Journal of Sound and Vibration (1998) **213**(1), 193–200

A RAYLEIGH–RITZ SUBSTRUCTURE SYNTHESIS METHOD IN PHYSICAL CO-ORDINATES FOR DYNAMIC ANALYSIS OF STRUCTURES

M. Scheble, C. N. Strizzolo and J. Converti

Department of Nuclear Engineering, Centro Atómico Bariloche, S. C. de Bariloche, Río Negro, Argentina

(*Received* 23 *October* 1996, *and in final form* 22 *December* 1997)

1. INTRODUCTION

A new method is presented that retains all the advantages of the more elaborate existing Rayleigh-Ritz methods such as the revised approach presented by Jen *et al*. [1], but with two additional important advantages.

The first advantage is related to simplicity of order reduction by using standard "assembling" techniques similar to those used in FEM. Thus, assembling and order reduction is simpler and numerically more efficient than singular value decomposition. This results from the use of Boolean instead of full real matrices and shows up as improved precision in large complex structures. The second advantage is related to computational cost. The Rayleigh–Ritz type methods lead to full matrices **K** and **M** to be diagonalized. The method presented yields sparse matrices that can be solved at much lower cost by sparse matrix techniques developed for FEM. The characteristic feature of this approach is to work in physical instead of generalized co-ordinates. The general steps of the procedure will be developed and commented upon in section 2. The main features are described below.

(1) Decomposition of the whole structure in simple structural components and adoption of suitable models for each component (set of shape functions) from a library. The stiffness and mass matrices in generalized co-ordinates can then be determined in a systematic manner.

(2) Transformation of stiffness and mass matrices from generalized to physical co-ordinates and enforcement of geometric boundary conditions, both internal and external, by using an "assembling" technique similar to FEM. This procedure will ensure admissibility of the remaining independent global shape functions.

(3) Solution of the eigenvalue problem for the stiffness and mass matrices of the whole structure obtained in the previous step by sparse matrix techniques.

(4) Back transformation from physical to generalized co-ordinates. This procedure can be done independently for each substructure and results consequently in low computational costs.

The method has been validated by case studies. The results show an excellent agreement with the corresponding exact theoretical solutions when available and, in other cases, with the results of alternative methods.

2. DESCRIPTION OF THE METHOD

2.1. *The Rayleigh–Ritz Method*

For a continuum system, an exact analytical solution is not always possible and an approximate solution via some kind of structural discretization is undertaken. In the

0022–460X/98/210193 + 08 $25.00/0/sv971489

Rayleigh–Ritz method, one assumes a finite series solution of a trial displacement field (shape functions) [2–5],

$$\vec{u}(x, y, z) = \sum_{j=1}^{N} \vec{\phi}_j(x, y, z) p_j. \tag{1}$$

For a continuum, the kinetic energy may be expressed by the following integral, which may be a volume, a surface or a line integral depending upon the configuration of the structure and the space co-ordinate system used:

$$T = \tfrac{1}{2} \int \rho \dot{u}^2 \, \mathrm{d}V, \tag{2}$$

where ρ is the mass density. From equation (1), the displacement $\bar{u}$ can be differentiated to obtain velocities and substituting into equation (2) yields

$$T = \tfrac{1}{2} \sum_{j=1}^{N} \sum_{k=1}^{N} \dot{p}_j \dot{p}_k \int \rho \vec{\phi}_j \vec{\phi}_k \, \mathrm{d}V. \tag{3}$$

The mass matrix is derived from the kinetic energy terms in equation (3). The generalized mass matrix component is recognized to have the form

$$m_{jk} = \int \rho \vec{\phi}_j \vec{\phi}_k \, \mathrm{d}V. \tag{4}$$

The stiffness matrix is derived from the strain energy term in equation (5). For a continuum, the strain energy may be expressed by the integral

$$U = \tfrac{1}{2} \int \{\dot{\boldsymbol{\tau}}\}^{\mathrm{T}} \{\mathbf{e}\} \, \mathrm{d}V, \tag{5}$$

where $\{\boldsymbol{\tau}\}$ is the stress vector containing the six components of stress at a point and $\{\mathbf{e}\}$ is the strain vector of six components of strain at a point. For elastic materials (obeying Hooke's law) the linear relationship between stress and strain may be written as

$$\{\boldsymbol{\tau}\} = [\mathbf{G}]\{\mathbf{e}\}, \tag{6}$$

where $[\mathbf{G}]$ is a square symmetric matrix of coefficients that are dependent upon the elastic properties of the material. Replacing equation (6) in equation (5), the strain energy has the form

$$U = \tfrac{1}{2} \int \{\mathbf{e}\}^{\mathrm{T}} [\mathbf{G}] \{\mathbf{e}\} \, \mathrm{d}V. \tag{7}$$

