\left[f_g\left(\frac{\partial y}{\partial x}\right)\left(1 + b\frac{\partial}{\partial t}\right)\frac{\partial^3 y}{\partial x^3} + \mu\frac{\partial^2 y}{\partial t^2}\right.$$
$$\left. + EI\frac{\partial^2}{\partial x^2}\left[f_g\left(\frac{\partial y}{\partial x}\right)\right]\left(1 + b\frac{\partial}{\partial t}\right)\frac{\partial^2 y}{\partial x^2} = P\sin\omega t\delta_1(x - x_p). \quad (6)$$

In the case of a cantilever bar the end boundary conditions are the following:

$$\{y(x = 0, t) = 0, \quad (\partial y/\partial x)(x = 0, t) = 0, \quad \frac{\partial^2 y}{\partial x^2}(x = l, t) = 0, \quad \frac{\partial^3 y}{\partial x^3}(x = l, t) = 0, \quad (7)$$

where l is the length of a bar.

A fatigue crack, which can occur in a bar, is considered as an additional elastic non-linearity describing the regular process of opening and closing of the crack edges during the vibration. As in some other works [9, 11, 17], it is assumed that the flexural rigidity of the damaged cross-section $EI(x = x_d)$ is changed in accordance with the following mathematical expression:

$$EI(x = x_d) = EI_0[1 - \sigma \cdot \delta(x - x_d)], \tag{8}$$

where EI_0 is the rigidity in bending of undamaged section; σ is the crack parameter; $\delta(x - x_d)$ is a Dirac delta function; x_d is the co-ordinate of the damaged cross-section of the bar.

The crack parameter σ is a step function which describes the relationship between the flexural rigidity of the damaged cross-section and the bend angle $\varphi = \partial y/\partial x$. The mathematical form of this function is dependent on the location of the fatigue crack (left or right relative to neutral line x, Figure 1). In the case of a left crack location the crack parameter σ is described by the expression

$$\sigma = \begin{Bmatrix} 0, & \partial y/\partial x > 0; \\ \sigma_c, & \partial y/\partial x \leqslant 0, \end{Bmatrix}, \tag{9}$$

but in the case of right crack location—by the expression

$$\sigma = \begin{Bmatrix} 0, & \partial y/\partial x \leqslant 0; \\ \sigma_c, & \partial y/\partial x > 0, \end{Bmatrix}, \tag{10}$$

where $\sigma_c = (1 - I_d/I_0)$ is the measure of relative crack value; I_d is the second moment of the damaged cross-section (on condition that the crack is opened).

3. METHOD OF ANALYSIS

By taking account of the mathematical intricacy of the problem stated (two different non-linear functions in differential equations—geometrical non-linearity of the bar and piecewise linear non-linearity of the fatigue crack), it has been found expedient to solve the differential equation (6) subject to the expressions (7)–(10) on the specialized hybrid analog–digital computational complex (HCC) designed in Riga Technical University [19, 20]. This computational complex is predominantly intended for solution of complex nonlinear dynamic problems and consists of two parts.

The integration of non-linear differential equations is carried out on the high speed analog part of HCC. By the operation principle of the computational complex, the initial partial differential equation (6) is replaced by an equivalent set of finite difference equations. The simulation is based on the direct analogy method [21], which establishes a quantitative and qualitative correspondence between mechanical vibrations and current and voltage oscillations in the electrical model of the system to be analyzed. A great number of electrical modules being analogous to inertial, elastic and dissipative elements with linear and practically any nonlinear characteristics (piecewise linear, polynomial, relay, etc.) are developed. Units for input of external excitations are also designed by modular principle. In the course of the simulation the electrical modules must be connected according to the structure of the initial mechanical system. Collections of standard circuit diagrams for solution in finite difference formulation of refined differential equations describing flexural, longitudinal and torsional vibrations of bars in view of various non-linear factors are developed [9, 19–21]. The analog part of HCC operates in the acoustical frequency range and therefore has very high speed of action, making it possible to obtain a great number of solutions in a comparatively short time.

