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FREE VIBRATION OF POLAR ORTHOTROPIC CIRCULAR PLATES OF VARIABLE THICKNESS WITH ELASTICALLY RESTRAINED EDGE

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Asymmetric vibrations of polar orthotropic circular plates of linearly varying thickness with elastic/rigid support are discussed on the basis of the classical plate theory. An approximate solution of the problem is obtained by the Rayleigh–Ritz method using functions based on static deflection of polar orthotropic circular plates. This type of approximating functions has a faster rate of convergence as compared to the polynomial co-ordinate functions. Frequency parameters of the plate have been obtained for different values of taper parameter, flexibility parameter and rigidity ratio in the first three modes of vibration. A comparison of results for special cases of isotropy, uniform thickness and of axisymmetric vibrations obtained by other techniques such as the finite element method and the Receptance method shows excellent agreement.

1. INTRODUCTION

Plates of uniform thickness are often encountered in engineering applications and their use in machine design, nuclear reactor technology, naval and aerospace structures is quite common. A considerable amount of work has been done on vibration of isotropic circular plates of variable thickness and are reported in [1–16] to mention a few. Quite a large number of workers have considered classical boundary conditions i.e. the edge is clamped or simply supported or free in those studies. In practice the actual conditions on a periphery often tend to be part way between these classical conditions and may correspond more closely to some form of elastic restraint. Therefore some of the research workers have studied vibration of plates with restrained edge conditions. Narita and Leissa [9, 10] have analysed isotropic circular plates having elastic edge constraints. Iglesias *et al*. [11] has studied vibrating circular plates of non-uniform thickness and variable rotational constraints along the edge. Irie *et al*. [12] have studied the free vibration of circular plates elastically restrained along some radial segments. Recently Azimi [13, 14] analysed the free vibration of isotropic circular plates restrained elastically on their boundaries by the Receptance method and has also reviewed much of the earlier work on circular plates of elastically restrained edge. Laura and Gutierrez [15] obtained the natural frequencies of a solid circular isotropic plate of linearly varying thickness with elastically restrained edge conditions for asymmetric vibrations. Pardoen [16] investigated the vibrational characteristics of isotropic circular plates and its stability using finite element method.

In recent years, the analysis of plates exhibiting anisotropic characteristics has received greater attention due to increasing use of fibre-reinforced materials specially in aerospace industries. A number of studies dealing with axisymmetric vibrations of plates possessing polar orthotropy (a special case of anisotropy) has been reported in [17–27]. A survey of

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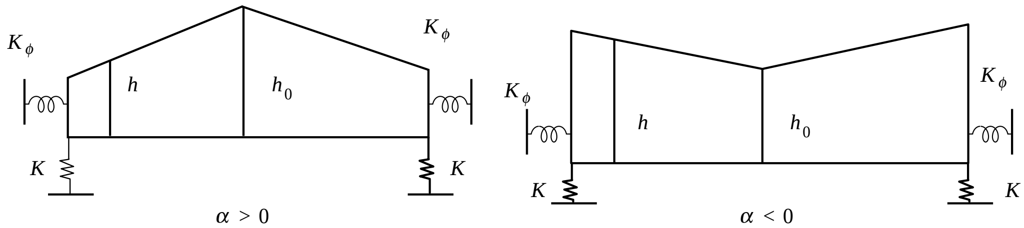

Figure 1

literature shows that no work has been done to study asymmetric vibrations of polar orthotropic circular plates of variable thickness with elastically restrained edge conditions, although an appreciable number of papers [17–27] have appeared dealing with axisymmetric vibration of polar orthotropic circular plates of variable thickness. Keeping the above facts in view, an attempt has been made to study the asymmetric vibrations of polar orthotropic circular plates with elastically restrained edge conditions.

2. ANALYSIS

A circular plate of radius a, variable thickness $h = h(r)$, elastically restrained against rotation and translation at the edge is shown in Figure 1, (r, θ) being the polar co-ordinates of a point in the neutral surface of the plate.

TABLE 1

Frequency parameter Ω as a function of rotational flexibility parameter K_ϕ and rigidity ratio E_θ/E_r $(=p^2)$ for polar orthotropic circular plate of linearly varying thickness for $D_{k0} = 1{\cdot}4$, $\nu_\theta = 0{\cdot}3$, $n = 0$

$K_\phi \backslash p^2$	0·50	0·75	1·00	2·00	5·00
			Ω_{00}		
0·0	3·5300	3·8486	4·1158	4·9404	6·5866
1·0	6·6286	6·9678	7·2518	8·1233	9·8387
10	7·0580	7·4170	7·7176	8·6393	10·4498
10^2	7·1079	7·4694	7·7722	8·7004	10·5239
∞	7·1134	7·4753	7·7783	8·7073	10·5323
			Ω_{01}		
0·0	22·9257	23·9016	24·7266	27·2953	32·4105
1·0	28·5809	29·6474	30·5511	33·3450	38·8403
10	30·1568	31·2721	32·2197	35·1485	40·9023
10^2	30·3605	31·4824	32·4363	35·3863	41·1797
∞	30·3817	31·5063	32·4611	35·4109	41·2083
			Ω_{02}		
0·0	59·1033	60·7071	62·0706	66·3711	74·8634
1·0	66·9414	68·6013	70·0298	74·5116	83·2951
10	70·2048	71·9206	73·4082	78·0785	87·2475
10^2	70·6697	72·3940	73·8914	78·6023	87·8496
∞	70·7156	72·4481	73·9470	78·6468	87·8881

Taper parameter $\alpha = 0{\cdot}3$.