Considering the set of shape functions adopted, it can be seen that each of them is characterized by its own strain distribution. If the strain vector related to the ith shape function is denoted by $\{\bar{\mathbf{e}}\}_i$; the total strain is found by superposition as

$$\{\mathbf{e}\} = \sum_{j=1}^{N} \{\bar{\mathbf{e}}\}_j p_j. \tag{8}$$

When this representation is replaced in equation (7) the strain energy integral becomes

$$U = \tfrac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} p_i p_j \int \{\bar{\mathbf{e}}\}_i^{\mathrm{T}} [\mathbf{G}] \{\bar{\mathbf{e}}\}_j \, \mathrm{d}V. \tag{9}$$

It is seen that the stiffness coefficient k_{ij} may be written as

$$k_{ij} = \int \{\bar{\mathbf{e}}\}_i^{\mathrm{T}} [\mathbf{G}] \{\bar{\mathbf{e}}\}_j \, \mathrm{d}V. \tag{10}$$

2.2. *Transformation to physical co-ordinates*

At this stage the method presented differs from the classical order reduction techniques employed in Rayleigh–Ritz approaches. A transformation to physical co-ordinates is adopted for every substructure. A convenient choice of physical co-ordinates must be done. These can be either components of displacements or rotations at specified points of the substructure. Physical co-ordinates of a structural component can be classified as internal or constraint. Constraint co-ordinates are first chosen and then as many internal coordinates to complete N (the order of approximation to the continuum is adopted). The order of the system muct be large enough to include at least all constraint coordinates required by the structural component. In general an increase of N results in a better approximation to the continuum.

The transformation to physical co-ordinates is achieved by means of a (N × N) matrix [**Φ**] whose columns are obtained by evaluating each of the physical coordinates resulting from the displacement field of every shape function $\bar{\varphi}_i(x, y, z)$. The co-ordinates in every substructure transform as

$$\{\mathbf{p}\} = [\mathbf{\Phi}]^{-1} \{\mathbf{x}\}. \tag{11}$$

The stiffness and matrices transform as

$$[\mathbf{M}] = [\mathbf{\Phi}^{\mathrm{T}}]^{-1} [\mathbf{m}] [\mathbf{\Phi}]^{-1}, \qquad [\mathbf{K}] = [\mathbf{\Phi}^{\mathrm{T}}]^{-1} [\mathbf{k}] [\mathbf{\Phi}]^{-1}. \tag{12, 13}$$

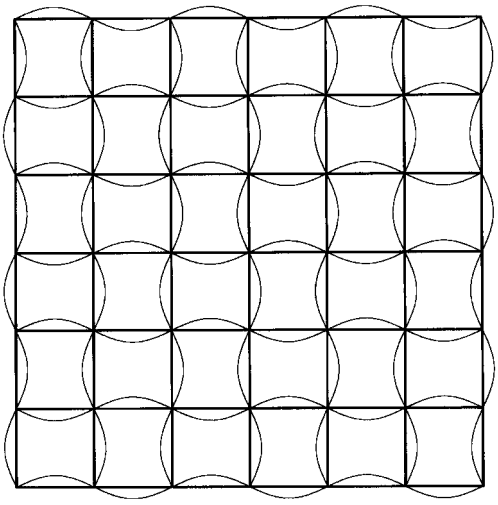

Figure 1. Mode 15 of complex beam structure.