The digital part of HCC consists of two types of computing machines: a universal personal computer (e.g. PC 486DX) and a specialized computer X6-4 with a wired-in program for the calculation of statistical characteristics of random processes. Such structure of the digital part makes it possible to use universal and specialized software for a personal computer in order to control the programming of the analog part and to process data (scaling, synthesis of regression equations, optimization, etc.). But the specialized computer X6-4 increases the rate and accuracy of calculations for random characteristics (correlation and autocorrelation functions, distribution density and probability distribution functions, power spectral density, etc.).

The quantitative estimation of accuracy in analog-digital simulation was carried out by the solution of a great number of test examples [19]. The results of test simulation have shown close agreement with exact and numerical solutions for systems with piecewise linear, polynomial and relay elastic–dissipative characteristic under harmonic and polyharmonic excitation.

The problem considered in this paper was solved assuming the parameters of equation (6) to be according to the following conditions:

$$\beta = b\omega_1 = 0{\cdot}009;\ l/h = 100;\ \varepsilon_1 = \omega_1 l^2\sqrt{\mu/(EI)} = 1{\cdot}875;\ \mu g l^3/(EI) = 0{\cdot}180;\ z_p = x_p/l = 0{\cdot}5$$

(here ω_1 is the first natural flexural frequency of the linearized beam in the undamaged state). The location of the damaged cross-section was taken near the restraint ($z_d = x_d/l = 0{\cdot}05$), where the origin of the fatigue crack is most probable because of the maximal level of dynamic stresses. Other parameters of equation (6), expressed in dimensionless form ($p = Pl^2/(EI)$; $v = \omega/\omega_1$; $\alpha = I_0/I_d$), have been varied within the limits of $p = 0{\cdot}01$–$0{\cdot}40$, $v = 0{\cdot}25$–$3{\cdot}00$ and $\alpha = 1{\cdot}0$–$1{\cdot}3$. Under such conditions the analysis of non-linear resonant regimes corresponding to the first flexural mode was possible.

4. SPECIAL FEATURES OF VIBRATIONS CAUSED BY THE INFLUENCE OF FATIGUE CRACK

Non-linear characteristic properties of vibrations may occur in the system under study even in the case of $\alpha = 1$ (undamaged bar) due to influence of geometrical non-linearity. Therefore a search for special features of vibration, conditioned by the appearance of a fatigue crack, is absolutely essential for developing new methods of nondestructive testing.

Non-linear properties of oscillations are particularly pronounced at super and subharmonic resonances, which are basically impossible in linear systems [22]. Figure 2 shows the domains on the co-ordinate plane p and α, within which the super and subharmonic regimes corresponding to the first flexural mode, may exist (domains are section-lined). Symbol j/i on the diagram denotes the ordinal number of the non-linear regime, which indicates how many natural periods of vibration (j) fall at i periods of external harmonic excitation.

As follows from the diagram, at $\alpha = 1$ (undamaged structure) the excitation of super- and subharmonic vibrations is possible only under the specific values of amplitudes p of external excitation, which must be in excess of threshold value $p_0^{(j/i)}$. With the propagation of a fatigue crack (elevation of parameter α) threshold values $p_0^{(j/i)}$ gradually decrease and in the limit reduce to zero. When the value of the fatigue crack exceeds some critical magnitude $\alpha_c^{(j/i)}$, the non-linear regime of the j/i order may be excited under any, however small, amplitude p of the external harmonic force. As to the upper bound of the domains

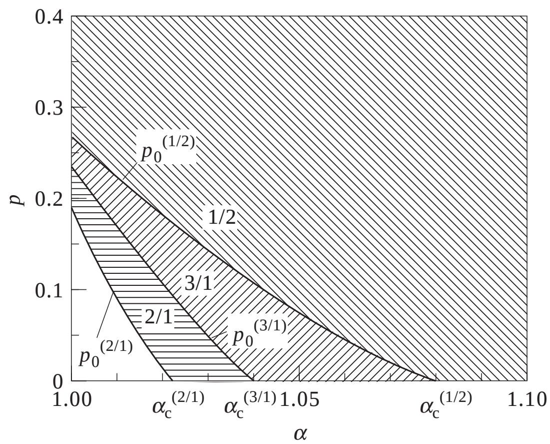

Figure 2. Domains for conditions of existence of 2/1 and 3/1 order superharmonic regimes and 1/2 order subharmonic regime (domains are section-lined). Notation: $p_0^{(j/i)}$, threshold value of amplitude of external harmonic force necessary for excitation of nonlinear regime of the j/i order; $\alpha_c^{(j/i)}$, critical magnitude of fatigue crack (for nonlinear regime of the j/i order).

of existence (for parameter p), its value in accordance with the theory [23] extends to infinity. But practically the upper bound is defined only by the conditions for the dynamic strength of the structure.