TABLE 2

Frequency parameter Ω as a function of rotational flexibility parameter K_ϕ and rigidity ratio E_θ/E_r $(=p^2)$ for polar orthotropic circular plate of linearly varying thickness for $D_{k0} = 1{\cdot}4$, $\nu_\theta = 0{\cdot}3$, $n = 0$

$K_\phi \backslash p^2$	0·50	0·75	1·00	2·00	5·00
			Ω_{00}		
0·0	4·5961	5·2236	5·7483	7·3565	10·5338
1·0	8·8279	9·3411	9·7804	11·1660	14·0063
10	11·1183	11·7004	12·1950	13·7396	16·8442
10^2	11·5020	12·1027	12·6132	14·2064	17·4058
∞	11·5477	12·1506	12·6631	14·2624	17·4740
			Ω_{01}		
0·0	31·9888	33·3640	34·5626	38·3517	46·0539
1·0	36·5070	37·9110	39·1340	42·9698	50·7260
10	42·2580	43·7914	45·1112	49·2661	57·5782
10^2	43·6597	45·2325	46·5936	50·8755	59·4562
∞	43·8428	45·4165	46·7814	51·0739	59·6640
			Ω_{02}		
0·0	81·5204	83·6799	85·6214	91·8278	104·3346
1·0	86·3140	88·5055	90·4866	96·7145	109·2695
10	95·5420	97·8702	99·9085	106·4580	119·5576
10^2	98·5340	100·8972	102·9978	109·7361	123·4025
∞	98·9578	101·3122	103·4128	110·1452	123·6343

Taper parameter $\alpha = -0{\cdot}3$.

TABLE 3

Frequency parameter Ω as a function of rotational flexibility parameter K_ϕ and rigidity ratio E_θ/E_r $(=p^2)$ for polar orthotropic circular plate of linearly varying thickness for $D_{k0} = 1{\cdot}4$, $\nu_\theta = 0{\cdot}3$, $n = 1$

$K_\phi \backslash p^2$	0·50	0·75	1·00	2·00	5·00
			Ω_{10}		
0·0	10·9143	11·0979	11·2763	11·9465	13·6678
1·0	15·2625	15·4606	15·6528	16·3708	18·1917
10	16·1710	16·3800	16·5828	17·3398	19·2567
10^2	16·2828	16·4932	16·6974	17·4599	19·3906
∞	16·2952	16·5060	16·7104	17·4736	19·4059
			Ω_{11}		
0·0	39·3153	39·7669	40·2048	41·8412	45·9995
1·0	46·1524	46·6330	47·0987	48·8354	53·2261
10	48·5337	49·0361	49·5228	51·3376	55·9166
10^2	48·8572	49·3629	49·8529	51·6801	56·2944
∞	48·8942	49·4004	49·8909	51·7193	56·3375
			Ω_{12}		
0·0	84·4310	85·1508	85·8480	88·4504	95·0358
1·0	93·2053	93·9494	94·6701	97·3572	104·1449
10	97·4771	98·2496	98·9971	101·7870	108·7648
10^2	98·1146	98·8917	99·6448	102·4542	109·5375
∞	98·1883	98·9663	99·7199	102·5288	109·6214

Taper parameter $\alpha = 0{\cdot}3$.

TABLE 4

Frequency parameter Ω as a function of rotational flexibility parameter K_ϕ and rigidity ratio E_θ/E_r $(=p^2)$ for polar orthotropic circular plate of linearly varying thickness for $D_{k0} = 1{\cdot}4$, $\nu_\theta = 0{\cdot}3$, $n = 1$

$K_\phi \backslash p^2$	0·50	0·75	1·00	2·00	5·00
			Ω_{10}		
0·0	15·6852	16·0622	16·4266	17·7804	21·1816
1·0	20·0346	20·3941	20·7418	22·0347	25·2895
10	23·9599	24·3501	24·7269	26·1240	29·6133
10^2	24·7678	25·1695	25·5574	26·9953	30·5850
∞	24·8670	25·2701	25·6595	27·1029	30·7065
			Ω_{11}		
0·0	54·9394	55·6818	56·4004	59·0767	65·8360
1·0	59·6581	60·4030	61·1238	63·8078	70·5803
10	67·2012	67·9908	68·7539	71·5925	78·7260
10^2	69·3470	70·1588	70·9431	73·8604	81·1787
∞	69·6264	70·4412	71·2286	74·1573	81·5206
			Ω_{12}		
0·0	116·5721	117·6888	118·7711	122·8049	133·0107
1·0	121·5183	122·63663	123·7184	127·7565	137·9737
10	132·3776	133·5415	134·6619	138·8493	149·3933
10^2	136·3517	137·5485	138·6958	142·9866	153·7603
∞	136·9001	138·0981	139·2509	143·5608	154·4603

Taper parameter $\alpha = -0{\cdot}3$.

TABLE 5

Frequency parameter Ω as a function of rotational flexibility parameter K_ϕ and rigidity ratio E_θ/E_r $(=p^2)$ for polar orthotropic circular plate of linearly varying thickness for $D_{k0} = 1{\cdot}4$, $\nu_\theta = 0{\cdot}3$, $n = 2$

$K_\phi \backslash p^2$	0·50	0·75	1·00	2·00	5·00
			Ω_{20}		
0·0	18·9931	19·8923	20·6519	22·9782	27·5159
1·0	24·1907	25·1972	26·0516	28·6552	33·6147
10	25·5575	26·6075	27·5051	30·2412	35·4764
10^2	25·7289	26·7897	27·6924	30·4491	35·7301
∞	25·7449	26·8014	27·7142	30·4689	35·7463
			Ω_{21}		
0·0	54·9773	56·4974	57·8072	61·8967	69·8880
1·0	62·6327	64·2050	65·5767	69·8764	78·0666
10	65·7342	67·3522	68·7791	73·2533	81·9554
10^2	66·1626	67·8001	69·2289	73·7443	82·5729
∞	66·2034	67·8291	69·2876	73·7837	82·5520
			Ω_{22}		
0·0	107·7197	109·8317	111·6821	117·5780	129·0883
1·0	117·1965	119·3310	121·2318	127·3341	138·6243
10	122·3699	124·5440	126·5083	132·8160	145·1883
10^2	123·1524	125·3560	127·3314	133·6953	146·4469
∞	123·2288	125·4159	127·4270	133·7472	146·1961

Taper parameter $\alpha = 0{\cdot}3$.

