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STOCHASTIC PARAMETER ESTIMATION OF NON-LINEAR SYSTEMS USING ONLY HIGHER ORDER SPECTRA OF THE MEASURED RESPONSE

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Methods for using fourth order spectral quantities to estimate the unknown parameters in non-linear, randomly excited dynamic systems are developed. Attention is focused on the case where only the response is measurable and the excitation is unmeasurable and known only in terms of a stochastic process model. The approach is illustrated through application to a non-linear oscillator with both non-linear damping and stiffness and with excitation modelled as a stationary Gaussian white noise process. The methods have applications in studies of the response of structures to random environmental loads, such as wind and ocean wave forces.

1. INTRODUCTION

Many mechanical and structural systems respond dynamically to random environmental loads, such as wind or wave forces. Examples include ships rolling in an irregular seaway, flexible buildings vibrating due to turbulent wind loading and jack-up offshore structures moving as a result of combined wave and wind loading. For the purposes of assessing the reliability of such structures it is important to predict their dynamic response, at the design stage. However, such predictions are invariably difficult. Whilst mass and stiffness parameters in the governing equations of motion can usually be computed with some accuracy, damping parameters are normally not quantifiable by theoretical means. For example, in the case of a ship rolling in waves the appropriate parametric form of damping is well established [1–5]. However, the damping arises from a very complex fluid–structure interaction between the waves and the ship motion, involving three-dimensional vortex shedding. Thus, a theoretical determination of the damping parameters by the use of computational fluid dynamics techniques is impractical.

As an alternative approach the damping parameters can be estimated by experimental means using model tests (e.g., see reference [4]). In such experimental work standard system identification techniques [3, 4, 6–8] can be applied, provided that simultaneous measurements of the system's excitation and response are available, to yield the required parameter estimates. A wide variety of appropriate system identification procedures are available for this purpose: these may be separated into time domain and frequency domain techniques [6]. In the time domain the identification methods are usually based upon least squares, maximum likelihood and related techniques. In the frequency domain one is usually concerned with spectral estimates and fitting models to estimated frequency response characteristics.

A difficulty in dealing with environmental loading, however, even at model scale, is that the actual forces experienced by the structure are unknown. Thus, referring to the ship

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rolling example again, if experiments are performed in a wave tank, using irregular waves to simulate real sea waves, then the wave motion can be measured but the actual wave moment experienced by the ship model, when it is moving, is unmeasurable. In this situation most of the techniques available in the system identification literature are inapplicable. Methods are required in which the estimation is based on response measurements only, together with a stochastic model of the excitation. Thus, one is faced with a continuous time stochastic estimation problem.

This stochastic estimation problem also arises when investigating the behaviour of structures once they have been built. Such investigations are often very desirable as a form of condition monitoring. Here again the forces on the structure are unmeasurable and one must work with the measured response alone, combined with some stochastic model of the excitation. For example, in the case of an offshore structure responding to wave loading, an on-line estimation of the parameters in the governing equations of motion, from measured responses, enables the parameters assumed in the design process to be corrected, leading to an improved assessment of the risk of failure. Such information can also be very useful in the implementation of active control systems and in providing information which can be used in the design of future structures.

For linear systems the stochastic estimation problem can be addressed through the application of normal spectral analysis methods. However, if significant non-linearities are present, then such spectral methods are inapplicable. There are a number of suitable time domain methods available, in the literature, for solving such non-linear problems [6]. However, rigorous testing of these techniques, through application to a particular type of non-linear oscillator excited by white noise, has shown that they usually give poor results in the case where the excitation is unknown [9]. Thus, new approaches are required, based on applications of results in stochastic process theory (e.g., see references [10, 11]).

It has been shown recently [12] that, for oscillators with both non-linear damping and stiffness, responding to wide-band random excitation, reasonably good parameter estimates can be obtained by combining a moment method with a second order spectral method [13]. The moment equation method offers a very good estimation of the stiffness parameters, and enables linear and non-linear contributions to be separated but is incapable of yielding estimates of the absolute level of the equivalent linear damping. On the other hand, good estimates of this level, and the intensity of the excitation, can be obtained by using the second order spectral method. Unfortunately, however, the spectral method does not provide a means of separating out the linear and non-linear contributions to the damping.

In order to estimate the absolute levels of both linear and non-linear damping coefficients, and not just the equivalent linear damping level, with accuracy, it is necessary to consider higher order statistical quantities relating to the *memory* of the response process. Higher order spectral quantities, or polyspectra, such as the bispectrum and the trispectrum [14] are appropriate for this purpose. They were originally introduced for the purpose of studying deviations from Gaussianity, in stationary random process, due to non-linearities. The concept of a spectral representation of higher-order moments of a stationary process was first proposed by Blanc-Lapierre and Fortet [15], and was further developed by Shiryaev [16] and Brillinger [17, 18]. Shiryaev [16] also considered a spectral representation for higher-order cumulants, attributed to Kolmogorov.

In this paper a general approach to the estimation of parameters in non-linear stochastic systems is presented and developed in some detail for the case where second and fourth order spectral quantities are employed. As an illustration, the problem of estimating the parameters in a non-linear oscillator, with linear plus quadratic damping and linear plus

cubic stiffness and with Gaussian wide-band random excitation is considered. The method is validated by applying it to some digitally simulated data.

2. THE ESTIMATION PROBLEM

A general form of the equations of motion of an n-degree-of-freedom system is

$$g_i(t; \boldsymbol{\lambda}) \equiv g_i[\mathbf{z}(t); \boldsymbol{\lambda}] = f_i(t; \boldsymbol{\theta}) \qquad i = 1, 2, \ldots, n, \tag{1}$$

where

$$\mathbf{z}(t) = [\mathbf{q}(t), \dot{\mathbf{q}}(t), \ddot{\mathbf{q}}(t)]^{\mathrm{T}}, \tag{2}$$

$\mathbf{q}$ is an n-vector of displacement response random process, $f_i(t; \boldsymbol{\theta})$ are random processes, $g_i(t; \boldsymbol{\lambda})$ are non-linear functions and $\boldsymbol{\lambda}$ and $\boldsymbol{\theta}$ are vectors of parameters.

Here both the response, $\mathbf{q}(t)$, and the excitation, $f_i(t; \boldsymbol{\theta})$, will be considered to be stationary, random processes. It will be supposed that sample functions of the displacement response processes can be measured over an interval of time $0 \leqslant t \leqslant T_o$. Where necessary sample functions of $\mathbf{z}(t)$ can be generated by numerically differentiating $\mathbf{q}(t)$. The excitation processes $f_i(t; \boldsymbol{\theta})$ will be supposed unmeasurable, and a stochastic model will be specified.

The elements of the parameter vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\theta}$ are treated here as unknowns, to be estimated from the response data. The estimation problem can be stated as follows: how can sample functions of the response be processed to yield estimates of $\boldsymbol{\lambda}$ and $\boldsymbol{\theta}$?

Here this problem will be solved through techniques based on the use of spectral relationships.

2.1. SPECIFICATION OF THE EXCITATION

Since sample functions of the excitation processes $f_i(t; \boldsymbol{\theta})$ can *not* be measured it is necessary to specify them in terms of a stochastic model. Specifically, here it will be supposed that $f_i(t; \boldsymbol{\theta})$ are zero-mean stationary processes, i.e.,

$$E\{f_i(t; \boldsymbol{\theta})\} = 0 \qquad i = 1, 2, \ldots, n, \tag{3}$$

where $E\{\cdot\}$ is the expectation operator, and that these processes are symmetric with respect to their means. Moreover it will be assumed that the g_i functions are odd with respect to the response variables, such that the response processes are also symmetric about their zero mean values.

Under these conditions only the even order statistics of the excitation and response processes are non-zero. Here it will be supposed that the second order and fourth order correlation (or cumulant) functions of $f_i(t; \boldsymbol{\theta})$ are known. The second order correlation functions (here equal to the second order cumulant functions, since the means are zero) are defined as

$$R^{(2)}_{ij}(\tau; \boldsymbol{\theta}) = E\{f_i(t; \boldsymbol{\theta}) f_j(t + \tau; \boldsymbol{\theta})\}, \tag{4}$$

and the fourth order correlation–cumulant functions may be defined by

$$\begin{aligned} K^{(4)}_{ijkl}(\tau_1, \tau_2, \tau_3; \boldsymbol{\theta}) = {} & R^{(4)}_{ijkl}(\tau_1, \tau_2, \tau_3; \boldsymbol{\theta}) - R^{(2)}_{ij}(\tau_1; \boldsymbol{\theta}) R^{(2)}_{kl}(\tau_3 - \tau_2; \boldsymbol{\theta}) \\ & - R^{(2)}_{ik}(\tau_2; \boldsymbol{\theta}) R^{(2)}_{jl}(\tau_3 - \tau_1; \boldsymbol{\theta}) - R^{(2)}_{il}(\tau_3; \boldsymbol{\theta}) R^{(2)}_{jk}(\tau_2 - \tau_1; \boldsymbol{\theta}), \end{aligned} \tag{5}$$

where

$$R^{(4)}_{ijkl}(\tau_1, \tau_2, \tau_3 ; \boldsymbol{\theta}) = E\{ f_i (t; \boldsymbol{\theta}) f_j (t + \tau_1 ; \boldsymbol{\theta}) f_k (t + \tau_2 ; \boldsymbol{\theta}) f_l (t + \tau_3 ; \boldsymbol{\theta})\} \tag{6}$$

are the fourth order correlation functions.

