Journal of Sound and Vibration (1998) **213**(1), 89–106

DYNAMICS OF NEARLY PERIODIC STRUCTURES

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(*Received* 2 *April* 1996, *and in final form* 22 *December* 1997)

An exact method for the analysis of nearly periodic structures is derived by using the U-transformation technique. This method can be applied to the investigation of the mode localization phenomena in nearly periodic systems other than for general structural analysis. When the number of subsystems approaches infinity, the present method is especially efficient. Worked examples are given to demonstrate the applications of the present method.

1. INTRODUCTION

Hodges [1] initiated the study of normal mode localization phenomena in periodic structures with extended disorders. Mead and Lee [2] used the characteristic receptance method to investigate the effect of disorder. Pierre and Dowell [3] presented a modified perturbation method for analysing the localization of vibrations due to structural irregularity. The wave transfer matrix method [4] was invoked to evaluate the effects of wave reflection and transmission, which result in localization. Recently the free and forced vibration of disordered layered systems was analysed by Vakakis *et al*. [5].

The U-transformation method was applied to the analysis of cyclic periodic structures [6] and some linear periodic structures [7] where the substructures have a symmetry plane. More recently the application of the U-transformation was extended to the study of the mode localization phenomena in the nearly periodic systems [8] which are made up of an infinite number of subsystems with one degree of freedom. In this paper, the U-transformation method is extended to vibration analysis of general disordered systems with multiple-coupling.

The method presented can be applied not only to the nearly cyclic periodic structures but also to the linear one although the subsystem may not be symmetrical. It is interesting to note that when the total number of substructures is approaching infinity, it becomes more convenient to solve the governing equation which does not become an ill-conditioned equation.

2. FORMULATION OF SOLUTION PROCEDURE FOR CYCLIC PERIODIC STRUCTURES

Let us consider a perfect cyclic periodic structure subjected to static loading [9]. The total potential energy of the structure considered may be expressed [10].

$$\Pi = \sum_{j=1}^{N} \pi_j, \tag{1}$$

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in which π_j denotes the potential energy of the jth substructure and N denotes the total number of substructures. In general, the potential energy may be defined as

$$\pi_j = \tfrac{1}{2}\{\bar{\boldsymbol{\delta}}\}_j^T[K]_{sub}\{\boldsymbol{\delta}\}_j - \tfrac{1}{2}(\{\bar{\boldsymbol{\delta}}\}_j^T\{\mathbf{F}\}_j + \{\boldsymbol{\delta}\}_j^T\{\bar{\mathbf{F}}\}_j), \tag{2}$$

where $[K]_{sub}$ denotes the stiffness matrix of the substructure; $\{\boldsymbol{\delta}\}_j$; $\{\mathbf{F}\}_j$ denote the displacement and loading vectors for the jth substructure, respectively, and the superior bar denotes complex conjugation. In equation (2) it is necessary to have the superior bar when deriving the variational equation with complex variables.

In general the displacement vector $\{\boldsymbol{\delta}\}_j$ is made up of the left, middle and right nodal displacement vectors, i.e.,

$$\{\boldsymbol{\delta}\}_j \equiv \begin{Bmatrix} \boldsymbol{\delta}_L \\ \boldsymbol{\delta}_M \\ \boldsymbol{\delta}_R \end{Bmatrix}_j, \tag{3}$$

in which $\{\boldsymbol{\delta}_L\}_j$, $\{\boldsymbol{\delta}_R\}_j$ denote the nodal displacement vectors on the common boundary between the $(j-1)$, $(j+1)$ and jth substructures, respectively, and $\{\boldsymbol{\delta}_M\}_j$ is the other nodal displacement vector in $\{\boldsymbol{\delta}\}_j$ except $\{\boldsymbol{\delta}_L\}_j$ and $\{\boldsymbol{\delta}_R\}_j$.

The continuity condition may be expressed as

$$\{\boldsymbol{\delta}_R\}_j = \{\boldsymbol{\delta}_L\}_{j+1}, \qquad j = 1, 2, \ldots, N, \tag{4}$$

where the subscript $N+1$ should be replaced by 1 due to cyclic periodicity.

Apply the U-transformation [9] to equations (1), (2) and (4); i.e., substituting the equation

$$\{\boldsymbol{\delta}\}_j = \frac{1}{\sqrt{N}} \sum_{m=1}^{N} \mathrm{e}^{\mathrm{i}(j-1)m\psi}\{\mathbf{q}\}_m, \qquad j = 1, 2, \ldots, N, \tag{5}$$

with $\psi = 2\pi/N$, $\mathrm{i} = \sqrt{-1}$, $\{\mathbf{q}\}_{N-m} = \{\bar{\mathbf{q}}\}_m$ and $\{\mathbf{q}\}_m = [q_L . q_M, q_R]^T$ into equations (1), (2) and (4) result in

$$\Pi = \sum_{m=1}^{N} \frac{1}{2}\{\bar{\mathbf{q}}\}_m^T[K]_{sub}\{\mathbf{q}\}_m - \sum_{m=1}^{N} \frac{1}{2}(\{\bar{\mathbf{q}}\}_m^T\{\mathbf{f}\}_m + \{\mathbf{q}\}_m^T\{\bar{\mathbf{f}}\}_m), \tag{6}$$

$$\{\mathbf{q}_R\}_m = \mathrm{e}^{\mathrm{i}m\psi}\{\mathbf{q}_L\}_m, \tag{7}$$

where

$$\{\mathbf{f}\}_m = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \mathrm{e}^{-\mathrm{i}(j-1)m\psi}\{\mathbf{F}\}_j, \qquad m = 1, 2, \ldots, N. \tag{8}$$

The continuity condition (7) may be rewritten as

$$\{\mathbf{q}\}_m = [T]_m\{\mathbf{q}_1\}_m, \tag{9}$$

with

$$\{\mathbf{q}_1\}_m = \begin{Bmatrix} \mathbf{q}_L \\ \mathbf{q}_M \end{Bmatrix}_m, \tag{10}$$

and

$$[T]_m = \begin{bmatrix} I_L & O \\ O & I_M \\ e^{im\psi} I_R & O \end{bmatrix}, \tag{11}$$

in which $[I_L], [I_M], [I_R]$ are unit matrices and their orders are in agreement with the dimensions of the vectors $\{\boldsymbol{\delta}_L\}$, $\{\boldsymbol{\delta}_M\}$ and $\{\boldsymbol{\delta}_R\}$.