TABLE 6

Frequency parameter Ω as a function of rotational flexibility parameter K_ϕ and rigidity ratio E_θ/E_r $(=p^2)$ for polar orthotropic circular plate of linearly varying thickness for $D_{k0} = 1{\cdot}4$, $v_\theta = 0{\cdot}3$, $n = 2$

$K_\phi \backslash p^2$	0·50	0·75	1·00	2·00	5·00
			Ω_{20}		
0·0	27·8049	29·2575	30·4945	34·3955	42·4325
1·0	32·2304	33·6945	34·9673	38·9947	47·1168
10	37·3452	38·9620	40·3590	44·7667	53·5393
10^2	38·5808	40·2327	41·6721	46·2099	55·2452
∞	38·7281	40·3931	41·8385	46·3914	55·4583
			Ω_{21}		
0·0	77·5427	79·9292	81·9354	88·1755	101·2830
1·0	82·4837	84·7496	86·7510	93·1802	106·0999
10	91·3208	93·7213	95·8215	102·6069	116·1554
10^2	94·1907	96·6111	98·7686	105·7308	119·7154
∞	94·5463	96·9972	99·1654	106·1413	120·1364
			Ω_{22}		
0·0	149·7226	152·9927	155·7348	164·1822	182·8680
1·0	155·0021	158·0218	160·7358	169·6814	187·6123
10	167·0437	170·2216	173·0145	182·3525	201·1283
10^2	172·0161	175·1612	178·0114	187·5679	207·1651
∞	172·6472	175·8515	178·7366	188·2802	207·7349

Taper parameter $\alpha = -0{\cdot}3$.

The maximum kinetic energy of the plate is given by

$$T_{\max} = \frac{1}{2}\rho\omega^2 \int_0^a \int_0^{2\pi} hw^2 r \, \mathrm{d}\theta \, \mathrm{d}r \tag{1}$$

where w is the transverse deflection, ρ the mass density, and ω the frequency in rad/s. The maximum strain energy of the plate is given by

$$U_{\max} = \frac{1}{2}\int_0^a \int_0^{2\pi} \left[D_r \left\{ \left(\frac{\partial^2 w}{\partial r^2}\right)^2 + 2v_\theta \frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right\} + D_\theta \left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)^2 + D_k \left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)\right\}^2 \right] r \, \mathrm{d}r \, \mathrm{d}\theta + \frac{1}{2} ak \int_0^{2\pi} w^2(a,\theta)\, \mathrm{d}\theta + \frac{1}{2} ak_\phi \int_0^{2\pi} \left(\frac{\partial w(a,\theta)}{\partial r}\right)^2 \mathrm{d}\theta \tag{2}$$

k and $1/k_\phi$ are the translational and rotational flexibility of the springs and flexural rigidities of the plate are

$$D_r = \frac{E_r h^3}{12(1 - v_r v_\theta)}, \qquad D_\theta = \frac{E_\theta h^3}{12(1 - v_r v_\theta)}, \qquad D_k = \frac{G_{r\theta} h^3}{3}.$$

The Ritz method requires that the functional

$$J(w) = U_{\max} - T_{\max} = \frac{1}{2}\int_0^a \int_0^{2\pi} \left[D_r \left\{ \left(\frac{\partial^2 w}{\partial r^2} \right)^2 + 2\nu_\theta \frac{\partial^2 w}{\partial r^2} \left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} \right) \right\} \right.$$

$$\left. + D_\theta \left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} \right)^2 + D_k \left\{ \frac{\partial}{\partial r} \left(\frac{1}{r}\frac{\partial w}{\partial \theta} \right) \right\}^2 \right] r \, \mathrm{d}r \, \mathrm{d}\theta$$

$$+ \frac{1}{2} ak \int_0^{2\pi} w^2(a, \theta) \, \mathrm{d}\theta + \frac{1}{2} ak_\phi \int_0^{2\pi} \left(\frac{\partial w(a, \theta)}{\partial r} \right)^2 \mathrm{d}\theta - \frac{1}{2}\rho\omega^2 \int_0^a \int_0^{2\pi} hw^2 r \, \mathrm{d}\theta \, \mathrm{d}r \qquad (3)$$

be minimized. Introducing the non-dimensional variables $\bar{W} = w/a$, $R = r/a$, we assume the deflection function as

$$\bar{W} = W(R) \cos n\theta = \cos n\theta \sum_{i=1}^{m} A_i F_i \qquad (4)$$

where A_i are unknown constants and

$$F_i = (1 + \alpha_i R^4 + \beta_i R^{1+p}) R^{2(i-1)+n} \qquad (5)$$

TABLE 7

Frequency parameter Ω as a function of taper parameter α and rigidity ratio E_θ/E_r ($=p^2$) for polar orthotropic free circular plate of linearly varying thickness for $D_{k0} = 1{\cdot}4$, $\nu_\theta = 0{\cdot}3$ and $n = 0$

$\alpha \backslash p^2$	0·5	0·75	1·0	2·0	5·0
			Ω_{01}		
−0·5	8·8749	9·7959	10·9166	14·3936	21·3268
−0·3	7·9989	9·1521	10·1332	13·1859	19·3075
−0·1	7·5395	8·5290	9·3728	12·0060	17·3140
0·0	7·3185	8·2271	9·0031	11·4256	16·3294
0·1	7·1028	7·9331	8·6418	10·8594	15·3558
0·3	6·6971	7·3737	7·9508	9·7622	13·4502
0·5	6·3367	6·8679	7·3210	8·7395	11·6304
			Ω_{02}		
−0·5	44·3355	46·1507	47·7599	52·9933	63·8196
−0·3	40·9500	42·6037	44·0645	48·7737	58·5430
−0·1	37·5236	39·0168	40·3297	44·5347	53·2103
0·0	35·7981	37·2057	38·4432	42·3640	50·5078
0·1	34·0532	35·3819	36·5407	40·2212	47·7917
0·3	30·5205	31·6774	32·6747	35·8375	42·2710
0·5	26·8825	27·8562	28·6949	31·3199	36·5942
			Ω_{03}		
−0·5	103·9913	106·6243	109·0013	116·8769	132·5134
−0·3	96·0634	98·4822	100·6539	107·7259	122·0104
−0·1	87·9294	90·1389	92·1115	98·4786	111·2886
0·0	83·7899	85·8806	87·7502	93·6607	105·7537
0·1	79·5662	81·5601	83·3152	88·9034	100·1977
0·3	70·8898	72·6481	74·1702	79·0364	88·7237
0·5	61·7218	63·2148	64·5153	68·6007	76·6408

TABLE 8

Frequency parameter Ω as a function of taper parameter α and rigidity ratio E_θ/E_r $(=p^2)$ for polar orthotropic free circular plate of linearly varying thickness for $D_{k0}=1{\cdot}4$, $\nu_\theta=0{\cdot}3$ and $n=1$

