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FREE VIBRATION ANALYSIS OF A CANTILEVER BEAM CARRYING ANY NUMBER OF ELASTICALLY MOUNTED POINT MASSES WITH THE ANALYTICAL-AND-NUMERICAL-COMBINED METHOD

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The natural frequencies and the corresponding mode shapes of a uniform cantilever beam carrying "any number of" elastically mounted point masses are determined by means of the analytical-and-numerical-combined method (ANCM). One of the key points for the present method is to replace each spring–mass system (with spring constant $k_{m,v}$ and mass magnitude $m_{m,v}$) by a massless "effective" spring with spring constant $k_{eff,v} = k_{m,v}/(1 - \gamma_v^2)$. Where γ_v is the frequency ratio defined by $\gamma_v = \omega_{m,v}/\bar{\omega}$, in which $\omega_{m,v} = \sqrt{k_{m,v}/m_{m,v}}$ is the natural frequency of the vth spring–mass system with respect to the attached beam and $\bar{\omega}$ is the natural frequency of the "constrained" beam. The present method is much better than the conventional finite element method (FEM), since it consumes less than 30% of the CPU time required by the conventional FEM to achieve approximately the same accuracy of the lowest five natural frequencies of the "constrained" beam. It is also superior to the existing analytical (or semi-analytical) approaches, since the latter is available only for the eigenvalue problems with "one or two" elastically mounted point masses but the former (the ANCM) easily solves the eigenvalue problems with "any number of" spring-mass attachments. To confirm the reliability of the present method, all the results obtained from the ANCM were checked by those calculated with the conventional FEM. For this purpose two kinds of techniques were presented to derive the stiffness matrix and mass matrix of the associated finite "constrained" beam element: (i) increasing one degree of freedom for each spring–mass attachment and (ii) replacing each spring–mass attachment by a massless effective spring.

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1. INTRODUCTION

In the introduction of reference [1], Gürgöze indicated that, in our everyday situation, many systems may be modelled as a uniform beam or plate carrying various concentrated elements. There is a number of works dealing with the problem of free and forced transverse vibration of such a constrained beam or plate. The type of concentrated elements includes a tip mass on a cantilever beam [2, 3], many point masses and/or springs arbitrarily distributed along a beam or a plate [4–6]. For simplicity, the effects of shear deformation and rotatory inertia were neglected in the above-mentioned literature. Many researchers have devoted themselves to the study of a Timoshenko beam carrying various concentrated elements [7–9] and the effects of shear deformation and rotatory inertia can be found from these works. As shown by Jen and Magrab [10], the eigenvalue problems

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appearing in existing literature were solved with either the orthogonal expansion theory, the Rayleigh–Ritz/Galerkin method, or the Laplace transformation method. In general, the classical method for finding the natural frequencies of a vibrating system becomes intractable when a beam or plate is constrained by the elastically mounted point masses [11]. Hence, studies in this aspect appear to be fewer [10–16].

In theory, most of the approaches presented in the foregoing references may be extended to solve the eigenvalue problems for a uniform beam or plate carrying any number of concentrated elements. However, in practice, they are not available because of the complexity of the mathematical expressions. For this reason, the total number of concentrated elements (such as elastically mounted point masses, rigidly attached point masses, translational springs, and/or rotational springs) illustrated in the examples of references [1–16] is less than two.

To improve the last drawback of the existing approaches and to save computer time, Wu and Lin [17] presented the analytical-and-numerical-combined method (ANCM) to calculate the natural frequencies and mode shapes of a uniform Bernoulli beam carrying any number of point masses. Hereafter, the same technique was successfully used to solve the eigenvalue problems for a Timoshenko beam carrying any number of translational and rotational springs together with any number of point masses [18] and for a rectangular plate carrying any number of translational springs and point masses [19].

The objective of this paper is to try to employ the ANCM to do the free vibration analysis of a uniform cantilever beam carrying any number of elastically mounted point masses. Although the mode shapes of a vibrating system are also the important information that engineers hope to have in addition to the natural frequencies, most of the existing literature does not provide this information. However, reference [17–20] and the present paper do.

It has been shown in reference [17–20] that use of the ANCM to determine natural frequencies and mode shapes of a uniform beam or plate carrying any number of point masses, translational springs and/or rotational springs is quite effective. To demonstrate the fact that the effectiveness of the ANCM is not affected by the existence of any number of spring–mass systems, a uniform cantilever beam carrying three elastically mounted point masses, three rigidly attached point masses, and three translational springs with arbitrary magnitudes and locations was also illustrated. To confirm the reliability of the theory presented in this paper as well, all the numerical results obtained from the ANCM were checked with the corresponding ones obtained from the conventional FEM.

2. EQUATION OF MOTION FOR A CANTILEVER BEAM CARRYING ANY SPRING–MASS SYSTEMS

For the cantilever beam carrying v elastically mounted point masses $m_{m,v}$ ($v = 1 \sim r$) as shown in Figure 1 and neglecting the effects of shear deformation and rotatory inertia, the equation of motion of the whole vibrating system is given by [15]

$$EI\frac{\partial^4 y(x,t)}{\partial x^4} + \bar{m}\frac{\partial^2 y(x,t)}{\partial t^2} = \sum_{v=1}^{r} F_{m,v}(t)\delta(x - x_{m,v}), \tag{1}$$

where E is the Young's modulus, I is the moment of inertia of the beam's cross-sectional area, $\bar{m}$ is the mass per unit length of the beam, $y(x, t)$ is the transverse deflection of the beam at position x and time t, $F_{m,v}(t)$ is the interaction force between the vth sprung mass ($m_{m,v}$) and the beam, and $\delta(\cdot)$ is the Dirac delta function.

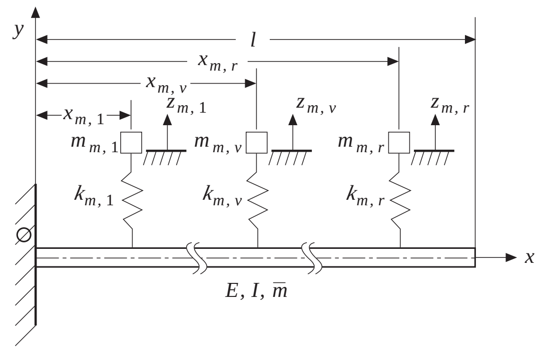

Figure 1. A cantilever beam carrying v elastically mounted point masses $m_{m,v}$ ($v = 1$–r).