Substituting equation (9) into equation (6) yields

$$\Pi = \sum_{m=1}^{N} \frac{1}{2} \{\bar{\mathbf{q}}_1\}_m^T [K]_m^* \{\mathbf{q}_1\}_m - \sum_{m=1}^{N} \frac{1}{2} (\{\bar{\mathbf{q}}_1\}_m^T \{\mathbf{f}\}_m^* + \{\mathbf{q}_1\}_m^T \{\bar{\mathbf{f}}\}_m^*), \tag{12}$$

where

$$[K]_m^* = [\bar{T}]_m^T [K]_{sub} [T]_m, \tag{13}$$

$$\{\mathbf{f}\}_m^* = [\bar{T}]_m^T \{\mathbf{f}\}_m. \tag{14}$$

In equation (12), the real and imaginary parts of $\{\mathbf{q}_1\}_m$ are independent variables. Substituting equation (12) into the variational equation

$$\delta \Pi = 0, \tag{15}$$

results in

$$[K]_m^* \{\mathbf{q}_1\}_m = \{\mathbf{f}\}_m^*. \tag{16}$$

When one considers harmonic vibration, the inertia force $\omega^2 [M]_{sub} \{\boldsymbol{\delta}\}_j$ must be added into the loading vector $\{\mathbf{F}\}_j$ and then equation (16) becomes

$$([K]_m^* - \omega^2 [M]_m^*) \{\mathbf{q}_1\}_m = \{\mathbf{f}\}_m^*, \tag{17}$$

where $\{\mathbf{q}_1\}_m$, $\{\mathbf{f}\}_m^*$ represent the amplitudes of corresponding vectors; ω denotes the circular frequency of the harmonic motion;

$$[M]_{sub}^* = [\bar{T}]_m^T [M]_{sub} [T]_m \tag{18}$$

and $[M]_{sub}$ denotes the mass matrix for a substructure. The frequency equation can be obtained by letting the characteristic determinant of $\{\mathbf{q}_1\}_m$ vanish, i.e.,

$$\det ([K]_m^* - \omega^2 [M]_m^*) = 0. \tag{19}$$

If the ω is not equal to any natural frequency, then the solution for $\{\mathbf{q}_1\}$ of equation (17) may be expressed as

$$\{\mathbf{q}_1\}_m = ([K]_m^* - \omega^2 [M]_m^*)^{-1} \{\mathbf{f}\}_m^*. \tag{20}$$

Substituting equations (9), (20), (14) and (8) into equation (5) results in

$$\{\boldsymbol{\delta}\}_j = \sum_{k=1}^{N} [\beta]_j, k \{\mathbf{F}\}_k, \tag{21}$$

where

$$[\beta]_{j,k} = \frac{1}{N} \sum_{m=1}^{N} e^{i(j-k)m\psi} [T]_m ([K]_m^* - \omega^2 [M]_m^*)^{-1} [\bar{T}]_m^T, \tag{22}$$

$[\beta]_{j,k}$ is referred to as harmonic influence coefficient matrix.

By letting N approach infinity, equation (22) becomes [6]

$$\lim_{N\to\infty} [\beta]_{j,k} = \frac{1}{2\pi}\int_0^{2\pi} e^{i(j-k)\theta}[B(\theta)]\,d\theta, \tag{23}$$

where

$$[B(\theta)] \equiv ([T]_m([K]_m^* - \omega^2[M]_m^*)^{-1}[\bar{T}]_m^T]_{m\psi=\theta}\,. \tag{24}$$

3. GOVERNING EQUATION FOR NEARLY PERIODIC STRUCTURES

Consider now the harmonic vibration of a nearly cyclic periodic structure with n disordered substructures, say $j_1, j_2, \ldots, j_n$th substructures. Their stiffness and mass matrices may be denoted as $([K]_{sub} + [\Delta K]_{j_m})$ and $([M]_{sub} + [\Delta M]_{j_m})$ $(m = 1, 2, \ldots, n)$, respectively, where $[\Delta K]_{j_m}$, $[\Delta M]_{j_m}$ denote the disordered matrices in stiffness and mass matrices for the j_mth substructure. The disordered term $(-[\Delta K]_{j_m} + \omega^2[\Delta M]_{j_m})\{\boldsymbol{\delta}\}_{j_m}$ may be treated as the loading vector for perfect cyclic periodic structure. We can invoke the principle of superposition. The expanded governing equation may be expressed as

$$\{\boldsymbol{\delta}\}_{j_i} = \sum_{m=1}^{n}[\beta]_{j_1 j_m}(\omega^2[\Delta M]_{j_m} - [\Delta K]_{j_m})\{\boldsymbol{\delta}\}_{j_m} + \{\boldsymbol{\delta}\}_{j_i}^*,$$

$$\{\boldsymbol{\delta}\}_{j_2} = \sum_{m=1}^{n}[\beta]_{j_2 j_m}(\omega^2[\Delta M]_{j_m} - [\Delta K]_{j_m})\{\boldsymbol{\delta}\}_{j_m} + \{\boldsymbol{\delta}\}_{j_2}^*,$$

$$\cdots\cdots\cdots\cdots$$

$$\{\boldsymbol{\delta}\}_{j_n} = \sum_{m=1}^{n}[\beta]_{j_n j_m}(\omega^2[\Delta M]_{j_m} - [\Delta K]_{j_m})\{\boldsymbol{\delta}\}_{j_m} + \{\boldsymbol{\delta}\}_{j_n}^*, \tag{25}$$

where matrix $[\beta]$ has been defined in equation (22) or (23); and $\{\boldsymbol{\delta}\}_{j_m}^*$ denotes the solution for the ordered structure subjected to the same loading as that acting on the nearly periodic structure, i.e.,

$$\{\boldsymbol{\delta}\}_{j_m}^* = \sum_{k=1}^{N}[\beta]_{j_m k}\{\mathbf{F}\}_k\,. \tag{26}$$

Usually many components in the vector $(\omega^2[\Delta M]_{j_m} - [\Delta K]_{j_m})\{\boldsymbol{\delta}\}_{j_m}$ will vanish. The unknowns of the governing equation should be in agreement with the independent nodal displacements on the right sides of equations (25). Therefore the governing equation is only a part of equation (25).