$\alpha\backslash p^2$	0·5	0·75	1·0	2·0	5·0
			Ω_{11}		
−0·5	24·3093	24·9862	25·6404	28·0707	34·1754
−0·3	22·4324	23·0162	23·5814	25·6870	31·0064
−0·1	20·5425	21·0352	21·5129	23·2981	27·8367
0·0	19·5919	20·0399	20·4746	22·1016	26·2524
0·1	18·6373	19·0411	19·4333	20·9038	24·6692
0·3	16·7149	17·0328	17·3419	18·5056	21·5089
0·5	14·7784	15·0141	15·2438	16·1115	18·3696
			Ω_{12}		
−0·5	72·8372	73·8472	74·8240	78·4746	87·7654
−0·3	67·1461	68·0379	68·9046	72·1427	80·4064
−0·1	61·3432	62·1214	62·8762	65·7016	72·9119
0·0	58·3911	59·1120	59·8116	62·4309	69·1171
0·1	55·3985	56·0623	56·7066	59·1197	65·2842
0·3	49·2629	49·8125	50·3462	52·3467	57·4671
0·5	42·8543	43·2897	43·7126	45·2988	49·3633
			Ω_{13}		
−0·5	145·5289	146·9409	148·2920	153·3504	166·2272
−0·3	134·3249	135·5703	136·7881	141·3274	152·9886
−0·1	122·8145	123·9202	124·9916	129·0027	139·1939
0·0	116·9239	117·9560	118·9573	122·7030	132·1874
0·1	110·9251	111·8836	112·8136	116·2913	125·0974
0·3	98·5219	99·3315	100·1169	103·0536	110·5147
0·5	85·3683	86·0257	86·6634	89·0476	95·1027

$p^2=E_\theta/E_r$ and α_i, β_i are unknown constants and n is a non-negative integer. Using non-dimensional variables $\bar{W}$ and R along with the relation (4) the functional $J(w)$ given by (3) becomes

$$J(W)=\frac{\pi}{2}D_{r0}\left[\int_0^1(1-\alpha R)^3\left[\left(\frac{\mathrm{d}^2W}{\mathrm{d}R^2}\right)^2+2\nu_\theta\frac{\mathrm{d}^2W}{\mathrm{d}R^2}\left(\frac{1}{R}\frac{\mathrm{d}W}{\mathrm{d}R}-\frac{n^2W}{R^2}\right)\right.\right.$$
$$\left.+p^2\left(\frac{1}{R}\frac{\mathrm{d}W}{\mathrm{d}R}-\frac{n^2W}{R^2}\right)^2+n^2D_{k0}\left\{\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{W}{R}\right)\right\}^2\right]R\,\mathrm{d}R$$
$$\left.+KW^2(1)+K_\phi\left(\frac{\mathrm{d}W(1)}{\mathrm{d}R}\right)^2-\Omega^2\int_0^1(1-\alpha R)W^2R\,\mathrm{d}R\right] \tag{6}$$

where $h=h_0(1-\alpha R)$; α is the taper parameter and h_0 the thickness of the plate at the centre and

$$D_{r0}=\frac{E_rh_0^3}{12(1-\nu_r\nu_\theta)},\qquad D_{k0}=\frac{D_k}{D_{r0}},\qquad \Omega^2=\frac{a^4\omega^2\rho h_0}{D_{r0}},\qquad K=\frac{a^3k}{D_{r0}},\qquad K_\Phi=\frac{ak_\Phi}{D_{r0}}.$$

The minimization of the function $J(W)$ given by (6) requires

$$\frac{\partial J(W)}{\partial A_i} = 0, \qquad i = 1(1)m \tag{7}$$

which leads to a system of homogeneous equations in A_i, $i = 1(1)m$ and its non-trivial solution leads to the frequency equation.

$$|A - \Omega^2 B| = 0 \tag{8}$$

where $A = [a_{ij}]$ and $B = [b_{ij}]$ are square matrices of order m and

$$a_{ij} = \int_0^1 (1-\alpha R)^3 \left[F_i'' F_j'' + 2\nu_\theta F_i'' \left(\frac{F_j'}{R} - \frac{n^2 F_j}{R^2} \right) + p^2 \left(\frac{F_i'}{R} - \frac{n^2 F_i}{R^2} \right) \left(\frac{F_j'}{R} - \frac{n^2 F_j}{R^2} \right) \right.$$
$$\left. + n^2 D_{k0} \left(\frac{F_i'}{R} - \frac{F_i}{R^2} \right) \left(\frac{F_j'}{R} - \frac{F_j}{R^2} \right) \right] R \, \mathrm{d}R$$
$$+ K F_i(1) F_j(1) + K_\Phi F_i'(1) F_j'(1) \tag{9}$$

TABLE 9

Frequency parameter Ω as a function of taper parameter α and rigidity ratio E_θ/E_r $(=p^2)$ for polar orthotropic free circular plate of linearly varying thickness for $D_{k0} = 1{\cdot}4$, $\nu_\theta = 0{\cdot}3$ and $n = 2$

$\alpha\backslash p^2$	0·5	0·75	1·0	2·0	5·0
			Ω_{20}		
−0·5	5·2908	6·1072	6·7389	8·4267	10·9788
−0·3	4·8994	5·6145	6·1680	7·6419	9·8659
−0·1	4·5274	5·1445	5·6203	6·8812	8·7768
0·0	4·3518	4·9212	5·3583	6·5140	8·2443
0·1	4·1862	4·7074	5·1067	6·1555	7·7229
0·3	3·8919	4·3203	4·6451	5·4892	6·7272
0·5	3·6760	4·0138	4·2687	4·9085	5·8381
			Ω_{22}		
−0·5	41·8740	43·6511	45·2245	50·3135	60·9242
−0·3	38·2625	39·8481	41·2519	45·7542	55·0793
−0·1	34·6238	36·0288	37·2625	41·1837	49·2634
0·0	32·7953	34·1125	35·2601	38·9008	46·3356
0·1	30·9645	32·1884	33·2517	36·5852	43·4042
0·3	27·2821	28·3215	29·2132	31·9876	37·5112
0·5	23·5799	24·4200	25·1397	27·2955	31·6239
			Ω_{23}		
−0·5	101·4945	104·2083	106·6512	114·5960	130·7326
−0·3	93·2203	95·6570	97·8589	104·9304	119·1329
−0·1	84·7606	86·9530	88·9118	95·1358	107·7871
0·0	80·4579	82·5335	84·3662	90·2184	101·9203
0·1	76·1095	78·0501	79·7622	85·0859	96·0067
0·3	67·2020	68·8755	70·3342	74·9216	83·8644
0·5	57·9341	59·2997	60·4991	64·1276	71·5549