The corresponding second order spectral cumulant functions (here identical to the spectral functions) are given by

$$S^{(2)}_{ij}(\omega; \boldsymbol{\theta}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R^{(2)}_{ij}(\tau; \boldsymbol{\theta})\, \mathrm{e}^{-\mathrm{i}\omega\tau}\, \mathrm{d}\tau \tag{7}$$

and the fourth order spectral cumulant functions are given by (e.g., see reference [14])

$$\begin{aligned} P^{(4)}_{ijkl}(\omega_1, \omega_2, \omega_3 ; \boldsymbol{\theta}) = {} & S^{(4)}_{ijkl}(\omega_1, \omega_2, \omega_3 ; \boldsymbol{\theta}) - S^{(2)}_{ij}(\omega_1 ; \boldsymbol{\theta}) S^{(2)}_{kl}(\omega_3 ; \boldsymbol{\theta}) \delta(\omega_1 + \omega_2) \\ & - S^{(2)}_{ik}(\omega_3 ; \boldsymbol{\theta}) S^{(2)}_{jl}(\omega_2 ; \boldsymbol{\theta}) \delta(\omega_1 + \omega_3) \\ & - S^{(2)}_{il}(\omega_1 ; \boldsymbol{\theta}) S^{(2)}_{jk}(\omega_2 ; \boldsymbol{\theta}) \delta(\omega_2 + \omega_3), \end{aligned} \tag{8}$$

where

$$S^{(4)}_{ijkl}(\omega_1, \omega_2, \omega_3 ; \boldsymbol{\theta}) = \frac{1}{8\pi^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R^{(4)}_{ijkl}(\tau_1, \tau_2, \tau_3 ; \boldsymbol{\theta})\, \mathrm{e}^{-\mathrm{i}(\omega_1 \tau_1 + \omega_2 \tau_2 + \omega_3 \tau_3)}\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2\, \mathrm{d}\tau_3 \tag{9}$$

is the fourth order spectral function and $\delta()$ is Dirac's delta function. The relationship between $P^{(4)}_{ijkl}(\omega_1, \omega_2, \omega_3 ; \boldsymbol{\theta})$ and $K^{(4)}_{ijkl}(\tau_1, \tau_2, \tau_3 ; \boldsymbol{\theta})$ is similar to that given by equation (9). It is noted that, if the excitation is Gaussian, the fourth order cumulant functions are zero.

The assumption of zero means and symmetry is made here to simplify the subsequent discussion. However, there is no difficulty in principal in extending the subsequent analysis to situations where the excitation and response processes are asymmetric, with non-zero means (see section 6).

3. GENERAL ESTIMATION METHOD

Suppose that the data is divided into N blocks ($r = 1, \ldots, N$), each of duration T ($T_0 = NT$), and that for each block the same transform is applied to both sides of equation (1), to obtain a relationship of the form

$$X_{i,r}(\rho; \boldsymbol{\lambda}) = Y_{i,r}(\rho; \boldsymbol{\theta}), \tag{10}$$

where ρ is the transformed variable, corresponding to t.

For each block, quantities can be formed by applying the same operations to both sides of equation (10), as follows:

$$V^k_{ij\cdots;r}(\rho_1, \rho_2, \ldots ; \boldsymbol{\lambda}) = H^k[X_{i,r}(\rho; \boldsymbol{\lambda})], \qquad W^k_{ij\cdots;r}(\rho_1, \rho_2, \ldots ; \boldsymbol{\theta}) = H^k[Y_{i,r}(\rho; \boldsymbol{\theta})], \tag{11}$$

where H^k ($k = a, b, \ldots$) are some suitable operations. For example, H^a could be a simple product, as follows:

$$V^a_{ij,r}(\rho_1, \rho_2 ; \boldsymbol{\lambda}) = X_{i,r}(\rho_1 ; \boldsymbol{\lambda}) X_{j,r}(\rho_2 ; \boldsymbol{\lambda}), \qquad W^a_{ij,r}(\rho_1, \rho_2 ; \boldsymbol{\theta}) = Y_{i,r}(\rho_1 ; \boldsymbol{\theta}) Y_{j,r}(\rho_2 ; \boldsymbol{\theta}). \tag{12}$$

Clearly $V^k = W^k$ and, taking expectations

$$\bar{V}^k_{ij\ldots}(\rho_1, \rho_2, \ldots; \boldsymbol{\lambda}) \equiv E\{V^k_{ij\ldots;r}(\rho_1, \rho_2, \ldots; \boldsymbol{\lambda})\} = \bar{W}^k_{ij\ldots}(\rho_1, \rho_2, \ldots; \boldsymbol{\theta})$$
$$\equiv E\{W^k_{ij\ldots;r}(\rho_1, \rho_2, \ldots; \boldsymbol{\theta})\}. \tag{13}$$

Since only stationary excitation and response processes are considered her, $\bar{V}^k$ and $\bar{W}^k$ are the same for all blocks: i.e., independent of r, as indicated.

The expectation $\bar{V}^k_{ij\ldots}$ can be estimated by averaging over the N blocks; thus,

$$\hat{\bar{V}}^k_{ij\ldots} = \frac{1}{N}\sum_{r=1}^{N} V^k_{ij\ldots;r}. \tag{14}$$

The unknown parameters can now be estimated by minimizing, in a least squares sense, the differences between the $\hat{\bar{V}}^k$ and $\bar{W}^k$. Thus, the cost function

$$J = \sum_k \sum_{i,j,\ldots} \sum_{r,s,\ldots} [\hat{\bar{V}}^k_{ij\ldots}(\rho_r, \rho_s, \ldots; \boldsymbol{\lambda}) - \bar{W}^k_{ij\ldots}(\rho_r, \rho_s, \ldots; \boldsymbol{\theta})]^2 \tag{15}$$

is minimized with respect to $\boldsymbol{\lambda}$ and $\boldsymbol{\theta}$. As indicated the cost function is evaluated by summing over a suitable range of ρ values.

To implement this method it is, of course, necessary to be able to evaluate $\bar{W}$, in terms of the known statistical parameter describing the excitation process.

4. SECOND ORDER SPECTRAL IDENTIFICATION METHOD

The second order spectral identification method [13] provides estimates of the vector parameters $\boldsymbol{\lambda}, \boldsymbol{\theta}$ by minimizing a cost function in which appears the second order spectral density of both the input $f_i(t; \boldsymbol{\theta})$ and the output $\mathbf{z}(t)$. Here the method is generalized through the use of a new and more direct formulation, based on the general framework given in the preceding section.

For this purpose the Fourier transform is chosen to convert equation (1) to a relationship of the form of equation (10). Thus, here $\rho = \omega$ and, introducing a data window, $D(t)$, one has

$$X_{i,r}(\omega; \boldsymbol{\lambda}) = \int_0^T g_{i,r}(t; \boldsymbol{\lambda}) D(t)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}t, \qquad Y_{i,r}(\omega; \boldsymbol{\theta}) = \int_0^T f_{i,r}(t; \boldsymbol{\theta}) D(t)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}t. \tag{16}$$

To form the V and W quantities in equation (11) the X and Y quantities are here multiplied by their complex conjugates and divided by $2\pi T$. Thus,

$$V^a_{i,j,r}(\omega) = \frac{1}{2\pi T} X_{i,r}(\omega; \boldsymbol{\lambda}) X^*_{j,r}(\omega; \boldsymbol{\lambda}), \qquad W^a_{i,j,r}(\omega) = \frac{1}{2\pi T} Y_{i,r}(\omega; \boldsymbol{\theta}) Y^*_{j,r}(\omega; \boldsymbol{\theta}). \tag{17}$$

On taking expectations of W one finds that

$$\bar{W}^a_{ij}(\omega; \boldsymbol{\theta}) = \frac{1}{2\pi T} \int_0^T \int_0^T R^{(2)}_{ij}(t - s; \boldsymbol{\theta}) D(s) D(t)\, \mathrm{e}^{\mathrm{i}\omega(t-s)}\, \mathrm{d}s\, \mathrm{d}t. \tag{18}$$

It is noted that this expectation can be calculated by numerical integration, since $R^{(2)}_{ij}$ is assumed to be known.

If the block length is significantly greater than the correlation time scale of the excitation then a standard analysis shows that (using equation (7))

$$\bar{W}^a_{ij}(\omega; \boldsymbol{\theta}) \approx \frac{k_2}{2\pi} \int_0^\infty R^{(2)}_{ij}(\tau; \boldsymbol{\theta})\, \mathrm{e}^{\mathrm{i}\omega\tau}\, \mathrm{d}\tau = k_2\, S^{(2)}_{ij}(\omega; \boldsymbol{\theta}), \tag{19}$$

where the constant k_2 is given by

$$k_2 = \frac{1}{T} \int_0^T D^2(t)\, \mathrm{d}t. \tag{20}$$

Thus, applying the general approach outlined in the previous section the parameter vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\theta}$ can be estimated, for relatively large block lengths, by minimizing

$$J_2 = \sum_{ij} \sum_r [\hat{\bar{V}}^a_{ij}(\omega_r\,; \boldsymbol{\lambda}) - k_2\, S_{ij}\,(\omega_r\,; \boldsymbol{\theta})]^2, \tag{21}$$

where $\hat{\bar{V}}$ is an estimate of the expectation of V, obtained by block averaging.

Although the method is easier to implement when the block length is relatively large it is noted that it can be applied for *any* block length.

As will be demonstrated later, through an example, the Fourier transform X (and hence V) can, for specific equations of motion, be related directly to the Fourier transforms of the measured displacement data, and its derivatives.

5. FOURTH ORDER SPECTRAL IDENTIFICATION METHOD

The Fourier transforms defined by equation (16) can be used as the basis of a fourth order spectral estimation method (due to the symmetry assumptions with regard to the excitation and response processes the corresponding third order spectra will be zero).