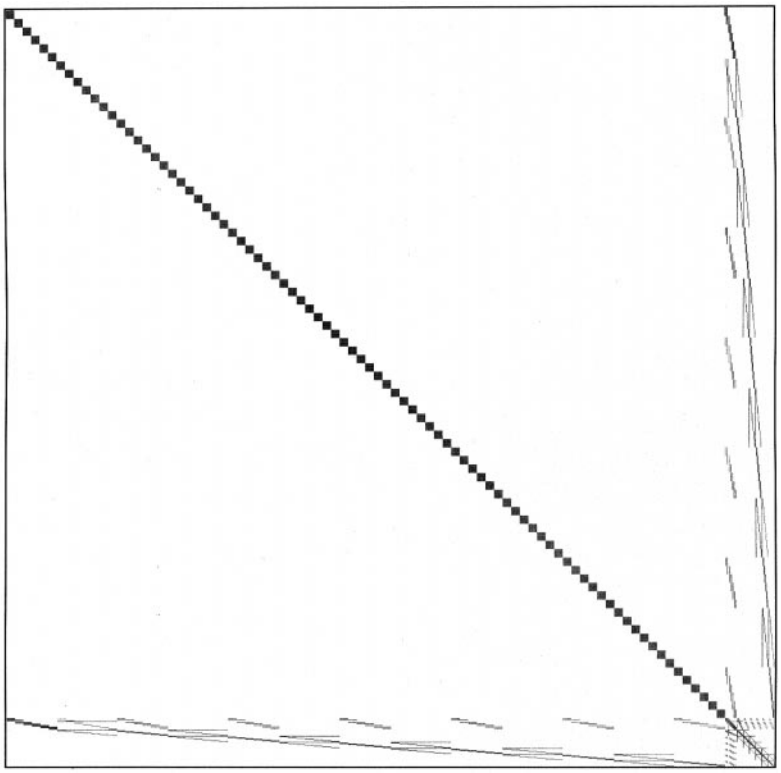

Figure 2. Portrait of resulting matrices of complex beam structure.

The dynamic resulting equation is

$$[\mathbf{M}]\{\ddot{\mathbf{x}}\} + [\mathbf{K}]\{\mathbf{x}\} = 0. \tag{14}$$

2.3. *Order reduction and assembling*

The method presented starts with an invariable set of shape functions chosen *a priori*, with all bounded conditions neglected at this stage. The stiffness and mass matrices can then be determined in a systematic manner. Enforcement of geometric boundary conditions, both internal and external and the corresponding order reduction is achieved in a simple systematic manner by an assembling procedure similar to FEM. The "exact identity method" described by Jen *et al*. [1] is achieved just by adopting the same shape function in the matching boundaries for problems of spatial dimension 2 or 3.

2.3.1. *Order reduction for a simple structure*. For a mechanical system whose behavior is described by a system of equations such as equation (14), order reduction results from external constraints when some of the physical variables of $\{\mathbf{x}\}$ are set equal to zero. This can be formalized as follows.

An array $\{\mathbf{x}^*\}$ is obtained by suppressing from $\{\mathbf{x}\}$ all those variables constrained. If $\{\mathbf{x}\}$ is of dimension N and q variables are reduced, $\{\mathbf{x}^*\}$ results and is of dimension $N - q$. This can be achieved by a Boolean matrix $[\mathbf{\Delta}]$ of dimension $(Nx(N - q))$ such that

$$\{\mathbf{x}^*\} = [\mathbf{\Delta}]^{\mathrm{T}}\{\mathbf{x}\} \qquad \text{and} \qquad \{\mathbf{x}\} = [\mathbf{\Delta}]\{\mathbf{x}^*\}. \tag{15}$$

Matrix $[\Delta]^T$ is obtained from the unitary matrix by removing those rows corresponding to constrained variables. Substituting equation (15) in equation (14) and premultiplying by $[\Delta]^T$, the resulting equation to be diagonalized is

$$[\mathbf{M}^*]\{\ddot{\mathbf{x}}^*\} + [\mathbf{K}^*]\{\mathbf{x}^*\} = 0, \tag{16}$$

where

$$[\mathbf{M}^*] = [\Delta]^T[\mathbf{M}][\Delta], \qquad [\mathbf{K}^*] = [\Delta]^T[\mathbf{K}][\Delta]. \tag{17, 18}$$

2.3.2. *Assembling complex structures*. The approximate models obtained for structural components can be extended to complex structures built from such components subject

TABLE 1

Frequency coefficients of a complex beam structure

	Characteristic angular frequency/$(EI/\rho AL^4)^{1/2}$	
Mode i	RRM in physical co-ordinates	theoretical solution
1	0·000000000	0
2	0·000000000	0
3	0·000000000	0
4	1·198960344	–
5	1·868306571	–
6	1·868306572	–
7	2·710979223	–
8	3·031195830	–
9	3·875174206	–
10	3·875174206	–
11	4·716672746	–
12	4·862330221	–
13	5·438447639	–
14	5·438447640	–
15	9·869604401	9·869604401
16	10·25281618	–
17	10·25281618	–
⋮	⋮	⋮
181	67·43119320	–
182	67·43119320	–
183	88·82643960	88·82643960
184	90·05431687	–
185	90·05431687	–
⋮	⋮	⋮
349	206·2385784	–
350	206·2385784	–
351	246·7403775	246·7401100
352	248·7844174	–
353	248·7844174	–
⋮	⋮	⋮
517	424·5432891	–
518	424·5432891	–
519	484·3049716	483·6106156
520	487·2800293	–
521	487·2800293	–
Total d.o.f.	987	∞

E = Young's modulus, I = area moment of inertia, ρ = mass density, A = cross-section area, L = length of beam (all these values refer to border beams).