For the point mass $m_{m,v}$ mounted on the spring with spring constant $k_{m,v}$ ($v = 1 \sim r$), the equation of motion for the sprung mass is

$$F_{m,v}(t) = -m_{m,v}\, \mathrm{d}^2 z_{m,v}/\mathrm{d}t^2 = -k_{m,v}[y_{m,v}(t) - z_{m,v}], \qquad v = 1, 2, \ldots, r, \tag{2}$$

or

$$m_{m,v}\ddot{z}_{m,v} + k_{m,v} z_{m,v} = k_{m,v} y_{m,v}(t), \qquad v = 1, 2, \ldots, r, \tag{3}$$

where $y_{m,v}(t)$ is the instantaneous transverse deflection of the constrained beam at the position $x = x_{m,v}$ where the vth sprung mass ($m_{m,v}$) is attached, and $\ddot{z}_{m,v}(t)$ and $z_{m,v}(t)$ are the instantaneous vertical acceleration and displacement of $m_{m,v}$ (at time t) respectively.

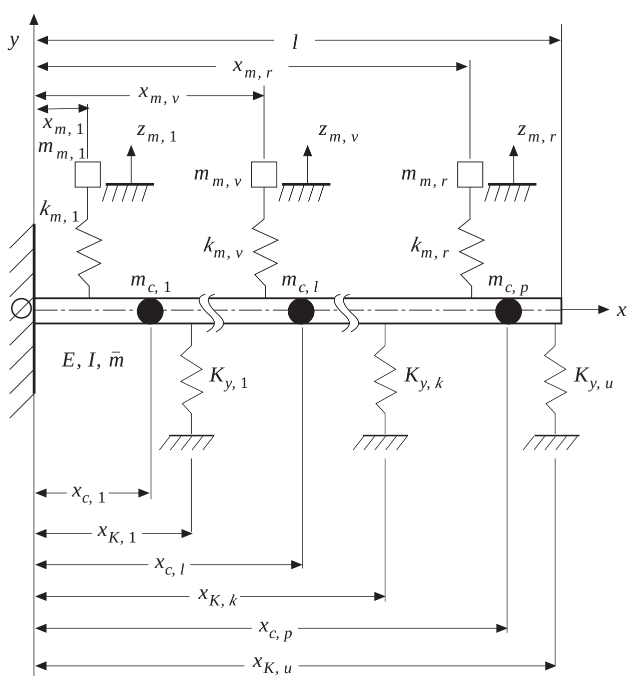

Figure 2. A uniform cantilever beam carrying r elastically mounted point mass $m_{m,v}$ ($v = 1$–r), p rigidly attached point mass $m_{c,l}$ ($l = 1$–p) and u linear springs $K_{y,k}$($k = 1$–u).

According to the expansion theorem [21] or the mode superposition methodology [22], the transverse deflection of the beam shown in Figure 1 may be assumed to be

$$y(x, t) = \sum_{i=1}^{n'} \bar{Y}_i(x) q_i(t), \tag{4}$$

where $\bar{Y}_i(x)$ represents the ith mode shape of the unconstrained beam (without any concentrated elements attached) [17], $q_i(t)$ is the generalized co-ordinate, and n' is the total mode number considered. Hence the value of $y_{m,v}(t)$ appearing in equation (3) may be represented by

$$y_{m,v}(t) = \sum_{i=1}^{n'} \bar{Y}_i(x) q_i(t) \delta(x - x_{m,v}) = \sum_{i=1}^{n'} \bar{Y}_i(x_{m,v}) q_i(t), \tag{5a}$$

where

$$\bar{Y}_i(x_{m,v}) q_i(t) = \bar{Y}_i(x) q_i(t) \delta(x - x_{m,v}). \tag{5b}$$

From equations (3) and (5) one sees that the particular solution of equation (3) takes the form

$$z_{m,v}(t) = \bar{z}_{m,v} \sum_{i=1}^{n'} q_i(t), \tag{6}$$

where $\bar{z}_{m,v}$ represents the amplitude of $z_{m,v}(t)$.

When the constrained beam (Figure 1) performs harmonic free vibration, one has

$$q_j(t) = \bar{q}_j \, e^{i\bar{\omega} t}, \qquad j = 1, 2, \ldots, n', \tag{7}$$

where $\bar{q}_j$ is the amplitude of the jth generalized co-ordinate $q_j(t)$ and $\bar{\omega}$ is the natural frequency of the constrained beam. To substitute equations (5)–(7) into equation (3) one obtains

$$z_{m,v}(t) = \frac{k_{m,v}}{k_{m,v} - m_{m,v}\bar{\omega}^2} \sum_{i=1}^{n'} \bar{Y}_i(x_{m,v}) q_i(t). \tag{8}$$

From equations (2) and (8) one finds that the external exciting force on the beam due to the existence of the elastically mounted attachment ($k_{m,v}$ plus $m_{m,v}$)

$$F_{m,v}(t) = -k_{eff,v} \sum_{i=1}^{n'} \bar{Y}_i(x_{m,v}) q_i(t). \tag{9}$$

where

$$k_{eff,v} = k_{m,v}(1/[1 - \gamma_v^2]), \qquad \gamma_v = \omega_{m,v}/\bar{\omega}, \tag{10a, b}$$

$$\omega_{m,v} = \sqrt{k_{m,v}/m_{m,v}}. \tag{11}$$

It is noted that $\omega_{m,v}$ defined by equation (11) represents the natural frequency of the vth sprung mass $m_{m,v}$ with respect to the still beam, and γ_v represents the ratio of the natural frequency of the spring–mass system ($\omega_{m,v}$) with respect to that of the constrained beam ($\bar{\omega}$).

Equation (9) is an important expression, since it shows that the effect of each spring–mass system ($k_{m,v}$ plus $m_{m,v}$) may be replaced by a general translational spring with effective spring constant $k_{eff,v}$ defined by equation (10a), and this is the key point of the present study.

Substituting equations (4) and (9) into equation (1), multiplying the resulting expression by $\bar{Y}_j(x)$ and then integrating the whole equation over the beam length ℓ, one obtains

$$\int_0^{\ell} \sum_{i=1}^{n'} \bar{Y}_j(x) EI \bar{Y}_i''''(x) q_i(t)\,\mathrm{d}x + \int_0^{\ell} \sum_{i=1}^{n'} \bar{Y}_j(x) \bar{m} \bar{Y}_i(x) \ddot{q}_i(t)\,\mathrm{d}x$$

$$= -\int_0^{\ell} \sum_{v=1}^{r} k_{eff,v} \bar{Y}_j(x_{m,v}) \sum_{i=1}^{n'} \bar{Y}_i(x_{m,v}) q_i(t)\,\mathrm{d}x. \tag{12}$$

If the mode shapes $\bar{Y}_i(x)$ ($i = 1 \sim n'$) are normalized with respect to $\bar{m}$, then application of the orthogonal properties of the normal mode shapes will reduce equation (12) to

$$\ddot{q}_j(t) + \omega_j^2 q_j(t) = -\sum_{v=1}^{r} \sum_{i=1}^{n'} k_{eff,v} \bar{Y}_j(x_{m,v}) \bar{Y}_i(x_{m,v}) q_i. \tag{13}$$

In the above equations, ω_j represents the natural frequency of the unconstrained beam.