For this purpose the W quantities in equation (11) may be defined as follows

$$W^b_{ijkl;r}(\omega_1, \omega_2, \omega_3\,; \boldsymbol{\theta}) = \frac{1}{8\pi^3 T}\, Y_{i;r}\,(\omega_1\,; \boldsymbol{\theta})\, Y_{j;r}\,(\omega_2\,; \boldsymbol{\theta})\, Y_{k;r}\,(\omega_3\,; \boldsymbol{\theta})\, Y^*_{l;r}\,(\omega_1 + \omega_2 + \omega_3\,; \boldsymbol{\theta}), \tag{22}$$

and the V quantity may be similarly defined. On taking expectations of this W quantity one finds that

$$\bar{W}^b_{ijkl}(\omega_1, \omega_2, \omega_3) = \frac{1}{8\pi^3 T} \int_0^T \int_0^T \int_0^T \int_0^T R^{(4)}_{ijkl}(s, t, u, v; \boldsymbol{\theta}) D(s) D(t) D(u) D(v)$$

$$\times\, \mathrm{e}^{-\mathrm{i}[\omega_1 s + \omega_2 t + \omega_3 u - (\omega_1 + \omega_2 + \omega_3) v]}\, \mathrm{d}s\, \mathrm{d}t\, \mathrm{d}u\, \mathrm{d}v. \tag{23}$$

It is shown in Appendix A that this W can be expressed as

$$\bar{W}^b_{ijkl}(\omega_1, \omega_2, \omega_3\,; \boldsymbol{\theta}) = k_4\, P_{ijkl}\,(\omega_1, \omega_2, \omega_3\,; \boldsymbol{\theta}) + S^{(2)}_{ij}(\omega_1\,; \boldsymbol{\theta}) S^{(2)}_{kl}(\omega_3\,; \boldsymbol{\theta}) \Delta_T\,(\omega_1 + \omega_2)$$

$$+ S^{(2)}_{ik}(\omega_3\,; \boldsymbol{\theta}) S^{(2)}_{jl}(\omega_2\,; \boldsymbol{\theta}) \Delta_T\,(\omega_1 + \omega_3)$$

$$+ S^{(2)}_{il}(\omega_1\,; \boldsymbol{\theta}) S^{(2)}_{jk}(\omega_2\,; \boldsymbol{\theta}) \Delta_T\,(\omega_2 + \omega_3), \tag{24}$$

where k_4 is defined by equation (A6) and the window $\Delta_T(\omega)$ is defined by (from equations (A17) and (A12))

$$\Delta_T(\omega) = \frac{1}{2\pi T}\left|\int_0^T D^2(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t\right|^2. \tag{25}$$

In the special case of a "box car" window ($D(t) = 1$), equation (25) gives

$$\Delta_T(\omega) = \frac{1}{2\pi T}\left|\int_0^T \mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t\right|^2 = \frac{T}{2\pi}\frac{\sin^2(\pi\Omega)}{(\pi\Omega)^2}, \tag{26}$$

where

$$\Omega = \omega T/2\pi, \tag{27}$$

and, according to equation (A6), $k_4 = 1$.

For other, common types of window k_4 and $\Delta_T(\omega)$ can be evaluated using equations (A6) and (25). For example, for the Hanning window

$$D(t) = \tfrac{1}{2}\left[1 - \cos\left(\frac{2\pi t}{T}\right)\right], \tag{28}$$

$k_4 = 35/128 = 0{\cdot}2734$ and

$$\Delta_T(\Omega) = \frac{T}{128\pi^3}(X^2 + Y^2), \tag{29}$$

where

$$X = \int_0^{2\pi}(1 - \cos u)^2 \cos(\Omega u)\,\mathrm{d}u \qquad Y = \int_0^{2\pi}(1 - \cos u)^2 \sin(\Omega u)\,\mathrm{d}u. \tag{30}$$

Expressions for X and Y are given in Appendix B.

In the general case it can be deduced, from equation (25) that

$$\int_{-\infty}^{\infty}\Delta_T(\omega)\,\mathrm{d}\omega = k_4, \tag{31}$$

independent of the value of T. In the limit, as T becomes very large, $\Delta_T(\omega)$, becomes concentrated at $\omega = 0$ and, using equation (31), it follows that

$$\Delta_T(\omega) \to k_4\,\delta(\omega). \tag{32}$$

Thus, equation (24) reduces to

$$\bar{W}^b_{ijkl}(\omega_1, \omega_2, \omega_3\,;\boldsymbol{\theta}) = k_4\,S_{ijkl}(\omega_1, \omega_2, \omega_3\,;\boldsymbol{\theta}). \tag{33}$$

On applying the general approach outlined earlier, for relatively large block lengths, $\boldsymbol{\lambda}$ and $\boldsymbol{\theta}$ can be estimated by minimizing

$$J_4^A = \sum_{ijkl}\sum_{rsu}[\hat{\bar{V}}^b_{ijkl}(\omega_r, \omega_s, \omega_u\,;\boldsymbol{\lambda}) - \bar{W}^b_{ijkl}(\omega_r, \omega_s, \omega_u\,;\boldsymbol{\theta})]^2, \tag{34}$$

where $\hat{\bar{V}}$ is an estimate of the expectation of

$$V^b_{ijkl;r}(\omega_1, \omega_2, \omega_3 ; \boldsymbol{\lambda}) = \frac{1}{8\pi^3 T} X_{i;r}(\omega_1 ; \boldsymbol{\lambda}) X_{j;r}(\omega_2 ; \boldsymbol{\lambda}) X_{k;r}(\omega_3 ; \boldsymbol{\lambda}) X^*_{l;r}(\omega_1 + \omega_2 + \omega_3 ; \boldsymbol{\lambda}) \quad (35)$$

obtained by block averaging, and $S_{ijkl}(\omega_1, \omega_2, \omega_3 ; \boldsymbol{\theta})$ is given by equation (9).

An alternative cost function can be obtained by replacing the spectral terms in equation (24) by estimates derived from the data. This is equivalent to defining new V and W as follows:

$$\hat{\bar{V}}^c_{ijkl;r}(\omega_1, \omega_2, \omega_3 ; \boldsymbol{\lambda}) = \hat{\bar{V}}^b_{ijkl;r}(\omega_1, \omega_2, \omega_3 ; \boldsymbol{\lambda}) - \frac{1}{k_2^2}\begin{bmatrix} \hat{\bar{V}}^a_{ij}(\omega_1 ; \boldsymbol{\lambda}) \hat{\bar{V}}^a_{kl}(\omega_3 ; \boldsymbol{\lambda}) \Delta_T(\omega_1 + \omega_2) \\ + \hat{\bar{V}}^a_{ik}(\omega_3 ; \boldsymbol{\lambda}) \hat{\bar{V}}^a_{jl}(\omega_2 ; \boldsymbol{\lambda}) \Delta_T(\omega_1 + \omega_3) \\ + \hat{\bar{V}}^a_{il}(\omega_1 ; \boldsymbol{\lambda}) \hat{\bar{V}}^a_{jk}(\omega_2 ; \boldsymbol{\lambda}) \Delta_T(\omega_2 + \omega_3) \end{bmatrix}, \quad (36)$$

$$\bar{W}^b_{ijkl}(\omega_1, \omega_2, \omega_3 ; \boldsymbol{\lambda}) = k_4 P_4(\omega_1, \omega_2, \omega_3 ; \boldsymbol{\lambda}). \quad (37)$$

The cost function then becomes

$$J_4^B = \sum_{ijkl} \sum_{rsu} [\hat{\bar{V}}^c_{ijkl}(\omega_r, \omega_s, \omega_u ; \boldsymbol{\lambda}) - k_4 P_{ijkl}(\omega_r, \omega_s, \omega_u ; \boldsymbol{\theta})]^2. \quad (38)$$

In the evaluation of either J_4^A or J_4^B, considerable reduction in computational effort may be achieved by restricting the frequencies to the principal domain, defined by [19]

$$\begin{cases} \omega_1 \geqslant \omega_2 \geqslant \omega_3 \\ \omega_1 + \omega_2 + \omega_3 \geqslant 0 \\ \omega_1 + \omega_2 + 2\omega_3 \geqslant 0 \\ \text{exactly only one frequency from } \omega_1, \omega_2, \omega_3 < 0. \end{cases} \quad (39)$$

6. EXTENSION TO ASYMMETRIC EXCITATION AND RESPONSE

Although it was assumed in section 2 that all the excitation and response processes are symmetric with respect to zero mean values, the approach outlined in section 3 is actually applicable in the more general case where the means are non-zero and the excitation and response processes are asymmetric. In particular, the system parameters can still be estimated by minimizing the cost functions J_2 and J_4^A, as defined by equations (21) and (34), respectively. However, it will be necessary to introduce extra parameters in the vector of excitation parameters, $\boldsymbol{\theta}$.

As an example, if the means of the excitation processes are non-zero, then these unknown quantities can be included in $\boldsymbol{\theta}$. Then, using the second order spectral identification method, the quantities $\hat{\bar{V}}^a_{ij}$ appearing in the cost function J_2, defined by equation (21), will contain spikes at zero frequency, corresponding to zero-frequency spikes in the excitation. The spikes in $\hat{\bar{V}}^a_{ij}$ will be finite, due to the use of finite block lengths in their generation. A minimization of J_2 will then lead to estimates of the excitation means. Similarly, finite spikes related to the excitation means will occur in the quantities $\hat{\bar{V}}^b_{ijkl}$ which appear in the cost function J_4^A, defined by equation (34), again allowing an estimation of these means through a minimization of this function.

In the more general case where further parameters are required to specify the degree of asymmetry of the excitation, or there are system parameters relating to asymmetries in

the systems dynamic characteristics, it is unlikely that satisfactory estimation will be achievable using only second and fourth order spectra. In this case it will be advantageous to also consider third order spectra (bispectra) and to minimize a cost function, J_3, relating to such spectra, defined by analogy with the definitions given earlier, relating to the second and fourth order spectra. In this situation the required parameters could be estimated by simultaneous minimizing J_2, J_3 and J_4, through the introduction of an overall cost function, as indicated by equation (15).

Clearly more computational effort will be required to deal with asymmetric excitations and responses and, with the introduction of further parameters, the overall accuracy of the estimation may decrease. We intend to investigate this issue in depth, in a future paper.

7. APPLICATION TO A NON-LINEAR OSCILLATOR

As an illustration a non-linear oscillator with the following equation of motion will be considered:

$$\ddot{x} + a_1 \dot{x} + n_1 \dot{x}|\dot{x}| + a_2 x + n_2 x^3 = f(t; \boldsymbol{\theta}). \tag{40}$$

Such an equation is often used to model the dynamic behaviour of ships rolling in random waves [4]. The linear-plus-quadratic form of damping adopted here has been shown to give an excellent fit to experimental ship rolling data (e.g., see reference [1]). Also the linear-plus cubic stiffness component in equation (40) gives a good representation of actual ship restoring moment characteristics, for small to moderately large roll angles ($<35^\circ$ say). It is noted that equation (40) can be used to model a variety of other fluid–structure systems.