4. EXAMPLES

4.1. EXAMPLE 1

Consider a periodic mass spring system with two free ends as shown in Figure 1(a) where M_1 and M_2 denote the masses of two adjacent particles and k_c denotes the stiffness constant of the coupling spring. The subsystem with three degrees of freedom includes two particles and two springs as shown in Figure 2. Because there is no symmetry plane for the subsystem, we cannot create an equivalent system with cyclic periodicity by using the image

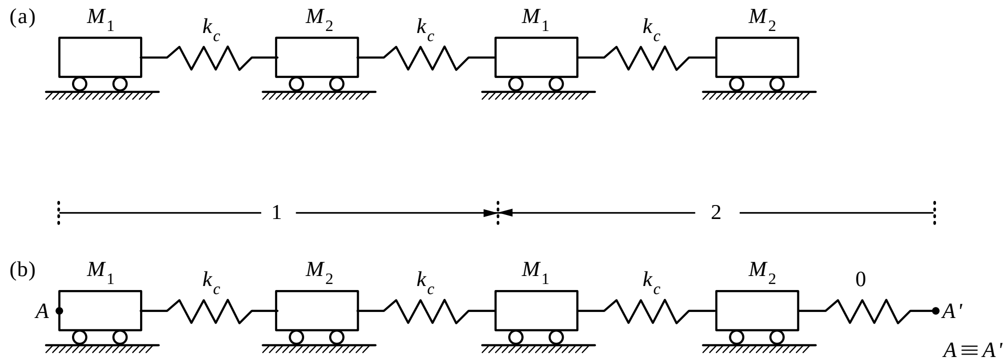

Figure 1. Periodic mass spring system with free extreme ends: (a) actural system; and (b) equivalent system with nearly cyclic periodicity.

method [7]. But the system considered can be treated as a nearly cyclic periodic one, if two extreme particles are imaginarily connected by a spring with zero stiffness as shown in Figure 1(b).

Consider now the natural vibration. It is aimed at deriving the frequency equation and demonstrating how the governing equation can be obtained.

The stiffness and mass matrices for the subsystem shown in Figure 2 may be obtained by means of the conventional expressions as

$$[K]_{sub} = \begin{bmatrix} k_c & -k_c & 0 \\ -k_c & 2k_c & -k_c \\ 0 & -k_c & k_c \end{bmatrix}, \tag{27}$$

$$[M]_{sub} = \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{28}$$

The displacement vector for the jth subsystem is

$$\{\boldsymbol{\delta}\}_j = \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_j, \qquad j = 1, 2, \ldots, N, \tag{29}$$

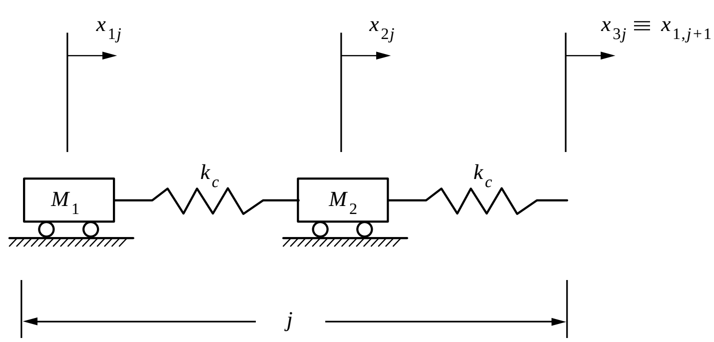

Figure 2. jth subsystem with its nodal displacements.

where X denotes the longitudinal displacement as shown in Figure 2. It indicates $\{\boldsymbol{\delta}_L\}$, $\{\boldsymbol{\delta}_M\}$, $\{\boldsymbol{\delta}_R\}$ are scalars, i.e.,

$$\{\boldsymbol{\delta}_L\}_j = X_{1j}, \qquad \{\boldsymbol{\delta}_M\}_j = X_{2j}, \qquad \{\boldsymbol{\delta}_R\}_j = X_{3j}. \tag{30}$$

The matrix $[T]_m$ shown in equation (11) takes the form of

$$[T]_m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ e^{im\psi} & 0 \end{bmatrix}. \tag{31}$$

Substituting equations (27), (28) and (31) into equations (13) and (18) results in

$$[K]^*_m = \begin{bmatrix} 2k_c & -k_c(1+e^{-im\psi}) \\ -k_c(1+e^{im\psi}) & 2k_c \end{bmatrix}, \tag{32}$$

$$[M]^*_m = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}. \tag{33}$$

Substituting equations (31)–(33) into equation (22) yields

$$[\beta]_{j,k} = \frac{1}{N}\sum_{m=1}^{N} e^{i(j-k)m\psi}\frac{1}{\Delta_m}\begin{bmatrix} 2k_c - M_2\omega^2 & k_c(1+e^{-im\psi}) & (2k_c - M_2\omega^2)\,e^{-im\psi} \\ k_c(1+e^{im\psi}) & 2k_c - M_1\omega^2 & k_c(1+e^{-im\psi}) \\ (2k_c - M_2\omega^2)\,e^{im\psi} & k_c(1+e^{im\psi}) & 2k_c - M_2\omega^2 \end{bmatrix}, \tag{34}$$

where

$$\begin{aligned}\Delta_m &\equiv \det\,[[K]^*_m - \omega^2[M]^*_m] \\ &= M_1M_2\omega^4 - 2k_c(M_1+M_2)\omega^2 + 2k_c^2(1-\cos m\psi).\end{aligned} \tag{35}$$

Equation (34) requires that ω is not any root of the frequency equation (19) for a cyclic periodic system (i.e., $[\Delta K] = [\Delta M] = [\mathbf{0}]$).

Noting that the second subsystem is the only disordered one and the disordered matrix $[\Delta K]_2$ is equal to the actual stiffness matrix of the disordered subsystem minus $[K]_{sub}$, the disordered matrices may be expressed as

$$[\Delta K]_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -k_c & k_c \\ 0 & k_c & -k_c \end{bmatrix} \tag{36}$$

and

$$[\Delta M]_2 = [0]. \tag{37}$$

For the present case the expanded governing equation (25) becomes

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_2 = -[\beta]_{2,2}[\Delta K]_2\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_2, \tag{38}$$

where $[\beta]_{2,2}$ and $[\Delta K]_2$ have been given as equations (34) and (36), respectively.

For the system under consideration, it is clear that

$$N = 2, \qquad \psi = \pi. \tag{39}$$

Substituting equations (35) and (39) into equation (34) and then multiplying by $[\Delta \mathbf{K}]_2$ shown in equation (36) yields

$$[\beta]_{2,2}[\Delta K]_2 = \begin{bmatrix} 0 & f(\Omega) & -f(\Omega) \\ 0 & g(\Omega) & -g(\Omega) \\ 0 & -h(\Omega) & h(\Omega) \end{bmatrix}, \tag{40}$$

where

$$f(\Omega) = \frac{1}{4(\Omega - 1)} - \frac{\eta}{4[\eta\Omega - (1+\eta)]}, \qquad g(\Omega) = \frac{1}{4(\eta\Omega - 1)} + \frac{1}{4[\eta\Omega - (1+\eta)]},$$

$$h(\Omega) = \frac{1}{4(\Omega - 1)} + \frac{\eta}{4[\eta\Omega - (1+\eta)]}, \tag{41}$$

with

$$\Omega = \frac{M_1\omega^2}{2k_c}, \qquad \eta = \frac{M_2}{M_1}. \tag{42}$$

Inserting equation (40) into equation (38), the governing equation can be obtained as

$$\begin{bmatrix} 1 + g(\Omega) & -g(\Omega) \\ -h(\Omega) & 1 + h(\Omega) \end{bmatrix} \begin{Bmatrix} X_2 \\ X_3 \end{Bmatrix}_2 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{43}$$