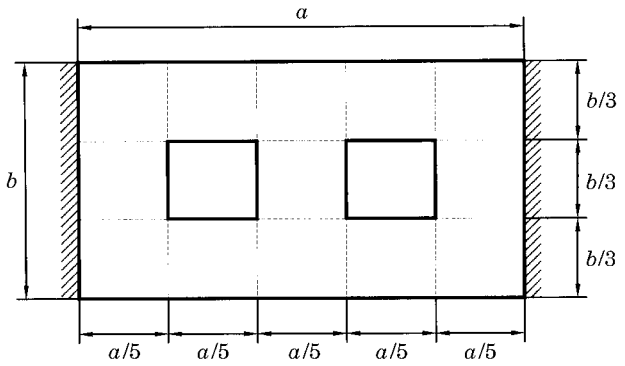

Figure 3. A C–SS–C–SS steel plate with dimensions $a = 13$ in, $b = 7$ in, thickness $h = 0{\cdot}045$ in and substructures indicated by dotted lines.

to internal constraints. The internal constraint is established in such a way that one or more of the physical variables is common to two or more of the structural components. For assembling complex structures, a Boolean matrix must be defined which establishes the correspondence between the local variables of each component and the variables chosen for the global structure [6]. This relation is expressed as

$$\{\mathbf{x}^{(i)}\} = [\mathbf{\Delta}^{(i)}]\{\mathbf{x}\}, \tag{19}$$

where $\{\mathbf{x}^{(i)}\}$ is an array with the local variables of each component. $\{\mathbf{x}\}$ is an array with the variables of the global structure and $[\mathbf{\Delta}^{(i)}]$ is the assembling matrix corresponding to the ith component. It should be noticed that these matrices are not only Boolean but sparse and need not full storage. $[\mathbf{M}^{(i)}]$ and $[\mathbf{K}^{(i)}]$ are the mass and stiffness matrices corresponding to the same component.

TABLE 2

Characteristic frequencies of lateral vibrations of a C–SS–C–SS plate with two cutouts

	Characteristic frequency (Hz)	
Mode, i	RRM in physical co-ordinates	Jen's Revised RRM
1	126·29	126·32
2	251·89	251·95
3	328·49	328·56
4	370·79	370·87
5	408·51	408·60
6	528·81	528·93
7	571·91	572·04
8	747·97	748·15
9	769·84	770·02
10	913·27	913·48
Total d.o.f.	327	327

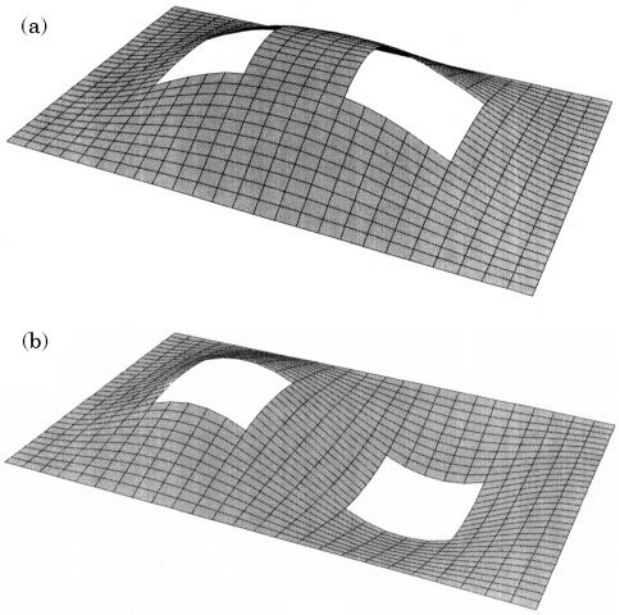

Figure 4. Mode shapes of the C–SS–C–SS steel plate (a) Mode 1; (b) Mode 2.

Replacing equation (19) in every substructure equation, premultiplying by $[\mathbf{\Delta}^{(i)}]^{\mathrm{T}}$ and adding up through all structural components, the global equation for the whole structure is obtained.

$$\sum_i [\mathbf{\Delta}^{(i)}]^{\mathrm{T}}[\mathbf{M}^{(i)}][\mathbf{\Delta}^{(i)}]\{\ddot{\mathbf{x}}\} + \sum_i [\mathbf{\Delta}^{(i)}]^{\mathrm{T}}[\mathbf{K}^{(i)}][\mathbf{\Delta}^{(i)}]\{\mathbf{x}\} = 0. \tag{20}$$

By solving the eigenvalue problem of equation (20), the natural frequencies and modes of the whole structure are obtained. The mass and stiffness matrices for the whole structure are sparse and sparse matrix techniques can be used for this purpose with the great benefit of reducing computational costs[7–9].