Here the vector of unknown system parameters is

$$\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \lambda_3, \lambda_4] = [a_1, n_1, a_2, n_2]^{\mathrm{T}}. \tag{41}$$

The linear-in-the-parameter nature of the equation of motion greatly facilitates the implementation of the methods discussed earlier. The implementation of the proposed methods does not require the linearity of the equation of motion with respect to the coefficients, provided that a general purpose minimization algorithm is used to minimize the cost functions. Here a standard algorithm, available within the software package MATLAB has been utilized [20].

It will be assumed here that $f(t; \boldsymbol{\theta})$ can be modelled as a Gaussian white noise. Thus,

$$R^{(2)}(\tau; \boldsymbol{\theta}) = E\{ f(t; \boldsymbol{\theta}) f(t + \tau; \boldsymbol{\theta}) \} = D^2 \delta(\tau), \tag{42}$$

where D is the "strength" of the excitation. $D^2 = 2\pi S_0$, where S_0 is the constant second order spectral level of the white noise process, that is

$$S^{(2)}(\omega) = S_0. \tag{43}$$

Here there is only one parameter, D (or S_0) relating the excitation and thus θ is scalar: i.e., $\theta = D$.

Due to the Gaussianity of the input process all the correlation–cumulant functions of the input process of order higher than two, and all the related spectral quantities, are zero: thus,

$$P^{(4)}(\omega_1, \omega_2, \omega_3 ; \boldsymbol{\theta}) = 0. \tag{44}$$

7.2. SECOND ORDER SPECTRAL IDENTIFICATION

A second order spectral input–output relationship can be derived fairly readily from equation (40) by following the procedure outlined earlier, in section 4.

Setting $y_0 = \ddot{x},\ y_1 = \dot{x},\ y_2 = \dot{x}|\dot{x}|,\ y_3 = x,\ y_4 = x^3$ the first of equations (16) can be expressed as (dropping the i subscript)

$$X_r(\omega; \boldsymbol{\lambda}) = \sum_{j=0}^{4} \lambda_j\, Y_{j;r}(\omega), \tag{45}$$

where $\lambda_0 = 1$ and $Y_{j;r}(\omega)$ are the finite length Fourier transforms of the corresponding y quantities: thus,

$$Y_{j;r}(\omega) = \int_0^T y_{j;r}(t) D(t)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}t \tag{46}$$

and the appropriate V quantity (see equations (17)) is

$$V_r^a(\omega; \boldsymbol{\lambda}) = \frac{1}{2\pi} X_r(\omega; \boldsymbol{\lambda}) X_r^*(\omega; \boldsymbol{\lambda}) = \sum_{j=0}^{4} \sum_{k=0}^{4} \lambda_j\, \lambda_k\, U_{jk;r}(\omega), \tag{47}$$

where

$$U_{jk;r}(\omega) = \frac{1}{2\pi} Y_j(\omega) Y_k^*(\omega). \tag{48}$$

The expectation of the appropriate W quantity is, from equation (19),

$$\bar{W}_r^a = k_2\, S_0 = k_2\, D^2/2\pi. \tag{49}$$

From equation (21) the unknown parameters can be estimated by minimizing the cost function

$$J_2 = \sum_r [\hat{\bar{V}}^a(\omega_r; \boldsymbol{\lambda}) - k_2\, S_0]^2, \tag{50}$$

where $\hat{\bar{V}}^a$ is the block average estimate of the expectation of V_r^a.

Some simplification is achieved by noting that, from equation (47)

$$\hat{\bar{V}}^a(\omega; \boldsymbol{\lambda}) = \sum_{j=0}^{4} \sum_{k=0}^{4} \lambda_j\, \lambda_k\, \hat{\bar{U}}_{jk}(\omega), \tag{51}$$

and, if the block length is significantly greater than the correlation time scale of the response

$$\bar{U}_{jk}(\omega) = k_2\, S_{jk}(\omega), \tag{52}$$

where $S_{jk}(\omega)$ is the cross spectrum for $y_k(t)$ and $y_j(t)$ (direct spectrum if $k = j$). Now the spectra relating to the velocity and acceleration responses, $\dot{x}(t)$ and $\ddot{x}(t)$ can be expressed in terms of the spectra relating to the displacement. If, for example, the spectra $S_{00}(\omega)$ relating to the acceleration response, $\ddot{x}$, are considered then

$$S_{00}(\omega) = \omega^4 S_{33}(\omega), \tag{53}$$

and hence, to a good approximation, if the number of blocks is large,

$$\hat{\bar{U}}_{00}(\omega) = \omega^4 \hat{\bar{U}}_{33}. \tag{54}$$

This removes the necessity for calculating the acceleration from the data. Similar arguments apply to the velocity but it is noted that this must, in any event, be calculated to evaluate y_2.

In the case where the data window is unity, expressions such as equation (52) can be derived more directly from the relationships

$$Y_{0;r}(\omega) = -\omega^2 Y_{3;r} \qquad \text{and} \qquad Y_{1;r}(\omega) = -i\omega Y_{3;r}. \tag{55}$$

However, these relationships are invalid for more general types of window and the validity of expressions such as those given in equation (53) then rests on the assumption that the block length is relatively large.

Using the fact that, from equation (52) the $U_{jk}(\omega)$ quantities are essential estimates of corresponding cross-spectra, $S_{jk}(\omega)$, the expression for $\hat{\bar{V}}^a$ can be written as

$$\begin{aligned}\hat{\bar{V}}^a(\omega; \boldsymbol{\lambda}) = {} & \hat{S}_X^{(2)}(\omega)\,[a_2^2 + \omega^4 - 2a_2\,\omega^2 + \omega^2 a_1^2] + n_1^2 \hat{S}_Y^{(2)}(\omega) + n_2^2 \hat{S}_Z^{(2)}(\omega) \\ & + 2(a_2 - \omega^2)\,(n_1\,\hat{S}_{XYr}^{(2)}(\omega) + n_2\,\hat{S}_{XZr}^{(2)}(\omega)) + 2n_1 n_2 \hat{S}_{YZr}^{(2)}(\omega) \\ & + 2\omega a_1\,(n_1\,\hat{S}_{XYi}^{(2)}(\omega) + n_2\,\hat{S}_{XZi}^{(2)}(\omega)).\end{aligned} \tag{56}$$

The additional subscripts, r and i, appearing in equation (56) denote, respectively, the real and imaginary parts of the cross-spectrum.

The number, N, of discrete frequencies, ω_i, at which $\hat{\bar{V}}^a$ is evaluated must be greater than the number of parameters to be estimated (here 5). The accuracy of the results obtained is, generally, not sensitive to the precise choice of N and ω_i. However, the chosen frequencies should concentrate around the dominant part of the response spectrum to ensure the most significant harmonic component of the response.

7.2. HIGHER ORDER SPECTRAL IDENTIFICATION

As will be demonstrated later through a presentation of numerical results obtained from simulated data, a significant improvement in estimation accuracy, over that obtainable using second order spectra, can be achieved by employing fourth order spectral quantities.

Using equations (35), (45) and (46) again the appropriate V quantity is

$$V_r^b(\omega_1, \omega_2, \omega_3; \boldsymbol{\lambda}) = \frac{1}{8\pi^3 T} \sum_{j=0}^{4} \sum_{k=0}^{4} \sum_{l=0}^{4} \sum_{m=0}^{4} \lambda_j\,\lambda_k\,\lambda_l\,\lambda_m\,U_{jklm;r}(\omega_1, \omega_2, \omega_3), \tag{57}$$

where

$$U_{jklm;r}(\omega_1, \omega_2, \omega_3) = \frac{1}{8\pi^3 T}\,Y_j(\omega_1)\,Y_k(\omega_2)\,Y_l(\omega_3)\,Y_m^*(\omega_1 + \omega_2 + \omega_3). \tag{58}$$

The expectation of the appropriate W quantity is, from equation (24)

$$\bar{W}^b(\omega_1, \omega_2, \omega_3; \theta) = S_0^2[\Delta_T(\omega_1 + \omega_2) + \Delta_T(\omega_1 + \omega_3) + \Delta_T(\omega_2 + \omega_3)]. \tag{59}$$

Thus, again, θ is here scalar and can be equated to D. The unknown parameters can now be evaluated by minimizing the cost function J_4^A, where here $\hat{\bar{V}}^b$ is given by equations (57) and (58) and $\bar{W}^b$ is given by equation (59).

As in the case of the estimation method based on second order spectra it is possible to avoid numerically differentiating the response displacement to obtain the acceleration response. For example

$$S_{0000}(\omega_1, \omega_2, \omega_3) = \omega_1^2\omega_2^2\omega_3^2\bar{\omega}^2 S_{3333}(\omega_1, \omega_2, \omega_3), \tag{60}$$

where $\bar{\omega} = \omega_1 + \omega_2 + \omega_3$. Hence, to a good approximation,

$$\hat{\bar{U}}_{0000}(\omega_1, \omega_2, \omega_3) = \omega_1^2\omega_2^2\omega_3^2\bar{\omega}^2 \hat{\bar{U}}_{3333}(\omega_1, \omega_2, \omega_3). \tag{61}$$

An alternative approach to estimation is to minimize the cost function defined by equation (37). Here, since the excitation is Gaussian, this reduces to

$$J_4^B = \sum_{rsu} [\hat{\bar{V}}^c(\omega_r, \omega_s, \omega_u ; \boldsymbol{\lambda})]^2, \tag{62}$$

where

$$\hat{\bar{V}}_c(\omega_r, \omega_s, \omega_u ; \boldsymbol{\lambda}) = \hat{\bar{V}}_b(\omega_r, \omega_s, \omega_u ; \boldsymbol{\lambda}) - \frac{1}{k_2^2}\begin{bmatrix} \hat{\bar{V}}^a(\omega_r ; \boldsymbol{\lambda})\hat{\bar{V}}^a(\omega_u ; \boldsymbol{\lambda})\Delta_T(\omega_r + \omega_s) \\ + \hat{\bar{V}}^a(\omega_u ; \boldsymbol{\lambda})\hat{\bar{V}}^a(\omega_s ; \boldsymbol{\lambda})\Delta_T(\omega_r + \omega_u) \\ + \hat{\bar{V}}^a(\omega_r ; \boldsymbol{\lambda})\hat{\bar{V}}^a(\omega_s ; \boldsymbol{\lambda})\Delta_T(\omega_s + \omega_u) \end{bmatrix}. \tag{63}$$

This has the advantage that the parameter vector $\boldsymbol{\lambda}$ can be estimated independently of the excitation parameter (here D).