Let the coefficient determinant of the simultaneous equation (43) be equal to zero and then expanding this determinant yields the frequency equation

$$4\eta^2\Omega^3 - 6\eta(1+\eta)\Omega^2 + 2(1 + 3\eta + \eta^2)\Omega - (1+\eta) = 0. \tag{44}$$

By using Newton's second law, it can be proved that equation (44) is the exact frequency equation of the actual system except for $\Omega = 0$. For the rigid body mode the disordered term $[\Delta K]_2\{\boldsymbol{\delta}\}_2$ vanishes and therefore the roots of equation (44) do not include zero.

4.2. EXAMPLE 2

Consider a nearly periodic system with infinite number of subsystems where only one subsystem, say the j_1th, is the disordered one as shown in Figure 3. This example is aimed at analysing the mode localization phenomena. The localized mode of a system with infinite number of subsystems is hardly affected by the conditions at infinity and so such

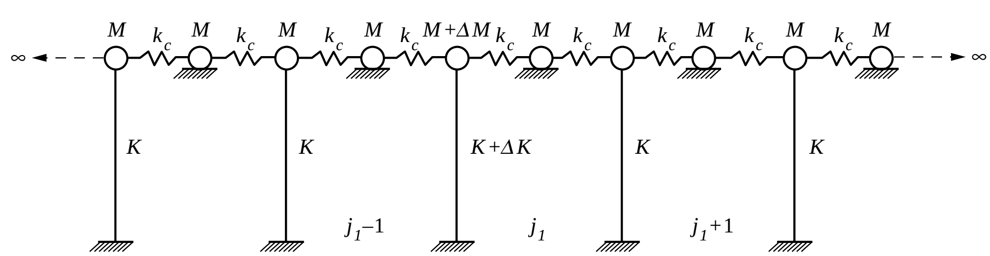

Figure 3. Nearly periodic system with infinite number of subsystems.

a system may be regarded as a cyclic periodic one. Then the harmonic influence coefficient matrix $[\beta]_{j,k}$ may be expressed as equation (23) instead of equation (22).

First, let us consider the harmonic vibration of the perfect cyclic periodic system shown in Figure 3 in which $\Delta K = \Delta M = 0$. The subsystem is shown in Figure 4. The subsystem stiffness and mass matrices are

$$[K]_{sub} = \begin{bmatrix} K + k_c & -k_c & 0 \\ -k_c & 2k_c & -k_c \\ 0 & -k_c & k_c \end{bmatrix}, \tag{45}$$

$$[M]_{sub} = \begin{bmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{46}$$

The vector $\{\boldsymbol{\delta}\}_i$ and matrix $[T]_m$ are identical to those shown in equations (29) and (31), respectively. Inserting equations (31), (45) and (46) into equations (13) and (18) results in

$$[K]^*_m = \begin{bmatrix} K + 2k_c & -k_c(1 + \mathrm{e}^{-\mathrm{i}m\psi}) \\ -k_c(1 + \mathrm{e}^{\mathrm{i}m\psi}) & 2k_c \end{bmatrix}, \tag{47}$$

$$[M]^*_m = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}. \tag{48}$$

The characteristic determinant is

$$\begin{aligned} \Delta_m &= \det\,([K]^*_m - \omega^2[M]^*_m) \\ &= M^2\omega^4 - (K + 4k_c)M\omega^2 + 2Kk_c + 2k_c^2(1 - \cos m\psi). \end{aligned} \tag{49}$$

The frequency equation (19) may be expressed as

$$M^2\omega^4 - (K + 4k_c)M\omega^2 + 2Kk_c + 2k_c^2(1 - \cos m\psi) = 0. \tag{50}$$

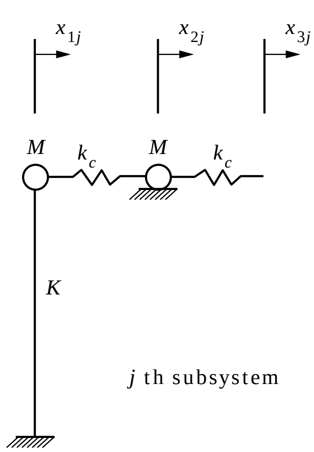

Figure 4. Ordered subsystem with its nodal displacements.

The above equation has the roots

$$\omega_{1m}^2 = \frac{K + 4k_c}{2M} - \frac{1}{2M}\sqrt{K^2 + 8k_c^2(1 + \cos m\psi)},$$

$$\omega_{2m}^2 = \frac{K + 4k_c}{2M} + \frac{1}{2M}\sqrt{K^2 + 8k_c^2(1 + \cos m\psi)} \quad m = 1, 2, \ldots, N. \tag{51}$$

The natural frequencies ω_{1m} and ω_{2m} are located at the first and second frequency bands, respectively. Let $[\omega_{1L}, \omega_{1U}]$ and $[\omega_{2L}, \omega_{2U}]$ denote the first and second pass bands. They can be obtained from equation (51) as

$$\omega_{1L}^2 = \frac{K + 4k_c}{2M} - \frac{1}{2M}\sqrt{K^2 + 16k_c^2}, \qquad \omega_{1U}^2 = \frac{2k_c}{M},$$

$$\omega_{2L}^2 = \frac{K + 2k_c}{M}, \qquad \omega_{2U}^2 = \frac{K + 4k_c}{2M} + \frac{1}{2M}\sqrt{K^2 + 16k_c^2}. \tag{52}$$

Inserting equations (47), (48) and (31) into equation (22) results in

$$[\beta]_{j,k} = \frac{1}{N}\sum_{m=1}^{N} \mathrm{e}^{\mathrm{i}(j-k)m\psi}[B(m\psi)], \tag{53}$$

where

$$[B(m\psi)] = \frac{1}{\Delta_m}\begin{bmatrix} 2k_c - M\omega^2 & k_c(1 + \mathrm{e}^{-\mathrm{i}m\psi}) & (2k_c - M\omega^2)\,\mathrm{e}^{-\mathrm{i}m\psi} \\ k_c(1 + \mathrm{e}^{\mathrm{i}m\psi}) & K + 2k_c - M\omega^2 & k_c(1 + \mathrm{e}^{-\mathrm{i}m\psi}) \\ (2k_c - M\omega^2)\,\mathrm{e}^{\mathrm{i}m\psi} & k_c(1 + \mathrm{e}^{\mathrm{i}m\psi}) & 2k_c - M\omega^2 \end{bmatrix}, \tag{54}$$

and Δ_m has been defined as in equation (49). Equation (54) requires $\Delta_m \neq 0$, i.e., ω lies in the stop band.