Well-founded conclusions about the intensity of super and subharmonic regimes as well as about possibilities of their utilization in diagnostic procedures may be obtained by the analysis of the amplitude–frequency characteristic (AFC), which graphically represents the mutual connections between the driving frequency v and the half-swing of oscillations u_0. As an example, Figure 3 shows the AFC, which has been plotted assuming $p = 0{\cdot}3$ and

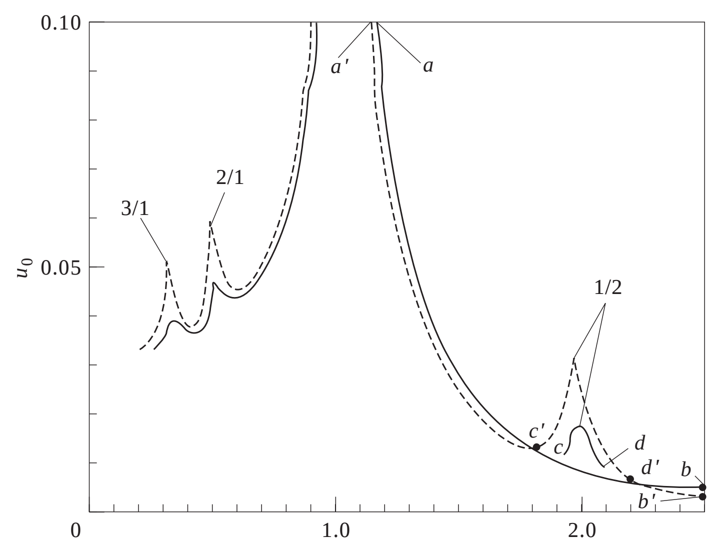

Figure 3. Amplitude–frequency characteristic for the first mode of bar flexural vibrations ($z = 0{\cdot}5$; $p = 0{\cdot}3$). The full line corresponds to the case of the undamaged bar ($\alpha = 1$); the dotted line shows resonance curve for bar with a crack ($\alpha = 1{\cdot}25$).

$z = 0{\cdot}5$. Full lines correspond to the case of an undamaged structure ($\alpha = 1$); dotted lines show the resonance curves for a bar with a crack ($\alpha = 1{\cdot}25$).

It is clear from the comparison of the AFC presented, that the appearance and further propagation of a fatigue crack facilitates the sharp amplification of all non-linear resonances (both superharmonic and subharmonic). Simultaneously the initial conditions necessary for excitation of subharmonic regimes change their values, and this fact also has an influence on the AFC. For example, in the undamaged structure ($\alpha = 1$) due to symmetry of elastic characteristic (3) the resonance curve *cd* of the 1/2 order subharmonic oscillations is the isolated one; therefore system set-up on this resonance curve is possible only under special, substantially different from zero initial values of displacement u and velocity $\dot{u}$ (under the zero and close to zero initial conditions the main oscillatory regime on the curve *ab* is realized). A propagated crack introduces an asymmetry (9) or (10) into the equivalent elastic characteristic, and in consequence of this the part $c'd'$ of the AFC corresponding to the subharmonic vibrations amalgamates to a common continuous curve with the main resonance curve $a'b'$. The subharmonic regime of the 1/2 order becomes the only possible (or the globally stable) regime in the corresponding frequency range [24]. At the same time the existence of globally stable subharmonic regimes in the initially undamaged structure is basically impossible.