7.3. STIFFNESS PARAMETER ESTIMATION USING MOMENTS

Previous work [12] has shown that a simple moments method can yield good estimates of the stiffness parameters, for a single-degree-of-freedom system such as that governed by equation (40). Full details of this method are given in reference [12] and therefore are not included here. However, some stiffness parameter estimates obtained by the moment method are presented here, for comparison with corresponding estimates obtained by the spectral methods.

7.4. VALIDATION THROUGH SIMULATION

To test the proposed approach, sample function of $X(t)$ were generated by numerically integrating equation (40), using a fourth order Runge–Kutta algorithm. Samples of the external Gaussian white noise were generated by using a method described by Roberts and

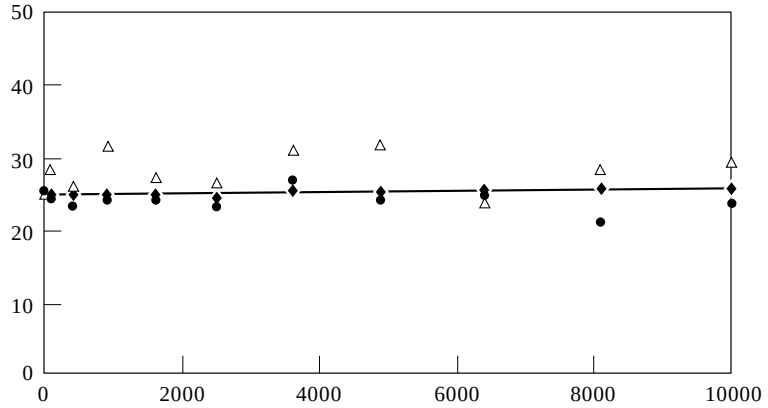

Figure 1. Estimates of a_2 for various levels of excitation. ——, true value: △, second order: ●, fourth order: ◆, moment method.

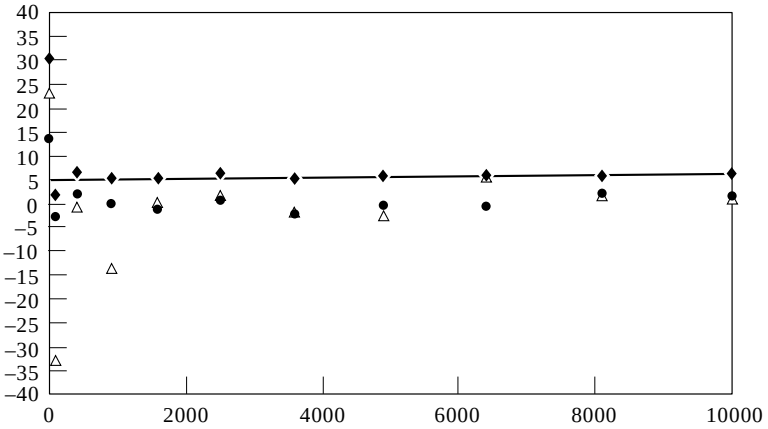

Figure 2. Estimates of n_2 for various levels of excitation. ——, true value: △, second order: ●, fourth order: ◆, moment method.

Spanos [21] and response time histories of duration 2000 s were computed, with a time step of 0·02 s. The system parameters chosen for the simulation study were $a_1 = 0{\cdot}5$, $n_1 = 0{\cdot}5$, $a_2 = 25$, $n_2 = 5$.

In applying the spectral estimation techniques each response history was processed in blocks of length 20·48 s. Some initial pilot studies revealed that the accuracy of the results was, generally, not sensitive to the precise choice of the number of discrete frequencies used to evaluate the cost function provided that this number is significantly in excess of the number of parameters to be estimated (here 5). Clearly, however, the chosen frequencies

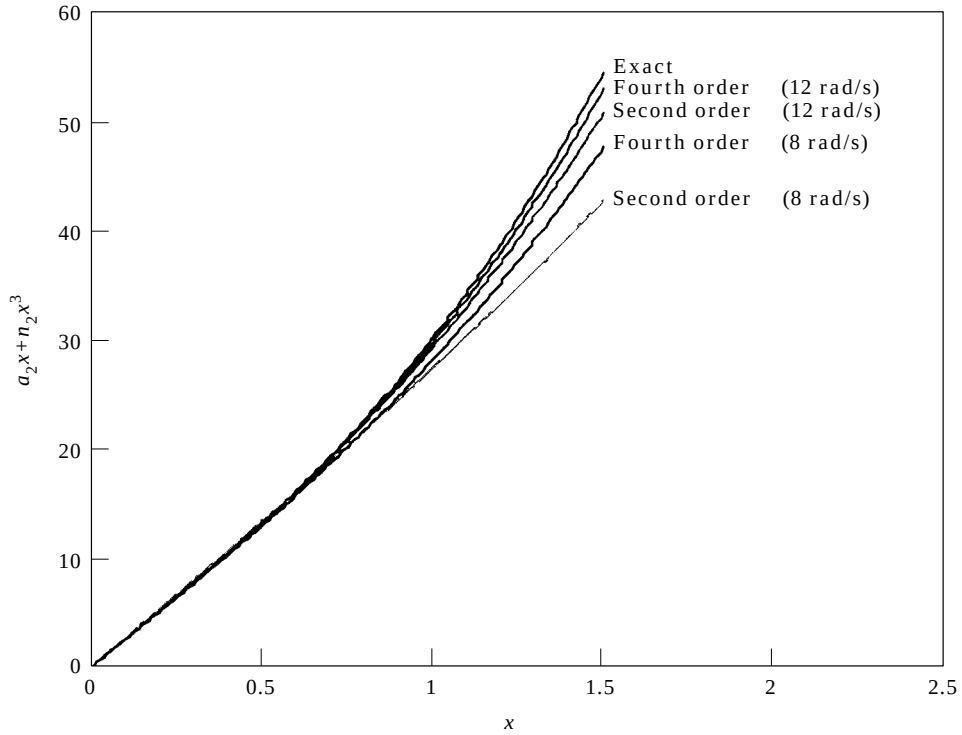

Figure 3. Comparison between estimated restoring force characteristics and the true characteristic; $S_0 = 2500$.

should concentrate around the dominant part of the spectrum of $x(t)$, in the case of the second order method, or the trispectrum of $x(t)$ in the case of fourth order estimation, to ensure that the most significant harmonic components of the response are included. A numerical evaluation of the trispectrum for the oscillator under examination showed that the range of frequency used for the minimization of J_2 is also appropriate for the minimization of J_4, the latter being confined to the principal domain defined by equation (39).

In order to reduce the computational effort involved in implementing the fourth order spectral method the number of frequencies included in the minimization procedure—i.e., the volume for the frequency points $(\omega_r, \omega_s, \omega_u)$—may be selected as the intersection between the principal domain and the region defined by

$$|\omega_1 + \omega_2| \leqslant \Delta\omega, \qquad |\omega_1 + \omega_3| \leqslant \Delta\omega, \qquad |\omega_2 + \omega_3| \leqslant \Delta\omega, \tag{64}$$

where $\Delta\omega = 2\pi/T$. Trial studies showed that this considerably reduced the computational effort without a significant loss of accuracy in the identification procedure.

Figures 1 and 2 show estimates of the linear and non-linear stiffness parameters for various levels of excitation, as measured by S_0. Here estimates obtained by the moment method are compared with those found by applying the second order and fourth order spectral methods. In both cases all the unknown parameters were simultaneously estimated. Generally the estimation is reasonably accurate, especially with regard to the linear term. It is evident that the moment method estimates are more accurate. The low accuracy of the estimate of n_2, at low excitation levels, simply reflects the fact that under these conditions the contribution of the non-linear stiffness is negligible.