Noting that the mth and $(N - m)$th terms of the series shown in equations (53) and (54) are a pair of conjugate matrices, we can replace the general term by its real part, i.e.,

$$[\beta]_{j,k} = \frac{1}{N}\sum_{m=1}^{N} \mathrm{Re}\,\{\mathrm{e}^{\mathrm{i}(j-k)m\psi}[B(m\psi)]\}. \tag{55}$$

By letting N approach infinity, equation (55) becomes

$$\lim_{N\to\infty} [\beta]_{j,k} = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\,\{\mathrm{e}^{\mathrm{i}(j-k)\theta}[B(\theta)]\}\,\mathrm{d}\theta. \tag{56}$$

Consider the case of $j = k$, i.e.,

$$\lim_{N\to\infty} [\beta]_{i,j} \equiv \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{bmatrix} = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\,\{[B(\theta)]\}\,\mathrm{d}\theta. \tag{57}$$

Substituting equations (54) and (49) into the right side of equation (57) results in

$$\beta_{11} = \beta_{33} = \frac{1}{\pi}\int_0^\pi \frac{2k_c - M\omega^2}{b(\omega) - 2k_c^2\cos\theta}\,\mathrm{d}\theta = \frac{2k_c - M\omega^2}{\mathrm{sgn}\,[b(\omega)]\sqrt{b^2(\omega) - 4k_c^4}},$$

$$\beta_{22} = \frac{1}{\pi}\int_0^\pi \frac{K + 2k_c - M\omega^2}{b(\omega) - 2k_c^2\cos\theta}\,\mathrm{d}\theta = \frac{K + 2k_c - M\omega^2}{\mathrm{sgn}\,[b(\omega)]\sqrt{b^2(\omega) - 4k_c^4}},$$

$$\beta_{12} = \beta_{21} = \beta_{23} = \beta_{32} = \frac{1}{\pi}\int_0^\pi \frac{k_c(1 + \cos\theta)}{b(\omega) - 2k_c^2\cos\theta}\,\mathrm{d}\theta$$

$$= \frac{1}{2k_c}\left[-1 + \frac{b(\omega) + 2k_c^2}{\mathrm{sgn}\,[b(\omega)]\sqrt{b^2(\omega) - 4k_c^4}}\right],$$

$$\beta_{13} = \beta_{31} = \frac{1}{\pi}\int_0^\pi \frac{(2k_c - M\omega^2)\cos\theta}{b(\omega) - 2k_c^2\cos\theta}\,\mathrm{d}\theta = \frac{2k_c - M\omega^2}{2k_c^2}\left[-1 + \frac{b(\omega)}{\mathrm{sgn}\,[b(\omega)]\sqrt{b^2(\omega) - 4k_c^4}}\right],$$

$$b^2(\omega) - 4k_c^4 > 0, \tag{58}$$

in which

$$b(\omega) = M^2\omega^4 - (K + 4k_c)M\omega^2 + 2Kk_c + 2k_c^2, \tag{59}$$

and the symbol "sgn" denotes the sign function.

Noting

$$b^2(\omega) - 4k_c^2 = M^4(\omega^2 - \omega_{1L}^2)(\omega^2 - \omega_{1U}^2)(\omega^2 - \omega_{2L}^2)(\omega^2 - \omega_{2U}^2), \tag{60}$$

the condition $b^2(\omega) - 4k_c^4 > 0$ is equivalent to that ω lies in the stop band.

Second, the nearly periodic system shown in Figure 3 is considered. If its natural frequency ω lies in stop band of a perfect periodic system, i.e., $\omega < \omega_{1L}$, $\omega_{1U} < \omega < \omega_{2L}$ or $\omega > \omega_{2U}$, the corresponding mode is a localized one.

The disordered stiffness and mass matrices for the j_1th subsystem shown in Figure 3 may be expressed as

$$[\Delta K]_{j_1} = \begin{bmatrix} \Delta K & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad [\Delta M]_{j_1} = \begin{bmatrix} \Delta M & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{61}$$

where the scalars ΔK and ΔM on the right sides of equation (61) denote the disorder in stiffness and mass for the j_1th subsystem.

Substituting equations (57), (58) and (61) into equation (25) with $\{\boldsymbol{\delta}\}_{j_1}^* = \{\mathbf{0}\}$ results in

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_{j_1} = \begin{bmatrix} (\omega^2\Delta M - \Delta K)\beta_{11} & 0 & 0 \\ (\omega^2\Delta M - \Delta K)\beta_{21} & 0 & 0 \\ (\omega^2\Delta M - \Delta K)\beta_{31} & 0 & 0 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_{j_1}. \tag{62}$$

The governing equation is the first one of equation (62), i.e.,

$$X_{1j_i} = (\omega^2\Delta M - \Delta K)\beta_{11}X_{1j_1}. \tag{63}$$

The frequency equation may be expressed as

$$(\omega^2 \Delta M - \Delta K)\beta_{11} = 1. \tag{64}$$

Substituting the first of equation (58) into equation (64) yields

$$(\omega^2 \Delta M - \Delta K)(2k_c - M\omega^2) = \mathrm{sgn}\,[b(\omega)]\sqrt{b^2(\omega) - 4k_c^4}, \tag{65}$$

in which $b(\omega)$ has been defined as equation (59). It may be verified directly that if $\omega < \omega_{1L}$ or $\omega > \omega_{2U}$ then $b(\omega) > 0$; if $\omega_{1U} < \omega < \omega_{2L}$ then $b(\omega) < 0$. Equation (65) may be rewritten as

$$(\omega^2 \Delta M - \Delta K)(2k_c - M\omega^2) = \sqrt{b^2(\omega) - 4k_c^4}, \qquad \omega < \omega_{1L} \quad \text{and } \omega > \omega_{2U} \tag{66a}$$

$$(\omega^2 \Delta M - \Delta K)(2k_c - M\omega^2) = -\sqrt{b^2(\omega) - 4k_c^4}, \qquad \omega_{1U} < \omega < \omega_{2L}. \tag{66b}$$

Let us discuss the relation between the disorder parameters and the roots of the frequency equations (66a, b).