The substitution of equation (7) into equation (13) will lead to the following equations of motion for a uniform beam carrying $v(=1\text{–}r)$ elastically mounted point masses (see Figure 1)

$$\omega_j^2 \bar{q}_j + \sum_{v=1}^{r} \sum_{i=1}^{n'} k_{eff,v} \bar{Y}_j(x_{m,v}) \bar{Y}_i(x_{m,v}) \bar{q}_i = \bar{\omega}_j^2 \bar{q}_j. \tag{14}$$

3. NATURAL FREQUENCIES AND MODE SHAPES OF THE CONSTRAINED BEAM

Equation (14) represents a set of n' simultaneous equations. For the convenience of obtaining the solution numerically, they are rewritten in the matrix form

$$[\mathbf{A}]\{\bar{\mathbf{q}}\} = \bar{\omega}^2[\mathbf{B}]\{\bar{\mathbf{q}}\} \tag{15}$$

where

$$[\mathbf{A}]_{n'\times n'} = [\diagdown \boldsymbol{\omega}^2 \diagdown]_{n'\times n'} + [\mathbf{A}']_{n'\times n'}, \qquad [\mathbf{B}]_{n'\times n'} = [\diagdown \mathbf{I} \diagdown]_{n'\times n'}$$

$$[\mathbf{A}']_{n'\times n'} = \sum_{v=1}^{r} k_{eff,v} [\bar{\mathbf{Y}}(x_{m,v})]_{n'\times n'}, \qquad [\bar{\mathbf{Y}}(x)]_{n'\times n'} = \{\bar{\mathbf{Y}}(x)\}_{n'\times 1}\{\bar{\mathbf{Y}}(x)\}^{\mathrm{T}}_{n'\times 1},$$

$$\{\bar{\mathbf{Y}}(x)\}_{n'\times 1} = \{\bar{\mathbf{Y}}_1(x) \quad \bar{\mathbf{Y}}_2(x) \quad \cdots \quad \bar{\mathbf{Y}}_{n'}(x)\}_{n'\times 1}, \qquad \{\bar{\mathbf{q}}\}_{n'\times 1} = \{\bar{q}_1 \quad \bar{q}_2 \quad \cdots \quad \bar{q}_{n'}\}_{n'\times 1}$$

$$[\diagdown \boldsymbol{\omega}^2 \diagdown]_{n'\times n'} = \lceil \omega_1^2 \quad \omega_z^2 \quad \cdots \quad \omega_{n'}^2 \rfloor_{n'\times n'}. \tag{16}$$

The symbols [ ], { } and ⌈ ⌋ appearing in equations (15) and (16) represent the square matrix, column vector and diagonal matrix, respectively.

According to equations (10a) and (10b), the effective spring constant $k_{eff,v}$ for each spring–mass system is a function of natural frequency ($\bar{\omega}$) of the constrained beam, so are the square matrices $[\mathbf{A}']$ and $[\mathbf{A}]$ defined by equation (16). Hence equation (15) cannot be solved with the general Jacobi method, in spite of the fact that it looks like a standard eigenvalue equation. To avoid this trouble, equation (15) is rewritten as

$$([\mathbf{A}] - \bar{\omega}^2[\mathbf{B}])\{\bar{\mathbf{q}}\} = \mathbf{0} \quad \text{or } [\mathbf{C}]\{\bar{\mathbf{q}}\} = \mathbf{0}, \tag{17a, b}$$

where

$$[\mathbf{C}] = [\mathbf{A}] - \bar{\omega}^2[\mathbf{B}]. \tag{18}$$

The non-trivial solution of equation (17) requires that

$$|C| = |[\mathbf{A}] - \bar{\omega}^2[\mathbf{B}]| = 0, \tag{19}$$

which is the frequency equation for the constrained beam. Here the half-interval method [23] is used to solve the eigenvalues $\bar{\omega}_j$ ($j = 1$–n') and then the corresponding eigenvectors $\{\bar{q}\}^{(j)}$ is obtained by substituting the values of $\bar{\omega}_j$ into equation (17).

Since the coefficient determinant of equation (17), $|C|$, is equal to zero for each specified eigenvalue $\bar{\omega}_j$, the corresponding eigenvector $\{\bar{\mathbf{q}}'\}^{(j)}$ is determined from the following equation

$$[\mathbf{C}']\{\bar{\mathbf{q}}'\}^{(j)} = -\{\mathbf{I}\}\bar{q}_k, \tag{20}$$

where $[\mathbf{C}']$ is a $(n' - 1) \times (n' - 1)$ square matrix obtained from the $n' \times n'$ square matrix $[\mathbf{C}]$ by eliminating the kth row and kth column, $\{\bar{\mathbf{q}}'\}^{(j)}$ is a $(n' - 1) \times 1$ column vector obtained from the $n' \times 1$ column vector by eliminating the kth row, $\{\mathbf{I}\}$ is a $(n' - 1) \times 1$ unit column vector defined by

$$\{\mathbf{I}\} = \{1 \quad 1 \quad \cdots \quad 1\}_{(n'-1)\times 1} \tag{21}$$

and $\bar{q}_k$ is the kth coefficient of the $n' \times 1$ column vector $\{\bar{\mathbf{q}}\}^{(j)}$.

Equation (20) is a general simultaneous equation, and various techniques may be used to solve the values of $\{\bar{\mathbf{q}}'\}^{(j)}$. For example,

$$\{\bar{\mathbf{q}}'\}^{(j)} = -[\mathbf{C}]'^{-1}\{\mathbf{I}\}\bar{q}_k. \tag{22}$$

Finally, the mode shapes of the "constrained" beam are determined by

$$\tilde{y}_j(x) = \sum_{i=1}^{n'} \bar{Y}_i(x)\bar{q}_i^{(j)} = \{\bar{\mathbf{Y}}(x)\}^{\mathrm{T}}\{\bar{\mathbf{q}}\}^{(j)}, \qquad j = 1, 2, \ldots, n'. \tag{23}$$

It is noted that all the $(n' - 1)$ coefficients of the $n' \times 1$ column vector $\{\bar{\mathbf{q}}\}^{(j)}$ appearing in equation (23) are exactly equal to those of the $(n' - 1) \times 1$ column vector $\{\bar{\mathbf{q}}'\}^{(j)}$ determined by equation (22), except the kth coefficient is equal to 1·0. The physical meaning of equation (22) is that the $n' \times 1$ eigenvector $\{\bar{\mathbf{q}}\}^{(j)}$ is obtained by expressing all its $(n' - 1)$ coefficients as the ratios between them and the kth coefficient $\bar{q}_k$. As stated above one may set $\bar{q}_k = 1{\cdot}0$ for simplicity, otherwise one may define the value of $\bar{q}_k$ with the following orthogonal relationship

$$\int_0^{\ell} \bar{m}\bar{q}_k^2\{\bar{\mathbf{q}}\}^{(j)\mathrm{T}}\{\bar{\mathbf{q}}\}^{(j)}\,\mathrm{d}x = 1{\cdot}0 \tag{24}$$

where $\bar{m}$ is the mass per unit length of the beam.