To obtain the spectral estimates shown in Figures 1 and 2 the upper limit of the frequency range used in the cost functions was 8 rad/s. Trials showed that improved

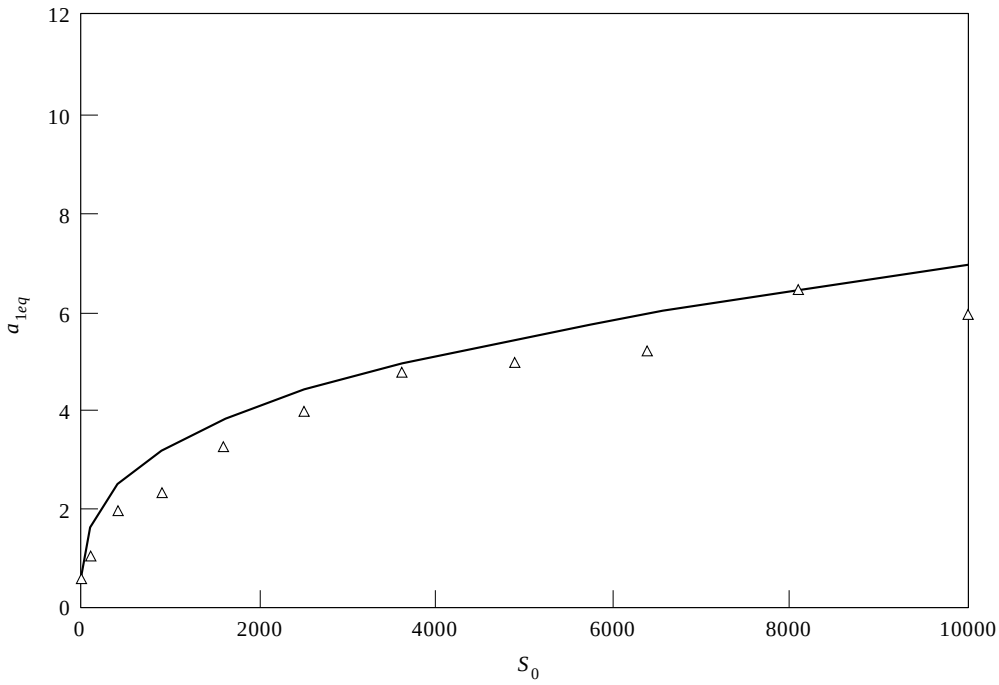

Figure 4. Estimates of a_{1eq} for various levels of excitation: ——, true value; △, estimated value.

accuracy, especially with respect to the non-linear stiffness coefficient was obtained by increasing this upper limit, at the expense of a significant increase in computational effort. However, a precise determination of these coefficients may not, in practice, be of paramount importance, so long as the overall restoring force is reasonably well estimated. In fact, a study of Figures 1 and 2 reveals evidence of a trade-off between the estimated values of a_2 and n_2 : when a_2 is low n_2 is high, and vice-versa. This suggests that the estimate of the restoring force, $f_R = a_2\, x + n_2\, x^3$, is fairly accurate. Figure 3 confirms that this is the case. Here the true restoring force, derived using the true parameters, is compared with the estimated restoring force, derived using the stiffness parameters estimated by spectral methods, for one value of S_0. Here the displacement range is the maximum obtained in the simulated responses. Good agreement is obtained: the fourth order spectral estimate being more accurate, as one would expect. Figure 3 also shows the degree of improvement in the accuracy of the estimation when the upper limit of the frequency range used is increased from 8 to 12 rad/s.

Previous work has shown that the second order spectral method is not capable of distinguishing between linear and non-linear damping contributions to the response [13]. However, it is able to give reasonable estimates of an equivalent linear damping [13]. These are obtained by replacing the non-linear damping force, $f_D = a_1\, \dot{x} + n_1\, \dot{x}|\dot{x}|$, by the linear force $f_D = a_{1eq}\, \dot{x}$, in the equation of motion, as far as the estimation procedure is concerned. Figure 4 shows a comparison between the estimated equivalent linear damping, as obtained by the second order spectral method, and the theoretical value, given by [21]

$$a_{1eq} = a_1 + n_1 \left(\frac{8}{\pi}\right)\sigma_{\dot{x}}, \tag{65}$$

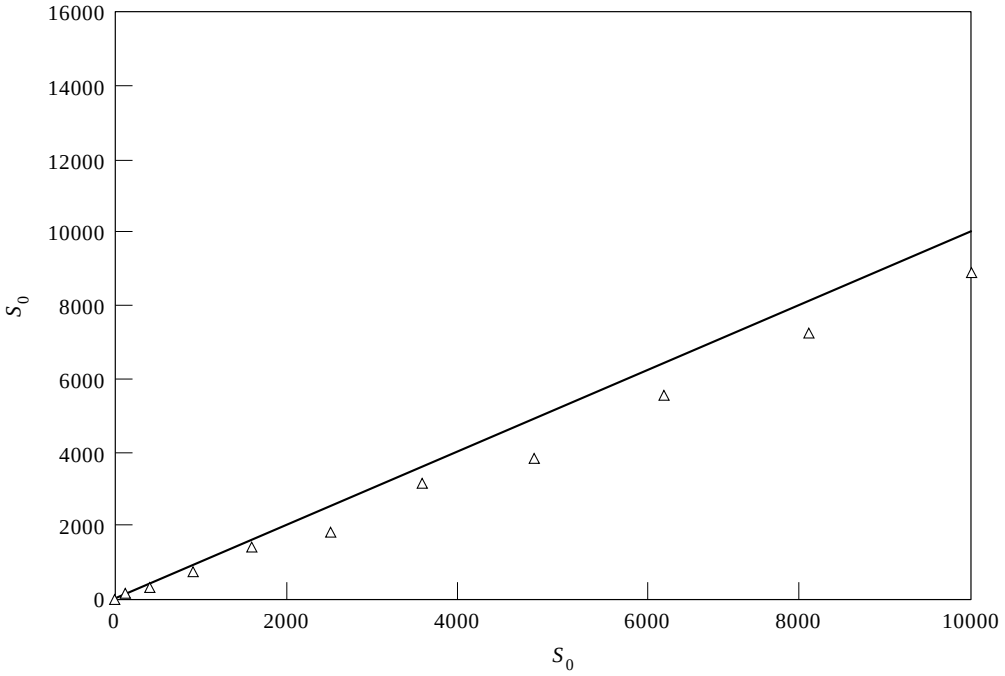

Figure 5. Comparison of estimated values of the excitation level with the true values: ——, true value; △, estimated value.

where $\sigma_{\dot{x}}$ is the standard deviation of $\dot{x}$. The upper frequency limit chosen for the cost function here, and for generating the further results to be discussed, was 8 rad/s. Here various equivalent linear damping estimates are plotted against the excitation level, S_0. The agreement is good, especially in view of the fact that the two definitions of equivalent linear damping involved here have a different origin.

The estimated values of the input spectral level, S_0, obtained by the second order spectral method, are plotted against the corresponding true values in Figure 5. Again a reasonably good level of agreement is obtained.

In Figure 6 estimates of the damping coefficients are shown: specifically, a_1 estimates are plotted against corresponding estimates of n_1. These estimates are derived from a number of simulation results, each for a spectral input level S_0 of 100, by using both the second order and fourth order spectral estimation methods. The second order estimates show a significant trade off, and are localized around a line corresponding approximately to equation (65), with a fixed value of equivalent linear damping. They are therefore of low accuracy, generally speaking. In contrast the fourth order estimation points lie in a small area very close to the target point, corresponding to the true parameter values, reflecting their high degree of accuracy. Here J_4 was minimized along the line corresponding to equation (65).

These results relate to properties of the second and fourth order cost functions. Figures 7 and 8 show contour maps for these two functions, respectively, for $S_0 = 100$. Here the cost function is regarded as a function of the two damping parameters, the other parameter values being fixed at their true values. The minimum of the second order spectral functional is localized along a valley, again corresponding approximatley to equation (65) with a fixed level of equivalent linear damping. On the other hand the fourth order cost function exhibits two minima localized symmetrically with respect to the origin of the damping parameters plane, one around the target point and the other about its reflection. Thus

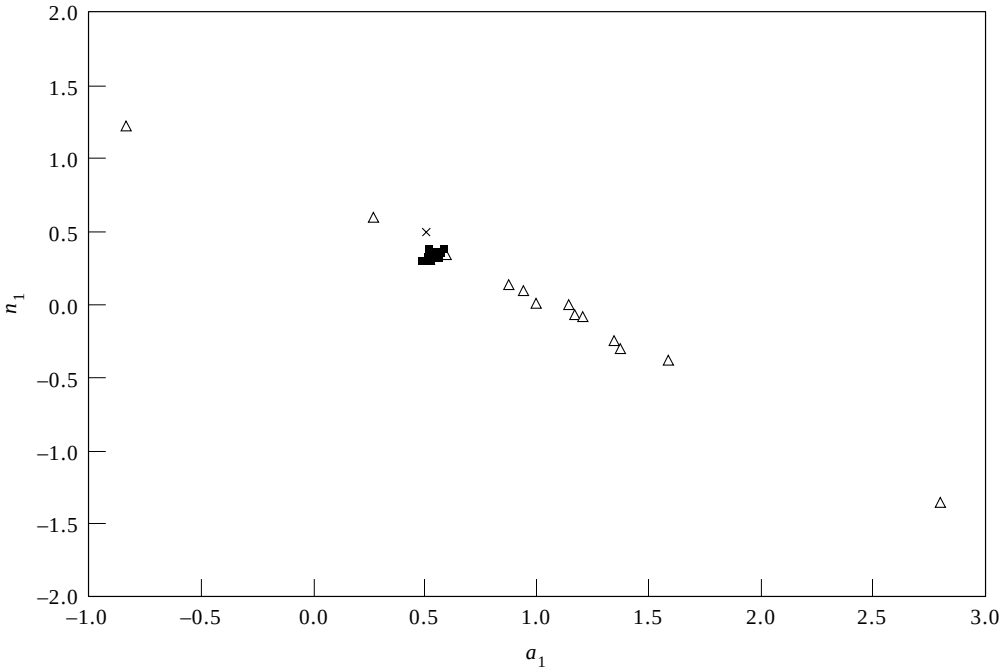

Figure 6. Variation of n_1 estimates with estimates of a_1: ×, true value; △, second order; ■, fourth order.

minimization of the fourth order cost function may give estimates close to either minimum. However, using the constraint that the equivalent linear damping is positive, and given by equation (65), the fourth order spectral estimation method may be employed without any ambiguity regarding the sign of the estimated parameters.

In the present investigation, based on simulated data, the effect of measurement noise has not been considered. However, it can be expected that the proposed methods will be robust with respect to such noise, provided that it is in a frequency range which is well separated from the range within which the dynamic response is dominant. In these circumstances the noise will be filtered out, through the exclusion of high frequency components in the cost functions. This conclusion is supported by some recent simulation studies using the second order spectral method for multi-degree-of-freedom systems. There noise of various levels was simulated [22] and it was found that, for noise to signal ratios below about 4%, its effect was negligible. Further support has been obtained from applying the second order method to some real data obtained from a model ship in a wave tank, rolling due to incident random waves. Here the measurement noise was at a low level, typical of that found in experimental work, and good results were obtained [9].