Consider the case of $\Delta M = 0$. The square of the equation (66a) or (66b) may be expressed as

$$(\Delta K)^2 M^2(\omega^2 - \omega_{1U}^2)^2 = M^4(\omega^2 - \omega_{1L}^2)(\omega^2 - \omega_{1U}^2)(\omega^2 - \omega_{2L}^2)(\omega^2 - \omega_{2U}^2). \tag{67}$$

Since we have no interest for the root $\omega = \omega_{1U}$, both sides of the above equation can be divided by $M^2(\omega^2 - \omega_{1U}^2)$, giving

$$(\Delta K)^2(\omega^2 - \omega_{IU}^2) = M^2(\omega^2 - \omega_{1L}^2)(\omega^2 - \omega_{2L}^2)(\omega^2 - \omega_{2U}^2). \tag{68}$$

Let

$$D(\omega^2) \equiv (\Delta K)^2(\omega^2 - \omega_{1U}^2), \qquad E(\omega^2) \equiv M^2(\omega^2 - \omega_{1L}^2)(\omega^2 - \omega_{2L}^2)(\omega^2 - \omega_{2U}^2). \tag{69}$$

The functions D and E of ω^2 may be qualitatively expressed by the curves shown in Figure 5. The longitudinal co-ordinates of the intersections represent the roots for ω^2 of equation (68).

It can be observed from Figure 5, that there are three roots altogether, and that they lie in three stop bands respectively. Since the domain of both equations (66a) and (66b) does not include all stop bands, it is not certain that the root of equation (68) is also the root of equation (66a) or (66b).

(1) When $\Delta K > 0$ and $\Delta M = 0$, only one root with $\omega > \omega_{2U}$ satisfies the frequency equation (66a). It is concluded that there is one localized mode with ω greater than ω_{2U}.

(2) When $\Delta K < 0$ and $\Delta M = 0$, two roots with $\omega < \omega_{1L}$ and $\omega_{1U} < \omega < \omega_{2L}$ satisfy equations (66a) and (66b), respectively, i.e., there are two localized modes with $\omega < \omega_{1L}$ and $\omega_{1U} < \omega < \omega_{2L}$.

Consider now the case of $\Delta K = 0$. Similarly, the following conclusion can be obtained.

(3) When $\Delta M > 0$ and $\Delta K = 0$, there are two localized modes with $\omega < \omega_{1L}$ and $\omega_{1U} < \omega < \omega_{2L}$. (4) When $\Delta M < 0$ and $\Delta K = 0$, there is one localized mode with $\omega > \omega_{2U}$.

The attenuation rate ξ for amplitude of localized mode may be defined as

$$\xi = \frac{X_{1,j_1+1}}{X_{1,j_1}} = \frac{X_{3,j_1}}{X_{1,j_1}}. \tag{70}$$

Substituting equation (62) into the above equation yields

$$\xi = \frac{\beta_{31}}{\beta_{11}}. \tag{71}$$

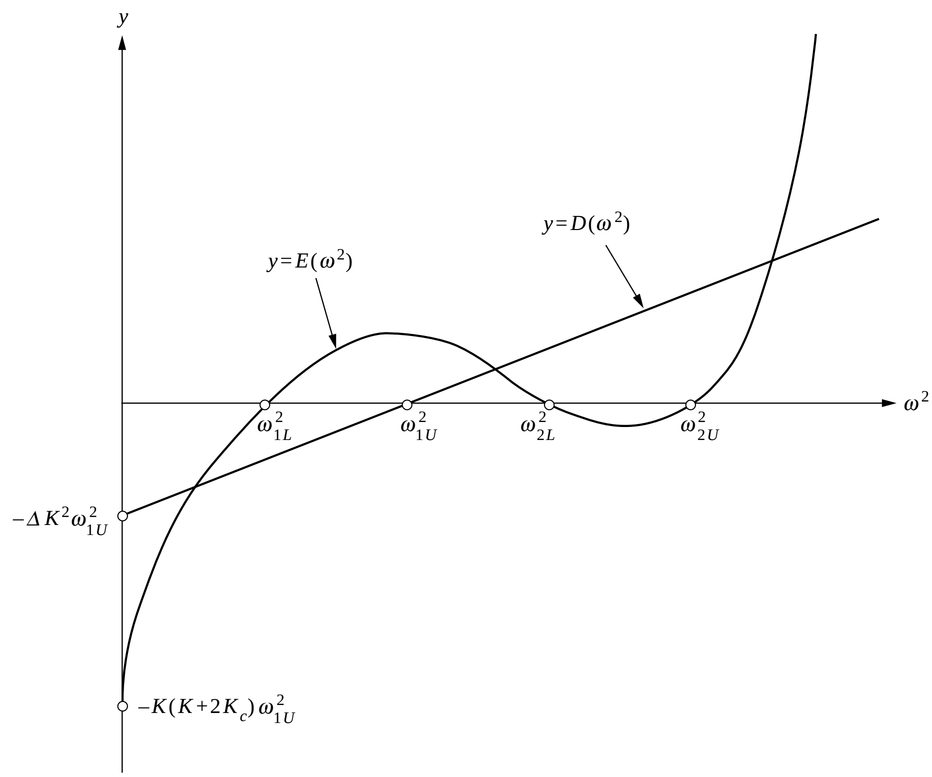

Figure 5. Curves of $D(\omega^2)$ and $E(\omega^2)$.

Inserting β_{11}, β_{31} shown in equation (58) into equation (71) results in

$$\xi = \frac{1}{2k_c^2}\left[b(\omega) - \operatorname{sgn}[b(\omega)]\sqrt{b^2(\omega) - 4k_c^4}\ \right], \tag{72}$$

in which ω is the natural frequency of the localized mode. The function $b(\omega)$ in equation (72) may be eliminated by using equation (65), i.e.,
when $\Delta K \neq 0$, $\Delta M = 0$,

$$\xi = \eta - \operatorname{sgn}(\eta)\sqrt{1+\eta^2},$$

$$\eta = \frac{\Delta K}{k_c}\left(1 - \frac{M\omega^2}{2k_c}\right), \tag{73}$$

when $\Delta M \neq 0$, $\Delta K = 0$,

$$\xi = \eta - \operatorname{sgn}(\eta)\sqrt{1+\eta^2},$$

$$\eta = -\frac{\omega^2 \Delta M}{k_c}\left(1 - \frac{M\omega^2}{2k_c}\right). \tag{74}$$

It is obvious that the magnitude of the attenuation rate is always less than one except when the disorder vanishes.