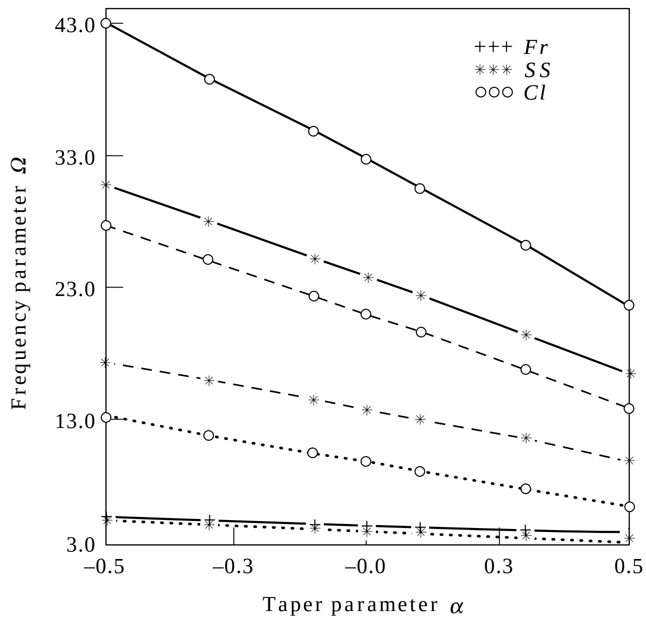

Figure 2. Frequency parameter in fundamental mode for $v_\theta = 0{\cdot}3$ and $E_\theta/E_r = 0{\cdot}5$ for $n = 0$ (---), $n = 1$ (–––) and $n = 2$ (——) LVT circular plates.

and

$$b_{ij} = \int_0^1 (1 - \alpha R) F_i F_j R \, \mathrm{d}R. \tag{10}$$

As each co-ordinate function (5) has to satisfy the boundary conditions [1, p. 14], we have

$$K_\phi \frac{\mathrm{d}W(1)}{\mathrm{d}R} = -(1-\alpha)^3 \left[\frac{\mathrm{d}^2 W}{\mathrm{d}R^2} + v_\theta \left(\frac{1}{R} \frac{\mathrm{d}W}{\mathrm{d}R} - n^2 \frac{W}{R^2} \right) \right]_{R=1} \tag{11}$$

$$KW(1) = (1-\alpha)^3 \left[\frac{\mathrm{d}}{\mathrm{d}R} \left(\frac{\mathrm{d}^2 W}{\mathrm{d}R^2} + \frac{1}{R} \frac{\mathrm{d}W}{\mathrm{d}R} - n^2 \frac{W}{R^2} \right) - \frac{1}{2} D_{k0} \left(\frac{1}{R} \frac{\mathrm{d}W}{\mathrm{d}R} - \frac{W}{R^2} \right) \right]_{R=1}. \tag{12}$$

The unknown constants α_i and β_i are determined using these boundary conditions as

$$\alpha_i = \frac{t_{12i} t_{23i} - t_{13i} t_{22i}}{t_{11i} t_{22i} - t_{12i} t_{21i}}, \qquad \beta_i = \frac{t_{21i} t_{13i} - t_{23i} t_{11i}}{t_{11i} t_{22i} - t_{12i} t_{21i}}$$

where

$$t_{11i} = K_\phi (2i + n + 2) + (1-\alpha)^3 ((2i + n + 2)(2i + n + 1) + v_\theta (2i + n + 2 - n^2))$$

$$t_{12i} = K_\phi (2i + n + p - 1) + (1-\alpha)^3 ((2i + n + p - 1)(2i + n + p - 2) + v_\theta (2i + n + p - 1 - n^2))$$

$$t_{13i} = K_\phi (2i + n - 2) + (1-\alpha)^3 ((2i + n - 2)(2i + n - 3) + v_\theta (2i + n - 2 - n^2))$$

$$t_{21i} = K - (1-\alpha)^3((2i+n+2)(2i+n+1)^2 - (1+n^2)(2i+n+2) + 2n^2 - 0{\cdot}5D_{k0}(2i+n+1))$$

$$t_{22i} = K - (1-\alpha)^3((2i+n+p-1)(2i+n+p-2)^2 - (1+n^2)(2i+n+p-1) + 2n^2 - 0{\cdot}5D_{k0}(2i+n+p-2))$$

$$t_{23t} = K - (1-\alpha)^3((2i+n-2)(2i+n-3)^2 - (1+n^2)(2i+n-2) + 2n^2 - 0{\cdot}5D_{k0}(2i+n-3)).$$

3. NUMERICAL RESULTS

The frequency equation (8) has been solved for various values of plate parameters such as taper parameter α, rigidity ratio E_θ/E_r, flexibility parameters K and K_ϕ. The numerical results have been computed for α $(=-0{\cdot}5(0{\cdot}2)0{\cdot}5;\ 0{\cdot}0)$, E_θ/E_r $(=0{\cdot}5, 0{\cdot}75, 1{\cdot}0, 2{\cdot}0, 5{\cdot}0)$ for the first three modes of vibration for lower axisymmetric and antisymmetric modes, i.e. $n = 0$, 1 and 2. The Poisson's ratio v_θ and shear modulus D_{k0} have been taken as 0·3 and 1·4 respectively. The rotational flexibility has been taken as K_ϕ $(=0, 10, 10^2, 10^3, 10^{20} \cong \infty)$. Special cases of classical boundary conditions i.e. simply supported, clamped and free have been considered by giving appropriate values to flexibility parameters, K and K_ϕ.