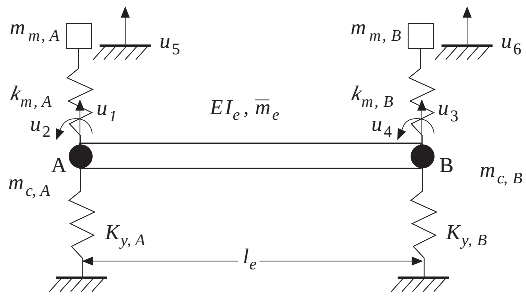

Figure 3. A constrained beam element carrying three kinds of concentrated attachments.

4. EQUATION OF MOTION FOR A CANTILEVER BEAM CARRYING VARIOUS CONCENTRATED ELEMENTS

In addition to the r elastically mounted point masses $m_{m,v}$ ($v = 1, 2, \ldots, r$) as shown in Figure 1. Figure 2 shows a cantilever beam further carrying p rigidly attached point masses $m_{c,1}(l = 1, 2, \ldots, p)$ and u translational linear springs with spring constants $K_{y,k}$ ($k = 1, 2, \ldots, u$). From equation (14) of this paper and equation (12) of reference [18] one may infer that the equations of motion of such a constrained beam are given by

$$\omega_j^2 \bar{q}_j + \sum_{v=1}^{r} \sum_{i=1}^{n'} k_{eff,v} \overline{Y}_j(x_{m,v}) \overline{Y}_i(x_{m,v}) \bar{q}_i + \sum_{k=1}^{u} \sum_{i=1}^{n'} K_{y,k} \overline{Y}_j(x_{K,k}) \overline{Y}_i(x_{K,k}) \bar{q}_i$$

$$= \bar{\omega}_j^2 \bar{q}_j + \bar{\omega}_j^2 \sum_{l=1}^{p} \sum_{i=1}^{n'} m_{c,l} \overline{Y}_j(x_{c,l}) \overline{Y}_i(x_{c,l}) \bar{q}_i, \qquad j = 1, 2, \ldots, n', \tag{25}$$

or

$$[\tilde{\mathbf{A}}]\{\bar{\mathbf{q}}\} = \bar{\omega}^2 [\tilde{\mathbf{B}}]\{\bar{\mathbf{q}}\}, \tag{26}$$

TABLE 1

The lowest five natural frequencies $\bar{\omega}_i$ ($i = 1$–5) for a cantilever beam carrying a spring–mass system

Location $x_1^* = x_{m,1}/\ell$	$k_1^* = \frac{k_{m,1}}{k_b}$	$m_1^* = \frac{m_{m,1}}{m_b}$	Methods	Natural frequencies (rad/s) $\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$	CPU time (s)
0·75	3·0	0·2	FEM	174·2097	322·1653	1415·5823	3964·9742	7767·0263	19·38
			ANCM	174·2097	322·1653	1415·5823	3964·8733	7766·7438	6·15
			Ref· [12]	174·2097	322·1653	1415·5823	3964·9742	7766·7438	—
1·0	100	0·5	FEM	128·6211	971·9848	2131·5152	4210·2821	7879·7358	19·93
			ANCM	128·6211	973·2345	2132·8475	4213·4020	7881·0164	6·39
			Ref· [1]	128·6211	—	—	—	—	—

Note: $\ell = 1{\cdot}0$ m; $k_b = EI/\ell^3 = 6{\cdot}34761 \times 10^4$ N/m; $m_b = \bar{m}\ell = 15{\cdot}3875$ kg.

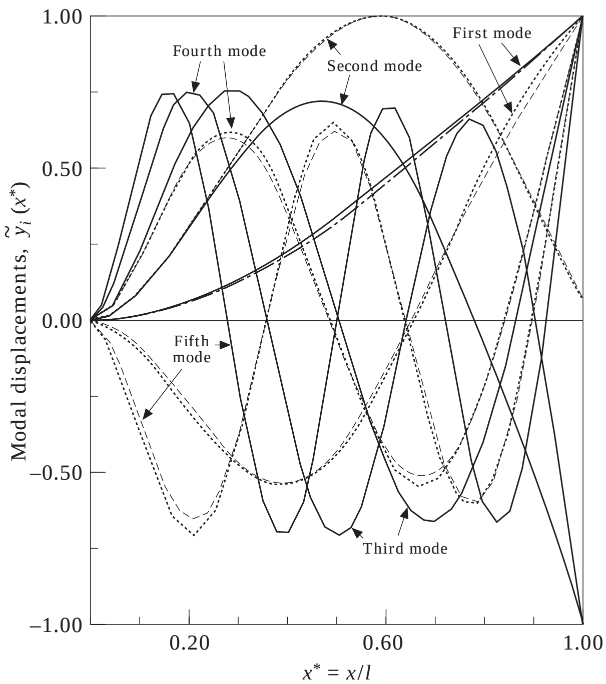

Figure 4. The lowest five mode shapes $\tilde{y}_i(x^*)(i = 1–5)$ for the cantilever beam carrying a spring–mass system at free end ($k_{m,l} = 100k_b = 6{\cdot}34761 \times 10^6$ N/m, $m_{m,l} = 0{\cdot}5m_b = 7{\cdot}69375$ kg). Key: ——, unconstrained beam; - - - -; constrained beam by ANCM; · · · ·, constrained beam by FEM.