8. CONCLUSIONS

A higher order spectra identification technique for estimating system and excitation process parameters from measurements of the response alone has been presented, within a general framework. Application to simulated data has shown that the proposed

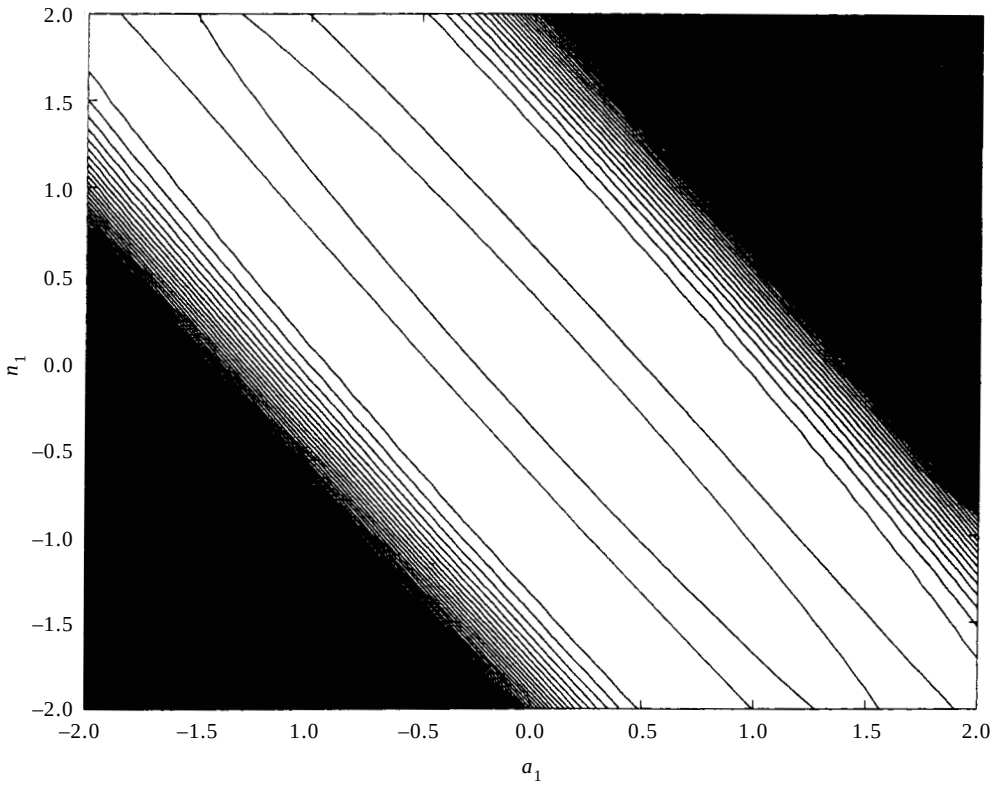

Figure 7. Contour map of the cost function used in the second order spectral estimation method.

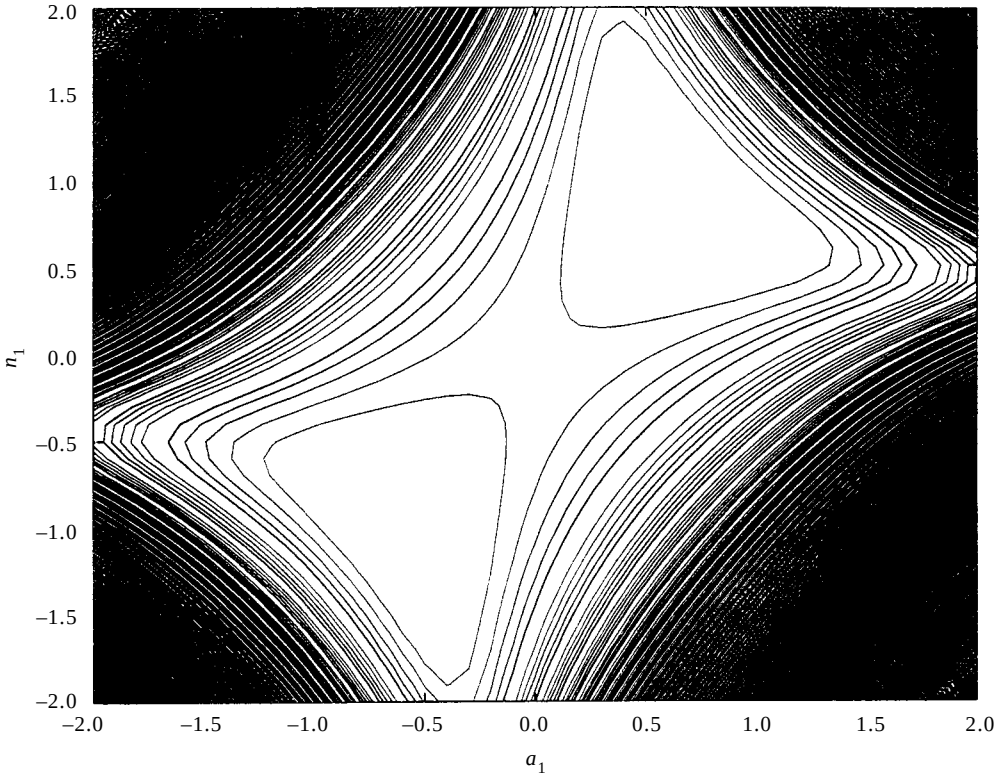

Figure 8. Contour map of the cost function used in the fourth order spectral estimation method.

technique permits the estimation of the unknown parameters with high accuracy. More specifically, a comparison between results obtained using both second and fourth order spectral estimation methods has shown that the use of higher order spectra has the significant advantage that it enables the separate contributions from the linear and non-linear damping components to be accurately determined. It has been demonstrated that the computational effort involved in the minimization of the appropriate cost functions can be significantly reduced by confining the calculation to a reduced region within the principal domain of the fourth order spectra.

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REFERENCES

1. J. B. Roberts 1985 *Journal of Ship Research* **29**, 127–138. Estimation of non-linear ship roll damping from free-decay data.
2. J. B. Roberts and N. M. C. Dacunha 1985 *Journal of Ship Research* **29**, 112–126. The roll motion of a ship in random beam waves: comparison between theory and experiment.

3. P. J. Gawthrop, A. Kountzeris and J. B. Roberts 1988 *Journal of Ship Research* **32**, 101–111. Parametric identification of non-linear ship roll motion from forced roll data.
4. A. Kountzeris, J. B. Roberts and P. J. Gawthrop 1990 *Transactions of the Royal Institution of Naval Architects* **132**, 253–266. Estimation of ship roll parameters from motion in irregular seas.
5. J. B. Roberts, A. Kountzeris and P. J. Gawthrop 1991 *International Shipbuilding Progress* **38**, 271–293. Parametric techniques for roll decrement data.
6. H. Unbehauen and G. P. Rao 1990 *Automatica* **26**, 23–35. Continuous time approaches to system identification—a survey.
7. H. J. Rice and J. A. Fitzpatrick 1991 *ASME Journal of Vibration and Acoustics* **113**, 132–140. The measurement of non-linear damping in single-degree-of-freedom systems.
8. H. J. Rice and J. A. Fitzpatrick 1991 *Journal of Sound and Vibration* **149**, 397–411. A procedure for the identification of linear and non-linear multi-degree-of-freedom systems.
9. A. Debonos 1993 *D. Phil. Thesis*, *University of Sussex*, *U.K.* Estimation of non-linear ship roll parameters using stochastic identification techniques.
10. J. B. Roberts, J. F. Dunne and A. Debonos 1992 *ASME Journal of Offshore Mechanics and Arctic Engineering* **114**, 114–121. Estimation of ship roll parameters in random waves.
11. J. B. Roberts, J. F. Dunne and A. Debonos 1994 *Probabilistic Engineering Mechanics* **9**, 83–93. Stochastic estimation methods for non-linear ship roll motion.
12. J. B. Roberts, J. F. Dunne and A. Debonos 1996 *Proceedings of the IUTAM Symposium on Advances in Non-linear Stochastic Mechanics*, *Trondheim*, *Norway* (A. Naess and S. Krenk, editors) 361–372. Dordrecht: Kluwer Academic Publishers. Parameter estimation for randomly excited non-linear systems: a method based on moment equations and measured response histories.
13. J. B. Roberts, J. F. Dunne and A. Debonos 1995 *Probabilistic Engineering Mechanics* **10**, 199–207. A spectral method for estimation of non-linear system parameters from measured response.
14. G. I. Schueller (editor) 1991 *Structural Dynamics*. Berlin: Springer Verlag.
15. A. Blanc-Lapierre and R. Fortet 1953 *Theories des Functions Aleatories*. Paris: Masson.
16. A. N. Shiryaev 1960 *Theory of Probability and its Applications* **5**, 265–284. Some problems in the spectral theory of higher order moments.
17. D. R. Brillinger 1965 *Annals of Mathematical Statistics* **36**, 1351–1374. An introduction to polyspectra.
18. D. R. Brillinger and M. Rosenblatt 1967 in *Spectral Analysis of Time Series* (B. Harris, editor) 189–232. New York: Wiley. Computation and Interpretation of kth Order Spectra.
19. V. Chandran and S. Elgar 1994 *IEEE Transactions on Signal Processing* **42**, 229–233. A general procedure for the derivation of principal domains of higher-order spectra.
20. MATLAB, The Math Works Inc., Natick, MA, U.S.A.
21. J. B. Roberts and P. D. Spanos 1990 *Random Vibration and Statistical Linearization*. Chichester, U.K.: Wiley.
22. J. B. Roberts and M. Battaini 1997 *Proceedings of the ESREL '97 International Conference on Safety and Reliability*, *Lisbon*, *Portugal* (G. Soares, editor) Vol. 2, pp. 1232–1330. Oxford: Pergamon.