4. DISCUSSION

The results have been presented in Tables 1–9 and Figures 2–7 for different values of E_θ/E_r $(=0{\cdot}5, 0{\cdot}75, 1{\cdot}0, 2{\cdot}0, 5{\cdot}0)$, flexibility parameter K_ϕ $(=0, 10, 10^2, 10^3, 10^{20} \cong \infty)$ and nodal diameter $n = 0$, 1 and 2 for linearly varying thickness circular plates. Table 1 shows the effect of E_θ/E_r and flexibility parameter K_ϕ on the frequency parameter Ω for taper parameter $\alpha = 0{\cdot}3$, when the plate is vibrating in the first three axisymmetric modes. It is

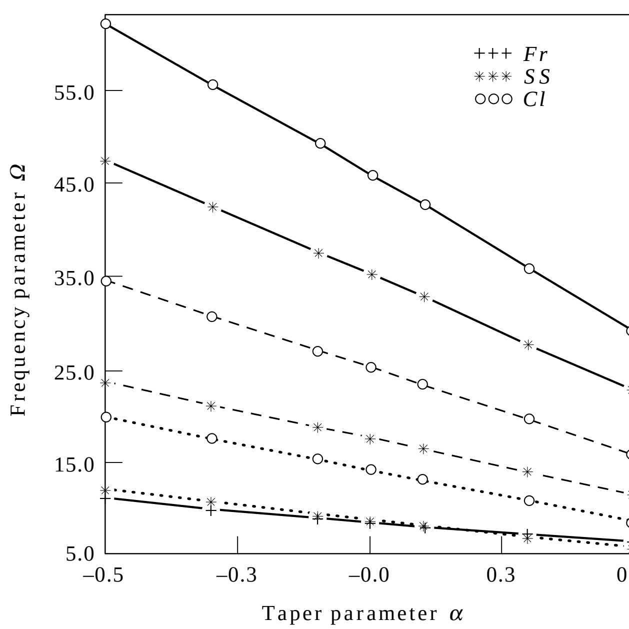

Figure 3. Frequency parameter in fundamental mode for $v_\theta = 0{\cdot}3$ and $E_\theta/E_r = 5{\cdot}0$ for $n = 0$ (---), $n = 1$ (–––) and $n = 2$ (——) LVT circular plates.

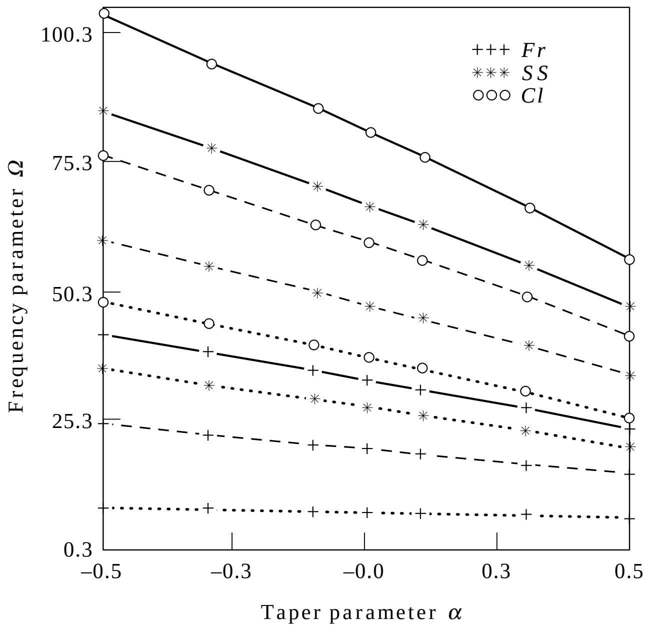

Figure 4. Frequency parameter in second mode for $\nu_\theta = 0{\cdot}3$ and $E_\theta/E_r = 0{\cdot}5$ for $n = 0$ (---), $n = 1$ (——) and $n = 2$ (——) LVT circular plates.

seen that frequency parameter Ω increases with the increase in orthotropy parameter E_θ/E_r, keeping all other plates parameter fixed. It can also be concluded that circumferential stiffened plates have greater frequency parameter as compared to the radially stiffened plates. The values of rotational flexibility parameter $K_\phi = 0$ and 10^{20} correspond to simply

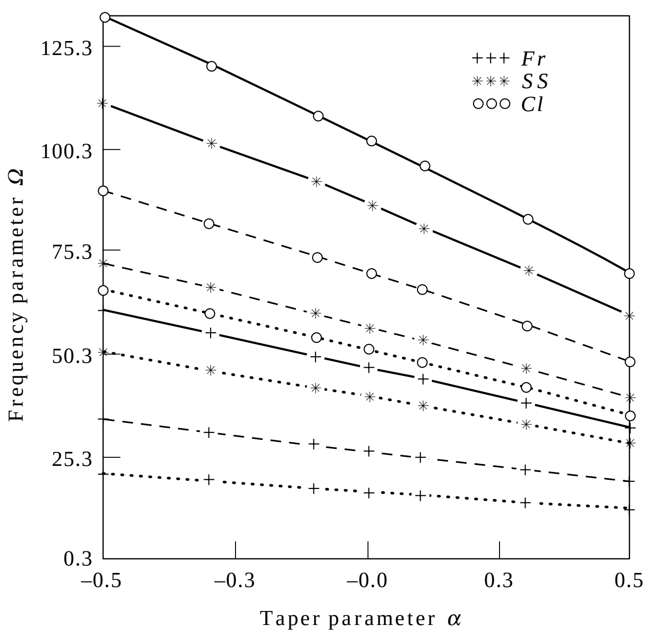

Figure 5. Frequency parameter in second mode for $\nu_\theta = 0{\cdot}3$ and $E_\theta/E_r = 5{\cdot}0$ for $n = 0$ (---), $n = 1$ (——) and $n = 2$ (——) LVT circular plates.