where

$$[\tilde{\mathbf{A}}]_{n' \times n'} = [\diagdown \boldsymbol{\omega}^2 \diagdown]_{n' \times n'} + [\mathbf{A}']_{n' \times n'} + [A^*]_{n' \times n'}, \qquad [\tilde{\mathbf{B}}]_{n' \times n'} = [\diagdown \mathbf{I} \diagdown]_{n' \times n'} + [\mathbf{B}']_{n' \times n'},$$

$$[A']_{n' \times n'} = \sum_{v=1}^{r} k_{eff,v}[\bar{\mathbf{Y}}(x_{m,v})]_{n' \times n'}, \qquad [\mathbf{A}^*]_{n' \times n'} = \sum_{k=1}^{u} K_{y,k}[\bar{\mathbf{Y}}(x_{K,k})]_{n' \times n'},$$

$$[\mathbf{B}']_{n' \times n'} = \sum_{l=1}^{P} m_{c,l}[\bar{Y}(x_{c,l})]_{n' \times n'}, \qquad [\bar{Y}(x)]_{n' \times n'} = \{\bar{\mathbf{Y}}(x)\}_{n' \times 1}\{\bar{\mathbf{Y}}(x)\}^{\mathrm{T}}_{n' \times 1}$$

$$\{\bar{\mathbf{Y}}(x)\}_{n' \times 1} = \{\bar{Y}_1(x) \quad \bar{Y}_2(x) \quad \cdots \quad \bar{Y}_{n'}(x)\}_{n' \times 1}$$

$$\{\bar{\mathbf{q}}\}_{n' \times 1} = \{\bar{q}_1 \quad \bar{q}_2 \quad \cdots \quad \bar{q}_{n'}\}_{n' \times 1}, \qquad [\diagdown \boldsymbol{\omega}^2 \diagdown]_{n' \times n'} = \lceil \omega_1^2 \quad \omega_2^2 \quad \cdots \quad \omega_{n'}^2 \rfloor_{n' \times n'}. \tag{27}$$

By means of the same technique as shown in the last section, one may determine the natural frequencies and the corresponding mode shapes of the constrained beam (Figure 2) from equation (26).

TABLE 2

The lowest five natural frequencies $\bar{\omega}_i$ $(i = 1\text{–}5)$ *for the cantilever beam carrying three spring–mass systems* $(k_{m,1} = 3k_b,\ m_{m,1} = 0{\cdot}2m_b;\ k_{m,2} = 4{\cdot}5k_b,\ m_{m,2} = 0{\cdot}5m_b;\ k_{m,3} = 6k_b,\ m_{m,3} = 1{\cdot}0m_b)$

Locations of the spring–mass systems x_1^*	x_2^*	x_3^*	Methods	Natural frequencies (rad/s) $\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$	CPU time (s)
0·1	0·4	0·8	FEM	102·7987	188·7388	248·6632	349·1174	1428·0327	19·18
			ANCM	102·7175	188·7607	248·5116	349·1476	1427·9722	5·05

Note: $\ell = 1{\cdot}0$ m; $k_b = EI/\ell^3 = 6{\cdot}34761 \times 10^4$ N/m; $m_b = \bar{m}\ell = 15.3875$ kg.

TABLE 3

The locations and magnitudes of the ten elastically mounted masses (*cf. Figure* 1)

Elastically mounted masses number	1	2	3	4	5	6	7	8	9	10
Locations $x_1^* = x_{m,1}/\ell$	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
Magnitudes of spring constants $k_i^* = k_i/k_b$	3·0	4·8	3·0	4·5	4·0	5·5	5·0	6·0	4·5	2·6
Magnitudes of point masses $m_i^* = m_i/m_b$	0·2	0·6	0·2	0·5	0·3	0·8	0·65	1·0	0·5	0·1

Note: $\ell = 1{\cdot}0$ m; $k_b = EI/\ell^3 = 6{\cdot}34761 \times 10^4$ N/m; $m_b = \bar{m}\ell = 15{\cdot}3875$ kg.

TABLE 4

The lowest five natural frequencies $\bar{\omega}_i$ $(i = 1\text{–}5)$ *for a cantilever beam carrying ten spring–mass systems shown in Table* 3

Methods	Natural frequencies (rad/s) $\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$	CPU time (s)
FEM	77·4452	162·8876	172·2835	181·0045	183·0103	22·08
ANCM	77·4453	162·8975	172·2979	181·0118	183·0263	8·98

TABLE 5

The locations and magnitudes of the three kinds of concentrated elements shown in Figure 7

Concentrated elements	Locations $x_i^* = x_i/\ell$ x_1^*	x_2^*	x_3^*	Magnitudes of spring constants $k_i^* = k_i/k_b$ k_1^*	k_2^*	k_3^*	Magnitudes of point masses $m_i^* = m_i/m_b$ m_1^*	m_2^*	m_3^*
Elastically mounted point masses $m_{m,i}$	0·1	0·4	0·8	3	4·5	6	0·2	0·5	1·0
Rigidly attached point masses $m_{c,j}$	0·2	0·5	0·9	—	—	—	0·6	0·3	0·5
Translational springs $K_{y,i}$	0·3	0·7	1·0	3	5	2	—	—	—

Note: $\ell = 1{\cdot}0$ m; $k_b = EI/\ell^3 = 6{\cdot}34761 \times 10^4$ N/m; $m_b = \bar{m}\ell = 15{\cdot}3875$ kg.

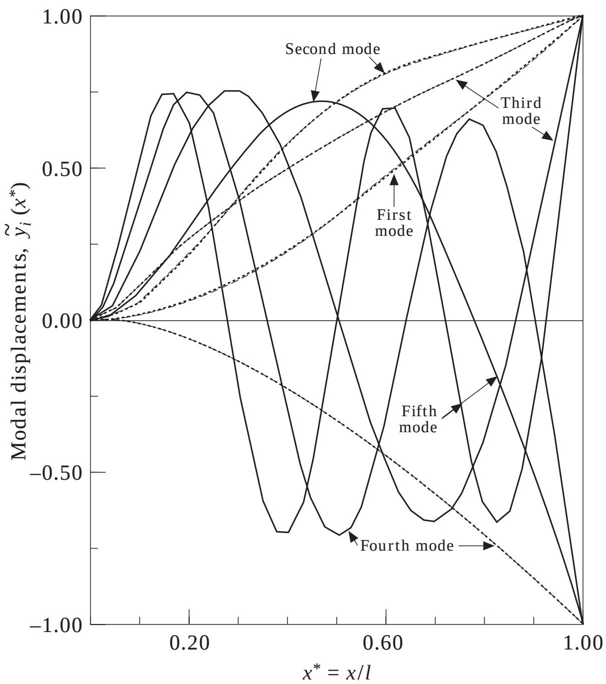

Figure 5. The lowest five mode shapes $\tilde{y}_i(x^*)$ (k = 1–5) for the cantilever beam carrying three spring-mass systems with spring constants: $k_{m,1} = 3k_b$, $k_{m,2} = 4{\cdot}5k_b$, $k_{m,3} = 6k_b$ and point masses: $m_{m,1} = 0{\cdot}2$ m_b, $m_{m,2} = 0{\cdot}5$ m_b, $m_{m,3} = 1{\cdot}0\mathrm{m}_b$, located at $x_1^* = x_{m,1}/\ell = 0{\cdot}1$, $x_2^* = x_{m,2}/\ell = 0{\cdot}4$, $x_3^* = x_{m,3}/\ell = 0{\cdot}8$. Key as for Figure 4.