By means of equation (19), the global modes in each structural component are obtained, and equation (11) yields the corresponding co-ordinates in the original basis (shape functions) and makes it possible to evaluate the modes in the whole continuum of the structure. This transformation back to generalized co-ordinates is done with low computational cost since the $[\mathbf{\Phi}]$ matrix of each component is of lower order as compared to the whole structure.

It must be emphasized that this method of order reduction is more efficient than singular value decomposition from a numerical point of view due to the Boolean character and sparsity of the matrices employed.

3. CASE STUDIES

Two different systems to validate the proposed method and at the same time highlight specific features of its application were chosen: a complex beam and a plate structure.

3.1. *A complex beam structure*

An Euler–Bernoulli beam model was developed by adopting Legendre polynomials as base functions, instead of monomial base functions generally used. Legendre polynomials

yield a diagonal component mass matrix and in some applications were found to be more convenient than monomials. For a base of N shape functions, the physical variables were chosen by dividing the beam length L in (N-1) equal spaces and adopting the normal displacements of the resulting points except those closer to the ends that are replaced by the angular displacements at both ends.

A complex frame of beams of equal length L such as shown in Figure 1 was adopted as benchmark. Several natural vibration modes can be obtained from the exact solutions of isolated beams from symmetry arguments. It should be noticed that the moment of inertia and mass per unit length of the internal beams of the frame are twice as large as for beams on the border. The maximum order of the Legendre polynomials adopted as base functions for each beam is fourteen. Geometric boundary conditions were enforced for displacements and rotations at both ends of the beams. Longitudinal vibrations were not considered just for the purpose of bench-marking.

The sparse structure of the resulting matrices for the structure is shown in Figure 2. This feature of the method is of great advantage for large structures since it benefits from very efficient sparse matrix techniques. The numerical results are shown in Table 1. It can be easily verified that all the approximate eigenvalues tend toward the exact theoretical values from above. One of the resulting modes is shown in Figure 1.

3.2. *A rectangular c–ss–c–ss plate with two cutouts*

A rectangular steel plate with two cutouts hinged on two opposite sides and clamped on the other two opposite boundaries as shown in Figure 3 is now considered. This case was analyzed by Jen *et al*. [1]. The plate was synthesized by using thirteen identical square plates. Legendre polynomials of order six in two space variables were used as shape functions for each substructure. The results and performance comparisons are shown in Table 2. In both analyses, the Young's modulus, mass density and Poisson ratio are considered to be $E = 30 \times 10^6$ lb/in^2, $\rho = 0{\cdot}283$ lb/in^3 and $v = 0{\cdot}3$, respectively. The first two mode shapes are illustrated in Figure 4.

4. CONCLUSIONS

The method presented in this paper to handle the problem of free vibrations of an undamped structure has proven to be very convenient for large structures and has the potential to tackle larger systems than previous variations of the Rayleigh–Ritz method.

REFERENCES

1. C. W. Jen, D. A. Johnson and F. Dubois 1995 *Journal of Sound and Vibration* **180**, 185–203. Numerical modal analysis of structures based on a revised substructure synthesis approach.
2. W. C. Hurty 1965 *American Institute of Aeronautics and Astronautics Journal* **3**, 678–684. Dynamic analysis of structural systems using component modes.
3. L. Meirovitch and A. L. Hale 1981 *American Institute of Aeronautics and Astronautics Journal* **9**, 940–947. On the substructure synthesis method.
4. L. Meirovitch and M. K. Kwak 1990 *American Institute of Aeronautics and Astronautics Journal* **28**, 1509–1516. Convergence of the classical Rayleigh–Ritz method and the finite element method.
5. L. Meirovitch and M. K. Kwak 1990 *American Institute of Aeronautics and Astronautics Journal* **29**, 1709–1719. Rayleigh–Ritz based substructure synthesis for flexible multibody systems.
6. T. J. Chung 1978 *Finite Element Analysis in Fluid Dynamics*. New York: McGraw Hill; Chapter 1.
7. S. Pissanetsky 1984 *Sparse Matrix Technology*. Academic Press.
8. B. N. Parlett 1980 *The Symmetric Eigenvalue Problem*. New Jersey: Prentice-Hall.
9. A. George and J. W. Liu 1981 *Computer Solution of Large Sparse Positive Definite Systems*. New Jersey: Prentice-Hall.