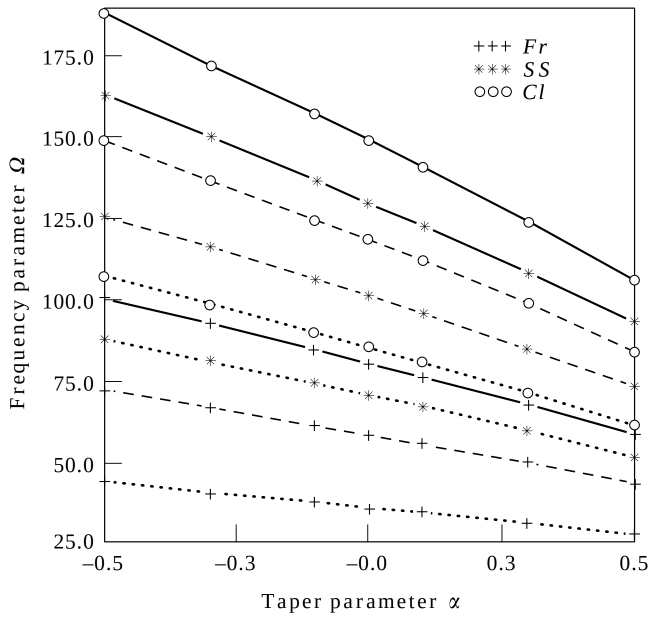

Figure 6. Frequency parameter in third mode for $\nu_\theta = 0{\cdot}3$ and $E_\theta/E_r = 0{\cdot}5$ for $n = 0$ (---), $n = 1$ (–––) and $n = 2$ (——) LVT circular plates.

supported and clamped classical boundary conditions respectively. The frequency parameter Ω_{0j} ($j = 0, 1, 2$) denotes the frequency parameter in the jth mode in axisymmetric mode of vibrations. The effect of flexibility parameter K_ϕ is found to be more pronounced in the range of zero to ten in all the three modes of vibration. Table 2 presents the frequency parameter Ω in an axisymmetric mode of vibrations for taper parameter

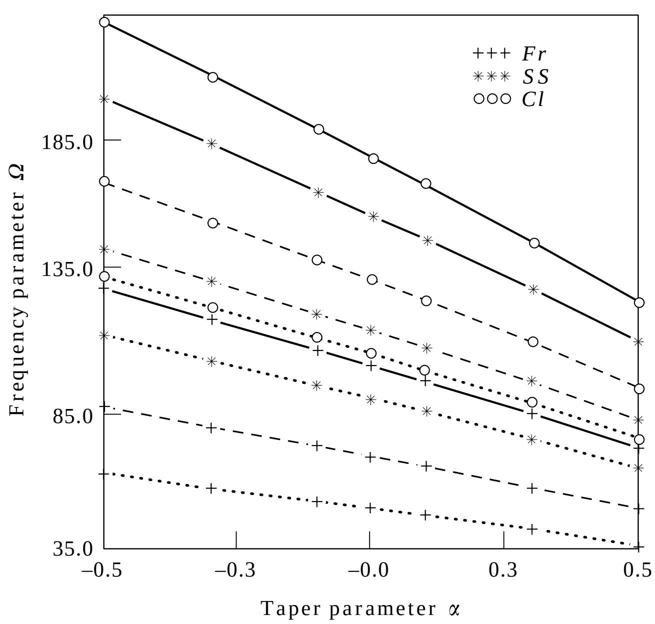

Figure 7. Frequency parameter in third mode for $\nu_\theta = 0{\cdot}3$ and $E_\theta/E_r = 5{\cdot}0$ for $n = 0$ (---), $n = 1$ (–––) and $n = 2$ (——) LVT circular plates.

TABLE 10

Comparison of frequency parameter Ω for uniform isotropic circular plate for $\nu_0 = 0{\cdot}3$ and different values of K_ϕ

K_ϕ	Ω_{00}	Ω_{01}	Ω_{02}	Ω_{10}	Ω_{11}	Ω_{12}	Ω_{20}	Ω_{21}	Ω_{22}
					$K = 0$				
	—	9·003*	38·443*	—	20·475*	59·812*	5·358*	35·260*	84·368*
0	—	9·0031	38·4432	—	20·4746	59·8116	5·3583	35·2601	84·3662
	—	13·513*	45·764*	—	26·298*	68·068*	8·050*	41·844*	93·256*
10	—	13·5129	45·7643	2·7735	26·2982	68·0681	8·0497	41·8438	93·2569
		14·539*	48·746*	—	28·126*	72·168*	8·693*	44·483*	98·431*
10^2	—	14·5388	48·7457	—	28·1262	72·1675	8·6933	44·4832	98·4336
	—	14·682*	49·218*	—	28·399*	72·860*	8·785*	44·904*	99·359*
∞	—	14·6820	49·2159	3·0948	28·3988	72·8781	8·7849	44·9031	99·3466
					$K = \infty$				
	4·935*	29·720*	74·156*	13·898*	48·479*	102·772*	25·613*	70·117*	134·290*
0	4·935	29·7200	74·1560	13·8982	48·4789	102·7733	25·6133	70·1170	134·2978
	8·752*	35·218*	80·685*	18·543*	54·516*	109·650*	30·848*	76·5442*	141·441*
10	9·7519	35·2190	80·6869	18·5438	54·5164	109·6529	30·8481	76·5442	141·4431
	10·019*	39·029*	87·488*	20·858*	59·710*	117·925*	34·226*	83·050*	151·126*
10^2	10·0192	39·0288	87·4900	20·8587	59·7102	117·9353	34·2259	83·0501	151·1094
	10·216*	39·771*	89·103*	21·260*	60·829*	120·077*	34·877*	84·584*	153·830*
∞	10·2158	39·7711*	89·1041	21·2604	60·8287	120·0792	34·8770	84·5826	153·8151

* Values are from [13, 14].

TABLE 11

Comparison of frequency parameter Ω for isotropic circular plate of linearly varying thickness in first two modes for $D_{k0} = 1{\cdot}4$, $\nu_\theta = 0{\cdot}33$

$\alpha\backslash\Omega$	Ω_{00}	Ω_{01}	Ω_{10}	Ω_{11}	Ω_{21}	Ω_{22}
			Fr-plate			
	9·43*	40·39*	21·55*	62·92*	5·625*	37·24*
−0·1	9·4352	40·3871	21·5468	62·9176	5·5217	37·2377
	9·07*	38·51*	20·51*	59·86*	5·35*	35·25*
0·0	9·0689	38·5070	20·5131	59·8591	5·2620	35·2425
	8·71*	36·62*	19·47*	56·77*	5·09*	33·25*
0·1	8·7106	36·6104	19·4760	56·7597	5·0125	33·2405
			SS-plate			
	5·25*	31·38*	14·79*	51·20*	27·28*	74·23*
−0·1	5·2495	31·3739	14·7845	51·1782	27·2823	74·0995
	4·98*	29·77*	13·94*	48·52*	25·65*	70·19*
0·0	4·9790	29·7544	13·9356	48·5123	25·6486	70·1499
	4·71*	28·16*	13·08*	45·84*	24·01*	66·17*
0·1	4·7076	28·1186	13·0764	45·8080	24·0056	66·1164
			Cl-plate			
	11·05*	42·15*	22·75*	64·44*	37·22*	90·00*
−0·1	11·0241	42·1231	22·7337	64·3339	37·2113	89·5003
	10·25*	39·81*	21·28*	60·89*	34·89*	84·82*
0·0	10·2158	39·7711	21·2604	60·8287	34·770	84·5826
	9·44*	37·47*	19·79*	57·31*	32·53*	79·70*
0·1	9·4084	37·3866	19·7704	57·2650	32·5204	79·5876

* Values are from [15].