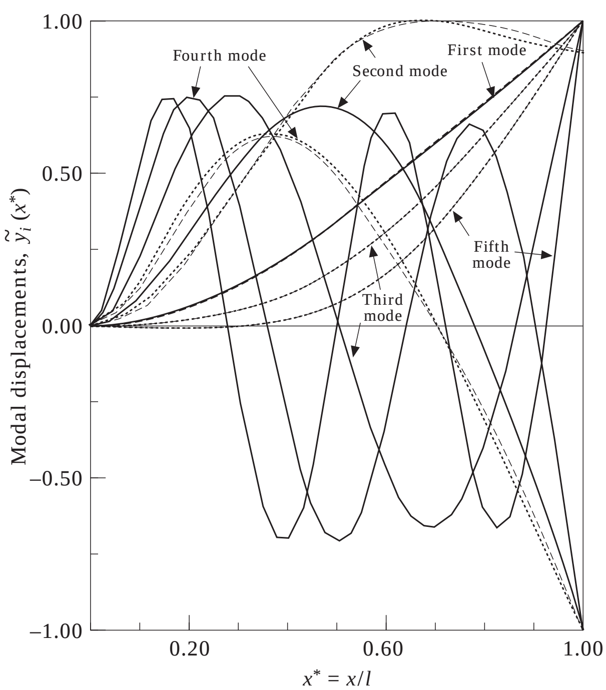

Figure 6. The lowest five mode shapes $\tilde{y}_i(x^*)$ (i = 1–5) for the cantilever beam carrying ten elastically mounted point masses with locations and magnitudes shown in Table 3. Key as for Figure 4.

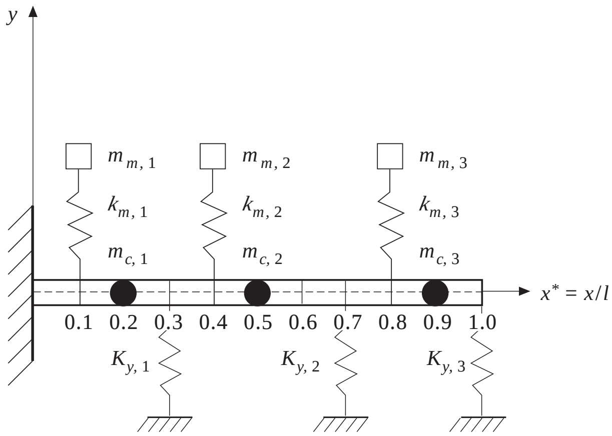

Figure 7. A cantilever beam carrying 3 elastically mounted point masses $m_{m,v}$ ($v = 1$–3), 3 rigidly attached point masses $m_{c,l}$ ($l = 1$–3) and three translational springs $K_{y,k}$ ($k = 1$–3).

TABLE 6

The lowest five natural frequencies $\bar{\omega}_i$ ($i = 1$–5) for a cantilever beam carrying three kinds of concentrated elements shown in Figure 7

	Natural frequencies (rad/s)					
Methods	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$	CPU time (s)
FEM	120·1370	188·8055	248·6301	274·0269	961·9332	18·18
ANCM	120·1499	188·8130	248·6481	274·0953	962·6189	5·36

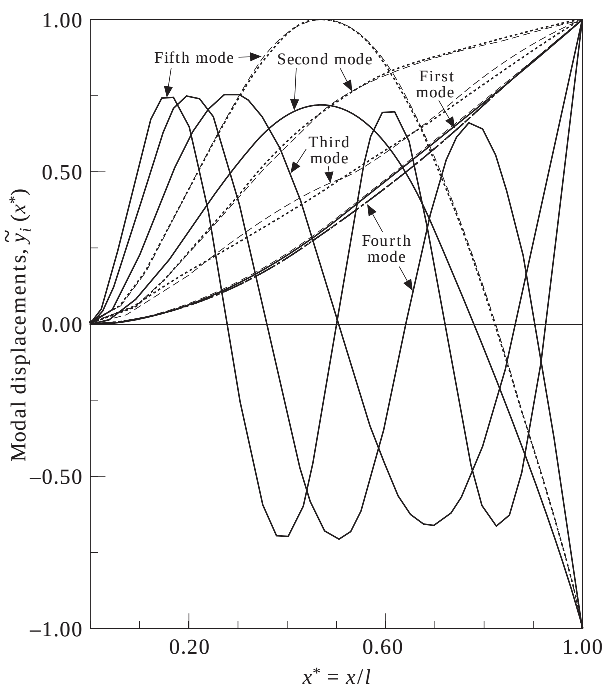

Figure 8. The lowest five mode shapes $\tilde{y}_i(x^*)$ ($i = 1$–5) for the cantilever beam carrying three elastically mounted point masses, three rigidly attached point masses and three translational springs with locations and magnitudes shown in Table 5.

5. ELEMENT MASS MATRIX AND STIFFNESS MATRIX FOR THE CONVENTIONAL FINITE ELEMENT ANALYSIS

In order to confirm the reliability of the presented theory and prove the effectiveness of the presented method, all the natural frequencies $\bar{\omega}_j$ ($j = 1$–n') obtained from the ANCM are checked with those obtained from the conventional FEM. The element property matrices required by the latter are derived below.