APPENDIX A

Using equation (5), the expression for W given by equation (24) can be rewritten as

$$W^{(4)}_{ijkl}(\omega_1, \omega_2, \omega_3\,;\boldsymbol{\theta}) = \alpha(\omega_1, \omega_2, \omega_3\,;\boldsymbol{\theta}) + \beta_{ij}\,(\omega_1, \omega_2\,;\boldsymbol{\theta})\gamma_{kl}\,(\omega_1, \omega_2, \omega_3\,;\boldsymbol{\theta}) + \beta_{ik}\,(\omega_1, \omega_3\,;\boldsymbol{\theta})\gamma_{jl}\,(\omega_1, \omega_3, \omega_2\,;\boldsymbol{\theta}) + \beta_{il}\,(\omega_2, \omega_3\,;\boldsymbol{\theta})\gamma_{jk}\,(\omega_3, \omega_2, \omega_1\,;\boldsymbol{\theta}). \tag{A1}$$

$$\alpha = \frac{1}{8\pi^3 T}\int_0^T\int_0^T\int_0^T\int_0^T K^{(4)}_{ijkl}(s, t, u, v;\boldsymbol{\theta})D(s)D(t)D(u)D(v) \times \mathrm{e}^{-\mathrm{i}[\omega_1 s + \omega_2 t + \omega_3 u - (\omega_1+\omega_2+\omega_3)v]}\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}v, \tag{A2}$$

$$\beta_{mn}(\omega_1, \omega_2; \boldsymbol{\theta}) = \frac{1}{\sqrt{8\pi^3 T}} \int_0^T \int_0^T R^{(2)}_{mn}(t-s; \boldsymbol{\theta}) D(t) D(s)\, \mathrm{e}^{-[\omega_1 s + \omega_2 t]}\, \mathrm{d}s\, \mathrm{d}t, \tag{A3}$$

$$\gamma_{mn}(\omega_1, \omega_2, \omega_3; \boldsymbol{\theta}) = \frac{1}{\sqrt{8\pi^3 T}} \int_0^T \int_0^T R^{(2)}_{mn}(t-s; \boldsymbol{\theta}) D(t) D(s)\, \mathrm{e}^{-[\omega_3 s - (\omega_1 + \omega_2 + \omega_3)t]}\, \mathrm{d}s\, \mathrm{d}t. \tag{A4}$$

The evaluation of the α term may be considerably simplified if the correlation time scale of the excitation is much smaller than the block length, T. Then, using the symmetry of the correlation–cumulant function, $K^{(4)}_{ijkl}$, one finds that (using a relationship similar to equation (9))

$$\begin{aligned}\alpha(\omega_1, \omega_2, \omega_3) &\approx \frac{k_4}{8\pi^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K^{(4)}_{ijkl}(\tau_1, \tau_2, \tau_3; \boldsymbol{\theta})\, \mathrm{e}^{-\mathrm{i}(\omega_1 \tau_1 + \omega_2 \tau_2 + \omega_3 \tau_3)}\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2\, \mathrm{d}\tau_3 \\ &= k_4\, P^{(4)}_{ijkl}(\omega_1, \omega_2, \omega_3; \boldsymbol{\theta}),\end{aligned} \tag{A5}$$

where

$$k_4 = \frac{1}{T} \int_0^T D^4(t)\, \mathrm{d}t. \tag{A6}$$

Turning to the evaluation of the β terms, on transforming variables to

$$\eta = t - s, \qquad \mu = \frac{t+s}{2}, \tag{A7}$$

equation (A3) may be rewritten as

$$\begin{aligned}\beta_{mn}(\omega_1, \omega_2; \boldsymbol{\theta}) = \frac{1}{\sqrt{8\pi^3 T}} \iint_{\mathfrak{R}} R^{(2)}_{mn}(\eta; \boldsymbol{\theta}) D\left(\mu - \frac{\eta}{2}\right) D\left(\mu + \frac{\eta}{2}\right) \\ \times \mathrm{e}^{-\mathrm{i}\omega_1 (\mu - \eta/2) - \mathrm{i}\omega_2 (\mu + \eta/2)}\, \mathrm{d}\eta\, \mathrm{d}\mu,\end{aligned} \tag{A8}$$

where the integration range, $\mathfrak{R}$, corresponds to that in equation (A3).

Again, if the excitation correlation time scale is small compared with the block length then a significant simplification is possible. Thus, over the range that $R^{(2)}_{mn}$ is significantly non-zero one has

$$D\left(\mu - \frac{\eta}{2}\right) \approx D(\mu), \qquad D\left(\mu + \frac{\eta}{2}\right) \approx D(\mu), \tag{A9}$$

and hence

$$\beta_{mn}(\omega_1, \omega_2; \boldsymbol{\theta}) \approx \frac{1}{2\pi} \int_{-T}^{T} R^{(2)}_{mn}(\eta; \boldsymbol{\theta})\, \mathrm{e}^{\mathrm{i}((\omega_1 - \omega_2)\eta/2)}\, \mathrm{d}\eta \times \frac{1}{\sqrt{2\pi T}} \int_0^T D^2(\mu)\, \mathrm{e}^{-\mathrm{i}(\omega_1 + \omega_2)\mu}\, \mathrm{d}\mu. \tag{A10}$$

Also

$$\frac{1}{2\pi} \int_{-T}^{T} R^{(2)}_{mn}(\eta; \boldsymbol{\theta})\, \mathrm{e}^{\mathrm{i}((\omega_1 - \omega_2)\eta/2)}\, \mathrm{d}\eta \approx \frac{1}{2\pi} \int_{-\infty}^{\infty} R^{(2)}_{mn}(\eta; \boldsymbol{\theta})\, \mathrm{e}^{\mathrm{i}((\omega_1 - \omega_2)\eta/2)}\, \mathrm{d}\eta = S^{(2)}_{mn}\left(\frac{\omega_1 - \omega_2}{2}\right). \tag{A11}$$

If $\Psi_T(\omega)$ is defined as

$$\Psi_T(\omega) = \frac{1}{\sqrt{2\pi T}} \int_0^T D^2(t)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}t, \tag{A12}$$

then, from equations (A9)–(A11) one has

$$\beta_{mn}(\omega_1, \omega_2 ; \boldsymbol{\theta}) = S^{(2)}_{mn}\left(\frac{\omega_1 - \omega_2}{2}\right)\Psi_T(\omega_1 + \omega_2). \tag{A13}$$

Now, for large T compared with τ_{cor} $\Psi_T(\omega_1 + \omega_2)$ will peak sharply at $\omega_1 \approx -\omega_2$ and the spectrum will be almost constant in the vicinity of this peak. Thus, one can approximate further, for the spectral term in equation (A12), by setting $\omega_1 = -\omega_2$. This gives

$$\beta_{mn}(\omega_1, \omega_2 ; \boldsymbol{\theta}) = S^{(2)}_{mn}(\omega_1)\Psi_T(\omega_1 + \omega_2). \tag{A14}$$

A similar analysis can be applied to approximate the $\gamma_{mn}(\omega_1, \omega_2, \omega_3 ; \boldsymbol{\theta})$ terms, defined by equation (4). The result is as follows:

$$\gamma_{mn}(\omega_1, \omega_2, \omega_3 ; \boldsymbol{\theta}) = S^{(2)}_{mn}(\omega_3)\Psi^*_T(\omega_1 + \omega_2). \tag{A15}$$

It follows that

$$\beta_{mn}(\omega_1, \omega_2 ; \boldsymbol{\theta})\gamma_{uv}(\omega_1, \omega_2, \omega_3 ; \boldsymbol{\theta}) = S^{(2)}_{mn}(\omega_1)S^{(2)}_{uv}(\omega_3)\Delta_T(\omega_1 + \omega_2), \tag{A16}$$

where

$$\Delta_T(\omega) = \Psi_T(\omega)\Psi^*(\omega). \tag{A17}$$

On combining equations (A1) and (A15) the following final result is obtained:

$$\begin{aligned} W^{(4)}_{ijkl}(\omega_1, \omega_2, \omega_3 ; \boldsymbol{\theta}) = {} & k_4\, P^{(4)}(\omega_1, \omega_2, \omega_3 ; \boldsymbol{\theta}) + S^{(2)}_{ij}(\omega_1 ; \boldsymbol{\theta})S^{(2)}_{kl}(\omega_3 ; \boldsymbol{\theta})\Delta_T(\omega_1 + \omega_2) \\ & + S^{(2)}_{ik}(\omega_3 ; \boldsymbol{\theta})S^{(2)}_{jl}(\omega_2 ; \boldsymbol{\theta})\Delta_T(\omega_1 + \omega_3) \\ & + S^{(2)}_{il}(\omega_1 ; \boldsymbol{\theta})S^{(2)}_{jk}(\omega_2 ; \boldsymbol{\theta})\Delta_T(\omega_2 + \omega_3). \end{aligned} \tag{A18}$$

APPENDIX B

$$X = \tfrac{3}{2}s_1 - s_2 - s_3 + \tfrac{1}{4}(s_4 + s_5) \qquad Y = \tfrac{3}{2}c_1 - c_2 - c_3 + \tfrac{1}{4}(c_4 + c_5), \tag{B1}$$

where

$$s_1 = \frac{\sin(2\pi\Omega)}{\Omega}, \qquad s_2 = \frac{\sin[2\pi(\Omega + 1)]}{\Omega + 1}, \qquad s_3 = \frac{\sin[2\pi(\Omega - 1)]}{\Omega - 1},$$

$$s_4 = \frac{\sin[2\pi(\Omega + 2)]}{\Omega + 2}, \qquad s_5 = \frac{\sin[2\pi(\Omega - 2)]}{\Omega - 2}, \tag{B2}$$

and

$$c_1 = \frac{1 - \cos(2\pi\Omega)}{\Omega}, \qquad c_2 = \frac{1 - \cos[2\pi(\Omega + 1)]}{\Omega + 1}, \qquad c_3 = \frac{1 - \cos[2\pi(\Omega - 1)]}{\Omega - 1},$$

$$c_4 = \frac{1 - \cos[2\pi(\Omega + 2)]}{\Omega + 2}, \qquad c_5 = \frac{1 - \cos[2\pi(\Omega - 2)]}{\Omega - 2}. \tag{B3}$$