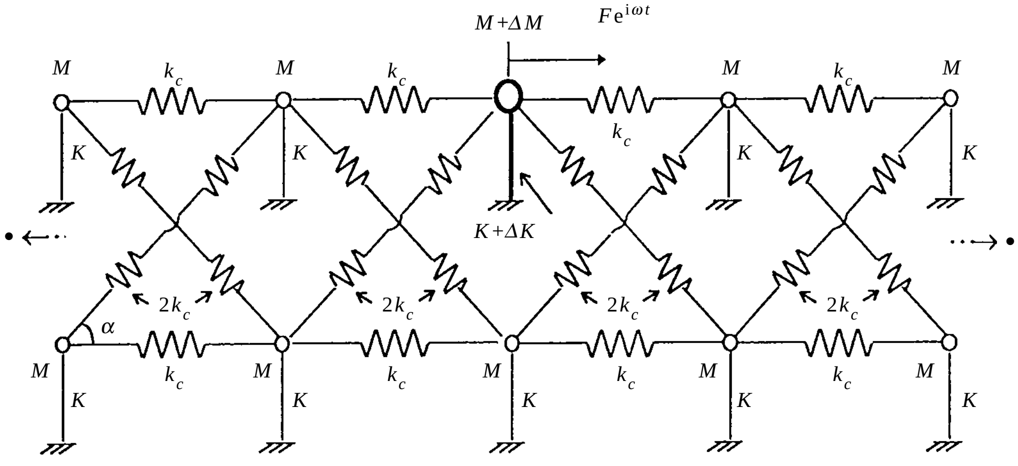

Figure 6. Two-coupling periodic system.

4.3. EXAMPLE 3

Consider now a two-degree-coupling periodic system with an infinite number of subsystems as shown in Figure 6 where the j_1th subsystem is the disordered one and a harmonic excitation $Fe^{i\omega t}$ acts on the first node of the disordered subsystem and $\alpha = \pi/4$. In this example, the solution for forced vibration will be worked out.

The system shown in Figure 6 also can be regarded as a cyclic periodic one, therefore the steady state solution can be found by using the formulas shown in sections 2 and 3.

A general subsystem, say, the jth one, is shown in Figure 7. Its displacement vector is defined as

$$\{\boldsymbol{\delta}\}_j = [x_1, x_2, x_3, x_4]_j^T, \tag{75}$$

where the components denote the nodal displacements as shown in Figure 7.

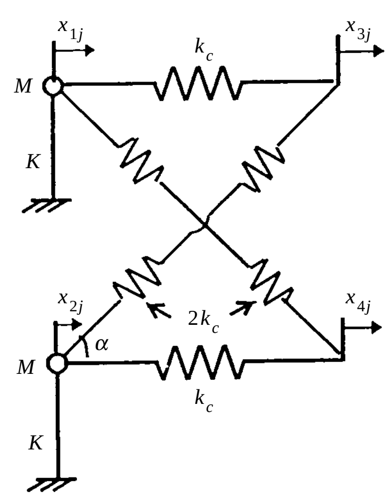

Figure 7. Subsystem and nodal displacements.

The stiffness and mass matrices for the subsystem are

$$[K]_{sub} = \begin{bmatrix} K+2k_c & 0 & -k_c & -k_c \\ 0 & K+2k_c & -k_c & -k_c \\ -k_c & -k_c & 2k_c & 0 \\ -k_c & -k_c & 0 & 2k_c \end{bmatrix}, \tag{76}$$

$$[M]_{sub} = \begin{bmatrix} M & 0 & 0 & 0 \\ 0 & M & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{77}$$

For this case, matrix $[T]_m$ shown in equation (11) becomes

$$[T]_m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ e^{im\psi} & 0 \\ 0 & e^{im\psi} \end{bmatrix}. \tag{78}$$

Substituting equations (76)–(78) into equations (13) and (18) yields

$$[K]^*_m = \begin{bmatrix} K+4k_c-2k_c\cos m\psi & -2k_c\cos m\psi \\ -2k_c\cos m\psi & K+4k_c-2k_c\cos m\psi \end{bmatrix}$$

$$[M]^*_m = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}, \qquad m = 1, 2, \ldots, N, \tag{79}$$

where $\psi = 2\pi/N$, and N denotes the total number for subsystems.

Substituting equation (79) into equation (19), the frequency equation for ordered periodic system can be obtained as

$$(K+4k_c-M\omega^2)(K+4k_c-M\omega^2-4k_c\cos m\psi) = 0, \qquad m = 1, 2, \ldots, N. \tag{80}$$

The root for ω^2 of the above equation can be expressed as

$$\omega_m^2 = \frac{1}{M}(K+4k_c-4k_c\cos m\psi), \qquad m = 1, 2, \ldots, N. \tag{81}$$

The lower and upper limits (ω_L and ω_U) of the pass band are

$$\omega_L^2 = \frac{K}{M}, \qquad \omega_U^2 = \frac{K+8k_c}{M}. \tag{82}$$

Inserting equations (78) and (79) into equation (22), results in

$$[\beta]_{j,k} = \frac{1}{N}\sum_{m=1}^{N}\begin{bmatrix} [A(m\psi)]\cos(j-k)m\psi & [A(m\psi)]\cos(j-k-1)m\psi \\ [A(m\psi)]\cos(j-k+1)m\psi & [A(m\psi)]\cos(j-k)m\psi \end{bmatrix}, \tag{83}$$

$$[A(m\psi)] \equiv [[K]_m^* - \omega^2[M]_m^*]^{-1}$$

$$= \frac{1}{\Delta(m\psi)}\begin{bmatrix} K + 4k_c - M\omega^2 - 2k_c \cos m\psi & 2k_c \cos m\psi \\ 2k_c \cos m\psi & K + 4k_c - M\omega^2 - 2k_c \cos m\psi \end{bmatrix}, \quad (84)$$

and

$$\Delta(m\psi) = (K + 4k_c - M\omega^2)^2 - 4k_c(K + 4k_c - M\omega^2)\cos m\psi. \quad (85)$$

By letting N approach infinity, equation (83) becomes

$$\lim_{N\to\infty} [\beta]_{j,k} = \frac{1}{2\pi}\int_0^{2\pi} \begin{bmatrix} [A(\theta)]\cos(j-k)\theta & [A(\theta)]\cos(j-k-1)\theta \\ [A(\theta)]\cos(j-k+1)\theta & [A(\theta)]\cos(j-k)\theta \end{bmatrix} \mathrm{d}\theta, \quad (86)$$

where

$$[A(\theta)] = \frac{1}{\Delta(\theta)}\begin{bmatrix} K + 4k_c - M\omega^2 - 2k_c \cos\theta & 2k_c \cos\theta \\ 2k_c \cos\theta & K + 4k_c - M\omega^2 - 2k_c \cos\theta \end{bmatrix}, \quad (87)$$

$$\Delta(\theta) = (K + 4k_c - M\omega^2)^2 - 4k_c(K + 4k_c - M\omega^2)\cos\theta. \quad (88)$$