5.1. INCREASING ONE DEGREE OF FREEDOM FOR EACH SPRUNG MASS

Figure 3 shows a constrained beam element carrying two elastically mounted point masses ($m_{m,A}$ and $m_{m,B}$), two rigidly attached point masses ($m_{c,A}$ and $m_{c,B}$), and two linear springs ($K_{y,A}$ and $K_{y,B}$), each located at the two nodes of the beam element (A and B). The mass matrix $[\mathbf{M}_e]$ and stiffness matrix $[\mathbf{K}_e]$ of such a constrained beam element are given by

$$[\mathbf{M}_e] = \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \left[\begin{array}{cccc|cc} M_{e11}+m_{c,A} & M_{e12} & M_{e13} & M_{e14} & 0 & 0 \\ M_{e21} & M_{e22} & M_{e23} & M_{e24} & 0 & 0 \\ M_{e31} & M_{e32} & M_{e33}+m_{c,B} & M_{e34} & 0 & 0 \\ M_{e41} & M_{e42} & M_{e43} & M_{e44} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & m_{m,A} & 0 \\ 0 & 0 & 0 & 0 & 0 & m_{m,B} \end{array}\right] \end{array} \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4, \\ u_5 \\ u_6 \end{array} \qquad (28)$$

$$[\mathbf{K}_e] = \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \left[\begin{array}{cccc|cc} K_{e11}+k_{m,A}+K_{y,A} & K_{e12} & K_{e13} & K_{e14} & -k_{m,A} & 0 \\ K_{e21} & K_{e22} & K_{e23} & K_{e24} & 0 & 0 \\ K_{e31} & K_{e32} & K_{e33}+k_{m,B}+K_{y,B} & K_{e34} & 0 & -k_{m,B} \\ K_{e41} & K_{e42} & K_{e43} & K_{e44} & 0 & 0 \\ \hline -k_{m,A} & 0 & 0 & 0 & k_{m,A} & 0 \\ 0 & 0 & -k_{m,B} & 0 & 0 & k_{m,B} \end{array}\right] \end{array} \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4, \\ u_5 \\ u_6 \end{array} \qquad (29)$$

In the last two equations the coefficients M_{eij} and K_{eij} ($i, j = 1$–4) are equal to those of the mass matrix and stiffness matrix for an unconstrained beam element respectively. One may find their values from general textbooks [22, 24].

5.2. REPLACING EACH SPRING–MASS SYSTEM BY AN EFFECTIVE SPRING

In section 2, it has been shown that the influence of a point mass $m_{m,v}$ elastically mounted by a spring with spring constant $k_{m,v}$ on the attached beam is the same as a linear spring with spring constant $k_{eff,v}$ defined by equations (10) and (11). In view of this fact, the mass matrix $[\mathbf{M}_e]$ and stiffness matrix $[\mathbf{K}_e]$ for the constrained beam element shown in Figure 3 may also be evaluated by

$$[\mathbf{M}_e] = \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \left[\begin{array}{cccc} M_{e11}+m_{c,A} & M_{e12} & M_{e13} & M_{e14} \\ M_{e21} & M_{e22} & M_{e23} & M_{e24} \\ M_{e31} & M_{e32} & M_{e33}+m_{c,B} & M_{e34} \\ M_{e41} & M_{e42} & M_{e43} & M_{e44} \end{array}\right] \end{array} \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array}, \tag{30}$$

$$[\mathbf{K}_e] = \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \left[\begin{array}{cccc} K_{e11}+k_{eff,A}+K_{y,A} & K_{e12} & K_{e13} & K_{e14} \\ K_{e21} & K_{e22} & K_{e23} & K_{e24} \\ K_{e31} & K_{e32} & K_{e33}+k_{eff,B}+K_{y,B} & K_{e34} \\ K_{e41} & K_{e42} & K_{e43} & K_{e44} \end{array}\right] \end{array} \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \tag{31}$$

The values of the coefficient M_{eij} and K_{eij} $(i, j = 1\text{–}4)$ in equations (30) and (31) are the same as those shown in equations (28) and (29).

By means of conventional FEM, the natural frequencies $\bar{\omega}_j$ and the corresponding mode shapes $\tilde{y}_j(x)$, $j = 1, 2, \ldots,$ for the constrained beam as shown in Figures 1 and 2 are determined from the following equation of motion

$$[\mathbf{M}]\{\ddot{\mathbf{u}}\} + [\mathbf{K}]\{\mathbf{u}\} = \mathbf{0}, \tag{32}$$

where $[\mathbf{M}]$ and $[\mathbf{K}]$ are the overall mass matrix and overall stiffness matrix for the whole constrained beam obtained from assembling the associated element mass matrices $[\mathbf{M}_e]$ and the element stiffness matrices $[\mathbf{K}_e]$ defined by equations (28)–(31) and imposing the specified boundary conditions respectively. In equation (32), $\{\ddot{\mathbf{u}}\}$ and $\{\mathbf{u}\}$ represent the node acceleration vector and displacement vector respectively.

For a researcher who is used to solving the eigenvalue problems with the half-interval method, two kinds of element property matrices given by equations (28)–(31) may all be suitable for him, but those given by equations (30) and (31) may be better because the total degrees of freedom of the whole vibrating system are not affected by the existence of any number of spring–mass systems. However, for the researcher who is familiar with the general Jacobi method [25], the element property matrices defined by equations (28) and (29) should be the only choice for him because the effective spring constants $k_{eff,A}$ and $k_{eff,B}$ appearing in equation (31) are a function of the unknown natural frequencies $\bar{\omega}_j$ $(j = 1, 2, \ldots)$. Since the effectiveness of the half-interval method and that of the Jacobi method have something to do with many factors, it is difficult to say which is better of the two kinds of element property matrices derived in this section.

6. NUMERICAL RESULTS AND DISCUSSIONS

The dimensional and physical properties for the cantilever beam studied here are: $\ell = 1{\cdot}0$ m, $d = 0{\cdot}05$ m, $E = 2{\cdot}069 \times 10^{11}\ N/m^2$, $\rho = 7{\cdot}8367 \times 10^3$ kg/m^3, $\bar{m} = \rho A = 15{\cdot}3875$ kg/m, $I = \pi d^4/64 = 3{\cdot}06796 \times 10^{-7}$ m^4, $m_b = \bar{m}\ell = 15{\cdot}3875$ kg, $k_b = EI/\ell^3 = 6{\cdot}34761 \times 10^4$ N/m. It is worthy of mentioning that m_b represents the total mass of the beam and k_b represents one third (1/3) of the spring constant of a clamped–free beam at free end. Since m_b and k_b are the important mass parameter and stiffness parameter of the cantilever beam, respectively, they are used as the bases of dimensionless parameters $m_i^*(=m_i/m_b)$ and $k_i^*(=k_i/k_b)$, $i = 1, 2, \ldots$, in the following discussions.

Extensive studies show that, for the present problem, the accuracy of the lowest five natural frequencies obtained from the ANCM by superposing five natural modes (i.e., $n' = 5$) is approximately equal to that obtained from the FEM by using twenty beam elements (i.e., $n_e = 20$). Therefore, the following comparisons are based on $n' = 5$ for the ANCM and $n_e = 20$ for the FEM. This criterion is the same as that of references [17] and [18].