Besides, the vibration spectrum of nonlinear regimes was analyzed. Figure 4 shows the time responses $u = f(\tau)$ and spectrograms for resonant points of the AFC. As follows from the spectrograms presented, in the undamaged structure ($\alpha = 1$) the vibration spectrum of the non-linear regime of odd order (e.g., the 3/1 order superharmonic regime) is the series of odd harmonic components 1/1, 3/1, 5/1, etc. At the same time the vibration spectrum of the non-linear regime of even order (e.g., superharmonic regime of the 2/1 order, subharmonic regime of the 1/2 order) is the series of both odd and even harmonic components (mixed spectrum). After the nucleation in a bar of a fatigue crack ($\alpha > 1$) the specified difference is lost, and the frequency spectrum of vibrations becomes the mixed one on all non-linear resonances (even on the non-linear resonance of the odd order). And what's more, the amplitude $u_{2/1}$ of even harmonic component, which additionally appears

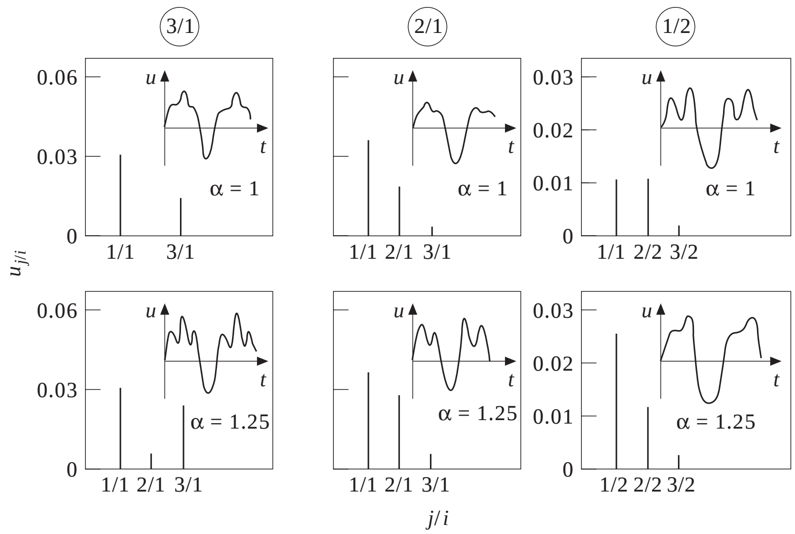

Figure 4. Time responses $u = f(\tau)$ and spectrograms for resonant points of the AFC (bar cross-section with the co-ordinate $z = 0{\cdot}5$).

in the frequency spectrum of the 3/1 order superharmonic vibrations, is directly related to the size α of the fatigue crack.

5. PROCEDURE OF VIBRATION TESTING

The above mentioned distinctions of dynamic behaviour form the theoretical basis of a new vibration procedure for the detection of fatigue cracks in geometrically nonlinear bars. Even the possibility to realize in the system the specific set of non-linear properties, which are impossible in an initially undamaged structure (self-excitation of subharmonic regimes, appearance of even-numbered harmonic components in frequency spectrum of 3/1 order superharmonic regime, etc.), may be used as a qualitative diagnostic sign indicating the presence of a fatigue crack in a testing bar. But a well-founded conclusion about the size of a fatigue crack and its location in a testing object may be done only by using quantitative algorithms for nondestructive testing.

The efficiency of the quantitative vibration procedure to a considerable extent is determined by the order j/i of the non-linear regime of vibration testing. As follows from the theoretical analysis, there is a close relationship between j/i and detection sensitivity of non-linear vibration parameters, their intensity and excitation condition. According to the search for an optimum performed by the methodology described in [19, 25], the most preferable is the superharmonic regime of the 3/1 order, which is characterized by the relatively small threshold values $p_0^{(3/1)}$ and $\alpha_c^{(3/1)}$ (Figure 2) as well by the global stability to possible variations of initial conditions. Besides, it is possible to increase sufficiently the detection sensitivity by use of spectral parameters of the 3/1 order superharmonic regime in the capacity of diagnostic signs.