Consider now the case of $j = k$ in (86), i.e.,

$$\lim_{N\to\infty} [\beta]_{j,j} \equiv \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} \\ \beta_{21} & \beta_{22} & \beta_{23} & \beta_{24} \\ \beta_{31} & \beta_{32} & \beta_{33} & \beta_{34} \\ \beta_{41} & \beta_{42} & \beta_{43} & \beta_{44} \end{bmatrix} = \frac{1}{2\pi}\int_0^{2\pi} \begin{bmatrix} [A(\theta)] & [A(\theta)]\cos\theta \\ [A(\theta)]\cos\theta & [A(\theta)] \end{bmatrix} \mathrm{d}\theta,$$

$$j = 1, 2, \ldots, N. \quad (89)$$

If the excitation frequency ω lies in the stop band, i.e., $\omega < \omega_L$ or $\omega > \omega_U$, then $\Delta(\theta) \neq 0$. The definite integral shown in equations (89), (87) and (88) is in existence. Substituting equations (87) and (88) into equation (89), results in

$$\beta_{11} = \beta_{22} = \beta_{33} = \beta_{44} = \frac{1}{b(\omega)} \cdot \frac{1}{2\pi}\int_0^{2\pi} \frac{b(\omega) - 2k_c\cos\theta}{b(\omega) - 4k_c\cos\theta}\,\mathrm{d}\theta,$$

$$\beta_{12} = \beta_{21} = \beta_{34} = \beta_{43} = \frac{1}{b(\omega)} \cdot \frac{1}{2\pi}\int_0^{2\pi} \frac{2k_c\cos\theta}{b(\omega) - 4k_c\cos\theta}\,\mathrm{d}\theta,$$

$$\beta_{13} = \beta_{31} = \beta_{24} = \beta_{42} = \frac{1}{b(\omega)} \cdot \frac{1}{2\pi}\int_0^{2\pi} \frac{(b(\omega) - 2k_c\cos\theta)\cos\theta}{b(\omega) - 4k_c\cos\theta}\,\mathrm{d}\theta,$$

$$\beta_{14} = \beta_{41} = \beta_{23} = \beta_{32} = \frac{1}{b(\omega)} \cdot \frac{1}{2\pi}\int_0^{2\pi} \frac{2k_c\cos^2\theta}{b(\omega) - 4k_c\cos\theta}\,\mathrm{d}\theta, \quad (90)$$

where

$$b(\omega) = K + 4k_c - M\omega^2. \quad (91)$$

The above integral can be expressed as the elementary functions, such as

$$\beta_{11} = \frac{1}{2}\left\{\frac{1}{b(\omega)} + \frac{1}{\operatorname{sgn}[b(\omega)]\sqrt{b^2(\omega) - (4k_c)^2}}\right\}. \tag{92}$$

For the system shown in Figure 6, the j_1th subsystem is the disordered one. The disordered matrices are

$$[\Delta K]_{j_1} = \begin{bmatrix} \Delta K & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad [\Delta M]_{j_1} = \begin{bmatrix} \Delta M & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{93}$$

and the loading vectors are

$$\{\mathbf{F}\}_{j_1} = \begin{Bmatrix} F \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \qquad \{\mathbf{F}\}_j = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \qquad j \neq j_1, \tag{94}$$

where F denotes the amplitude of the harmonic excitation.

For the present case, equations (25) and (26) become

$$\{\boldsymbol{\delta}\}_{j_1} = [\beta]_{j_1 j_1}(\omega^2[\Delta M]_{j_1} - [\Delta K]_{j_1})\{\boldsymbol{\delta}\}_{j_1} + \{\boldsymbol{\delta}\}^*_{j_1}, \tag{95}$$

$$\{\boldsymbol{\delta}\}^*_{j_1} = [\beta]_{j_1 j_1}\{\mathbf{F}\}_{j_1}. \tag{96}$$

Substituting equations (93) and (94) into equations (95) and (96), respectively, yields

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}_{j_1} = (\Delta M\omega^2 - \Delta K)x_{1j_1}\begin{Bmatrix} \beta_{11} \\ \beta_{21} \\ \beta_{31} \\ \beta_{41} \end{Bmatrix} + \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}^*_{j_1}. \tag{97}$$

and

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}^*_{j_1} = F\begin{Bmatrix} \beta_{11} \\ \beta_{21} \\ \beta_{31} \\ \beta_{41} \end{Bmatrix}, \tag{98}$$

where x_{1j_1} denotes the first component of the vector $\{\boldsymbol{\delta}\}_{j_1}$. The first equations of equations (97) and (98) are

$$x_{1j_1} = (\Delta M\omega^2 - \Delta K)\beta_{11}x_{1j_1} + x^*_{1j_1}, \tag{99}$$

$$x^*_{1j_1} = \beta_{11}F. \tag{100}$$

The solution for x_{1j_1} of equation (99) can be found as

$$x_{1j_1} = \frac{\beta_{11} F}{1 + (\Delta K - \Delta M\omega^2)\beta_{11}}, \tag{101}$$

where β_{11} has been defined as shown in equation (92), and x_{1j_1} is the displacement amplitude of the loaded node.

Substituting equations (98) and (101) into equation (97) gives

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}_{j_1} = \left[\frac{F}{1 + (\Delta K - \Delta M\omega^2)\beta_{11}}\right] \begin{Bmatrix} \beta_{11} \\ \beta_{21} \\ \beta_{31} \\ \beta_{41} \end{Bmatrix}. \tag{102}$$

In order to assess the accuracy of the above result, let us consider the special case of $k_c = 0$. The periodic system becomes many independent systems where each system is a single-degree-of-freedom one.

Substituting $k_c = 0$ and equation (91) into equation (90) yields

$$\beta_{11} = \frac{1}{K - M\omega^2}, \qquad \beta_{21} = \beta_{31} = \beta_{41} = 0, \tag{103}$$

and then equation (102) becomes

$$x_{1j_1} = \frac{F}{K + \Delta K - (M + \Delta M)\omega^2}, \qquad x_{2j_1} = x_{3j_1} = x_{4j_1} = 0. \tag{104}$$

This result is in agreement with that of the single-degree-of-freedom system with stiffness $K + \Delta K$ and mass $M + \Delta M$.

5. CONCLUSION

The method presented in this paper is applicable to the vibration analysis of nearly periodic structures which include both cyclic and linear periodic structures. The number of unknowns in the governing equation is in agreement with the total number of independent variables in all disordered terms. The present method is very useful for analysing the mode localization phenomena in nearly periodic structures with a large number of substructures and very small number of disorders. It is important that when the number of substructures approaches infinity, the governing equation does not become ill-conditioned.

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