6.1. RELIABILITY OF THE THEORY AND THE COMPUTER PROGRAMS

In the existing literature, only the case of a uniform beam or plate carrying one spring–mass system can be found [1, 11–16]. For example, reference [12] determined the natural frequencies of a cantilever beam carrying one elastically mounted point mass at $x_1^* = x_{m,1}/\ell = 0{\cdot}75$, and the dimensionless spring constant and point mass are: $k_1^* = k_{m,1}/k_b = 3{\cdot}0$ and $m_1^* = m_{m,1}/m_b = 0{\cdot}2$. A similar problem was also studied in reference [1], but the sprung mass was located at the free end (i.e., $x_1^* = 1{\cdot}0$) and the dimensionless magnitudes of spring constant and point mass were $k_1^* = 100{\cdot}0$ and $m_1^* = 0{\cdot}5$. For convenience of comparison, the above mentioned two cases are studied here by using the conventional FEM and the presented ANCM. The lowest five natural frequencies of the constrained cantilever beam, $\bar{\omega}_i$ $(i = 1\text{–}5)$ (rad/s), are shown in Table 1 and the corresponding mode shapes for the case of $x_1^* = x_{m,1}/\ell = 1{\cdot}0$ are shown in Figure 4. From Table 1 one sees that the values of $\bar{\omega}_i$ $(i = 1\text{–}5)$ obtained either from the ANCM, reference [12] or reference [1] are very close to those from the FEM. Besides, the lowest five mode shapes $\tilde{y}_i(x^*)$ $(i = 1\text{–}5)$ obtained from the ANCM are also very close to those from the FEM (see Figure 4). Hence, the reliability of the theory presented and the computer programs developed in this paper should be acceptable. It is noted that the eigenvalues presented in references [12] and [1] are the frequency coefficients $\overline{\beta}_i\ell$ and those shown in Table 1 are the natural frequencies $\bar{\omega}_i$, the relationship between them is given by $\bar{\omega}_i = (\overline{\beta}_i\ell)^2\sqrt{EI/\bar{m}\ell^4}$ $(i = 1, 2, \ldots)$.

6.2. A CANTILEVER BEAM CARRYING ANY NUMBER OF SPRING–MASS SYSTEMS

For the case of the cantilever beam carrying three sprung masses located at $x_1^* = 0{\cdot}1$, $x_2^* = 0{\cdot}4$ and $x_3^* = 0{\cdot}8$, respectively, Table 2 shows the lowest five natural frequencies of the constrained cantilever beam, $\bar{\omega}_i$ $(i = 1\text{–}5)$, obtained from the ANCM and those from the conventional FEM. It is evident that two sets of natural frequencies are very close to each other. The corresponding mode shapes are shown in Figure 5 and good agreements are also achieved. For the present case, the spring constants of the three spring-mass systems are $k_{m,1} = 3k_b$, $k_{m,2} = 4{\cdot}5k_b$ and $k_{m,3} = 6k_b$; and the magnitudes of the three sprung masses are $m_{m,1} = 0{\cdot}2m_b$, $m_{m,2} = 0{\cdot}5m_b$ and $m_{m,3} = 1{\cdot}0m_b$.

For the case of the cantilever beam carrying ten spring–mass systems, Table 3 shows the locations and magnitudes of the ten elastically mounted point masses. From Table 4 one can see that the first five natural frequencies of the constrained cantilever beam, $\bar{\omega}_i$ $(i = 1\text{–}5)$, obtained from the ANCM are also very close to those from the conventional FEM. The corresponding mode shapes are shown in Figure 6 and good agreements are also achieved.

6.3. A CANTILEVER BEAM CARRYING VARIOUS CONCENTRATED ELEMENTS

In addition to any number of spring–mass systems studied in the last subsection, the cantilever beam studied in this section further carries three rigidly attached point masses

$m_{c,l}$ ($l = 1$–3) and three translational springs $K_{y,k}$ ($k = 1$–3) as shown in Figure 7. The locations and magnitudes of the three kinds of concentrated elements are summarized in Table 5 and the lowest five natural frequencies $\bar{\omega}_i$ ($i = 1$–5) and the corresponding mode shapes $\tilde{y}_i(x^*)$ are shown in Table 6 and Figure 8, respectively. It is evident that the results obtained with the ANCM and those with the FEM are also in good agreement. Therefore, the ANCM presented in this paper is available for the free vibration analysis of a uniform beam carrying any number of various concentrated elements. The last column of Tables 1, 2, 4, and 6 shows the CPU time required by the ANCM and the FEM. It is noted that the ANCM consumes only 1/4 to 1/3 the CPU time required by the FEM and this is one of the reasons why the ANCM is better than the FEM. The computing machine used here is the IBM PC 486.

In Figures 4–6 and 8 the mode shapes of the ‘‘un-constrained’’ (pure) cantilever beam are represented by the solid lines (——), while those of the ‘‘constrained’’ beams are represented by the dotted lines (. . . . .) if they are obtained from the FEM and by the dashed lines (----) if they are obtained from the ANCM. By comparing the three kinds of curves one sees that the dotted lines and the associated dashed ones are almost coincident, which means that the results of ANCM and those of FEM are in good agreement. However, the mode shapes of the ‘‘unconstrained’’ cantilever beam are different from those of the ‘‘constrained’’ beams significantly. This phenomenon is stated below.

From each of the Figures 4–6 and 8 one sees that the node number for the ith mode shapes of the ‘‘un-constrained’’ cantilever beam (see the solid lines) is $i - 1$, i.e., zero node for the first mode shapes, one node for the second mode shapes, $\cdots$, and four nodes for the fifth mode shapes. The last statement is not true for the mode shapes of the ‘‘constrained’’ beams (see the dotted lines and the dashed lines) except the first mode shapes. For an ‘‘un-constrained’’ cantilever beam, its total mass is distributed uniformly along the beam length. The dynamic equilibrium of such a uniform beam in free vibration requires that the summation of the shear force at the (left) fixed end and the inertia forces of the upward deflected parts together with those of the downward deflected parts of the beam must be zero. Similarly, the summation of the bending moment at the (left) fixed end and the moments induced by the above mentioned inertia forces must also vanish. Of course, the last requirements for the dynamic equilibrium of an ‘‘un-constrained’’ cantilever beam must also be satisfied by the ‘‘constrained’’ beams carrying any number of concentrated elements. However, since the magnitudes and locations of the concentrated elements along the length of the ‘‘constrained’’ beams are arbitrary, the contribution of each concentrated element on the ‘‘dynamic equilibrium’’ of the whole beam is different from case to case. This is the reason why the mode shapes for various cases are different.

7. CONCLUSIONS

The analytical-and-numerical-combined method (ANCM) presented in this paper is available for the determination of natural frequencies and the associated mode shapes of a uniform beam carrying any number of various concentrated elements with reasonable accuracy. It is better than the conventional finite element method (FEM) in saving much computer time and also better than the existing analytical method for tackling problems with the total number of concentrated elements being more than two.

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