Figure 5 shows in graphical form the relationship between the spectral ratios $u_{2/1}/u_{1/1}$ and $u_{3/1}/u_{1/1}$, measured on the 3/1 superharmonic resonance, and crack value α. In spite of a relatively small amplitude of the harmonic component $u_{2/1}$, the increment of its changing with the elevation of the parameter α is much higher in comparison with the increment of changing of the harmonic $u_{3/1}$. According to the analysis performed, as the crack value α goes above 1% the amplitude of harmonic component $u_{2/1}$ varies by 10%, whereas the amplitude of harmonic $u_{3/1}$ changes by 5%. In other words, relative sensitivity of the diagnostic sign $u_{2/1}/u_{1/1}$ to the growth of a fatigue crack is about two times higher in

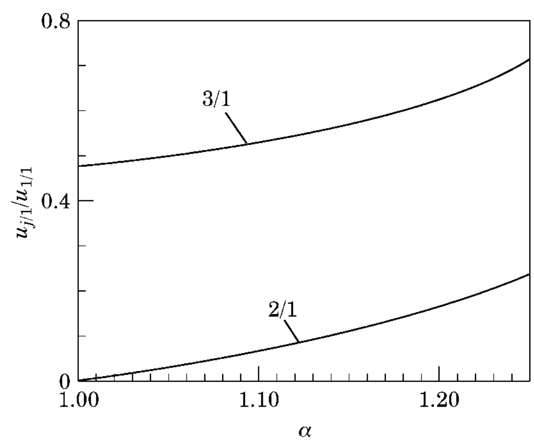

Figure 5. Graphs of spectral ratios $u_{2/1}/u_{1/1}$ and $u_{3/1}/u_{1/1}$ as functions of crack value α for 3/1 order superharmonic regime (bar cross-section with the co-ordinate $z = 0{\cdot}5$).

comparison with the sensitivity of diagnostic sign $u_{3/1}/u_{1/1}$. Besides, relative value of the harmonic component 2/1 (unlike of the 3/1 order harmonic) is simply not connected with geometrical nonlinearity of a bar; thanks to this the problem of suppression of interfering influence of geometrical nonlinearity on the results of vibration testing is solved without any additional control operation. Therefore it is proposed to carry out the vibration testing by recording the spectral ratio $u_{2/1}/u_{1/1}$ on the 3/1 order superharmonic resonance.

Detection of fatigue cracks in flexible bars by the proposed superresonant vibration method may be carried out by the following operational procedure. At first, the forced bending vibrations of a testing bar are excited at a frequency, which is three times smaller in comparison with the first natural resonant frequency ω_1. After that the Fourier spectrum of the bar flexural vibrations is analyzed. In case of the presence of a 2/1 order harmonic component in the spectral response of the system the conclusion of the presence of a fatigue crack in the tested object can be made. On the stage of quantitative vibration monitoring amplitudes of harmonic components $u_{2/1}$ and $u_{1/1}$ in frequency spectrum are measured. The size α of the fatigue crack is evaluated by the value of the spectral ratio $u_{2/1}/u_{1/1}$ using a preliminary constructed calibration curve $\alpha = f(u_{2/1}/u_{1/1})$ (similar curve is presented in Figure 5). Also, the procedure of vibration testing is sufficiently simplified thanks to the invariance of the spectral ratio $u_{2/1}/u_{1/1}$ to possible changes of the amplitude P of external harmonic excitation. The main advantage of the proposed superresonant method of non-destructive testing lies in the great detection sensitivity, which is about 10 times higher in comparison with familiar linear vibration procedures [4, 5].

6. CONCLUSIONS

The results presented in this paper show a new approach for the detection of fatigue cracks in flexible, geometrically non-linear beam-type structural elements. By mathematical simulation it is shown, that the appearance of a fatigue crack has a stimulating influence on the realization of a new set of non-linear properties, which are impossible in the initially undamaged structure (self-excitation of subharmonic regimes, appearance of even-numbered harmonic components in frequency spectrum of 3/1 order superharmonic regime). Utilization of these non-linear effects has made it possible to develop a new vibration method for the recognition of the response to a crack through interfering signals from the geometrical non-linearity of a bar. It is shown, that the detection sensitivity of the proposed non-linear method is about 10 times higher in comparison with the familiar linear vibration procedures. Therefore it is found expedient to introduce this non-linear algorithm of vibration monitoring into engineering practice.

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