$\alpha = -0{\cdot}3$. A comparison of Tables 2 and 3 shows that the frequency parameter for a plate with a thicker outer edge ($\alpha < 0$) is greater than that for a corresponding plate with a thinner outer edge ($\alpha > 0$). Tables 3 and 4 represent the frequency parameter Ω for the same plate parameter values considered above when the plate is vibrating in an antisymmetric mode. In an antisymmetric mode ($n = 1$) the frequency parameter is found to be greater than that for an axisymmetric mode of vibration, but the rate of increase of Ω is greater in this case. Similar conclusions can be drawn from the results given in

TABLE 12

Comparison of frequency parameter Ω with the exact solution and that of obtained by finite element method

	SS-plate			*Cl*-plate		
$n\backslash\Omega_{nj}$	Ω_{n0}	Ω_{n1}	Ω_{n2}	Ω_{n0}	Ω_{n1}	Ω_{n2}
$n\backslash 0$	4·9352*	29·7222*	74·1938*	10·2159*	39·7766*	89·1708*
	4·9352†	29·7200†	74·1961†	10·2158†	39·7711†	89·1041†
	4·9352	29·7200	74·1961	10·2158	39·7711	89·1041
	13·8983*	48·4867*	102·8465*	21·2611*	60·8441*	120·19581*
$n\backslash 1$	13·8982	48·4789†	102·7734†	21·2604†	60·8287†	120·0793†
	13·8982	48·4789	102·7734	21·2604	60·8287	120·0793
	25·6145*	70·1404*	134·4588*	34·8799*	84·6236*	154·0564*
$n\backslash 2$	25·6133†	70·1170†	134·4529†	34·8770†	84·5827†	153·8151†
	25·6133	70·1170	134·4529	34·8770	84·5827	153·8151

* and † are taken from [16] [1].

TABLE 13

Comparison of present method with the polynomial coordinate functions for $E_\theta/E_r = 5{\cdot}0$, $n = 1$ *and* $\nu_\theta = 0{\cdot}3$

Boundary conditions	No. of terms used in	Taper $\alpha = -0{\cdot}5$ I mode	II mode	III mode	Taper $\alpha = 0{\cdot}5$ I mode	II mode	III mode
Frequency		—	34·1754	87·7664	—	18·3697	49·3633
Fr-plate	Present method		5	9		8	11
	Polynomial co-ordinate method		10	12		10	14
Frequency		23·6515	72·2075	144·9910	11·0805	38·9389	81·1848
SS-plate	Present method	5	9	10	5	8	9
	Polynomial co-ordinate method	8	11	13	7	9	11
Frequency		34·3825	89·5274	168·5150	15·4606	47·2434	93·1302
Cl-plate	Present method	7	9	10	5	9	10
	Polynomial co-ordinate method	10	12	15	9	12	15

Tables 5 and 6 for antisymmetric mode $n = 2$. Table 7 gives the frequency parameter Ω for free edge circular plates for the first three modes of vibration in an axisymmetric case. The natural frequencies in fundamental mode of vibration are small and so are not given here. Tables 1–9 show that the frequency parameter Ω_{SS} for simply supported edge is smaller than Ω_{Cl}, for clamped edge, but greater than Ω_{Fr}, for free edge, all other plate parameter being fixed. Tables 8 and 9 give the frequency parameter Ω for the plates vibrating in antisymmetric modes for $n = 1$ and 2 respectively, when the plate is free. The frequency parameter is found to increase with the increase in nodal diameter n.

Figure 2 shows the effect of taper parameter α on the frequency parameter Ω for all the three boundary conditions, i.e. free (*Fr*), simply supported (*SS*) and Clamped (*Cl*) for nodal diameters $n = 0$, 1 and 2 corresponding to orthotropy parameter $E_\theta/E_r = 0{\cdot}5$ (i.e. when the plate is radially stiffened) in fundamental mode of vibration. The figure shows that frequency parameter Ω varies linearly as the taper α increases, i.e. $(= -0{\cdot}5(0{\cdot}2)0{\cdot}5)$. The rate of change of Ω depends upon the boundary conditions and the nodal diameter. The rate of increase of frequency parameter for the case of *SS* plate lower than that for the case of the *Cl* plate but greater than that for the *Fr* plate. The rate of change of Ω is found to increase with the increase in nodal diameter. Figure 3 is plotted for frequency parameter Ω vs taper parameter α for $E_\theta/E_r = 5{\cdot}0$ (when the plate is circumferentially stiffened) in the fundamental mode. Figures 2 and 3 show the effect of orthotropy parameter on the plate vibrating in the fundamental mode for all the three classical boundary conditions. The frequency parameter is found to increase with an increase in orthotropy parameter E_θ/E_r. It is seen that the rate of increase of Ω for circumferentially stiffened plates is greater compared to that of the radially stiffened plates. Figures 4 and 5 show frequency parameter Ω vs taper parameter α in a second mode of vibration for $n = 0$, 1 and 2 corresponding to orthotropy parameter $E_\theta/E_r = 0{\cdot}5$ and $5{\cdot}0$ respectively for all the three boundary conditions. In the second mode of vibration, it is observed that Ω decreases more rapidly with the increase in α as compared to the fundamental mode. A similar behaviour is observed when the plate vibrates in a third mode, as is presented in Figures 6 and 7.

A comparison of results has been given to show the versatility of the method employed here. Table 10 shows the comparison of the results with isotropic uniform circular plates with an elastically restrained edge obtained by Azimi [13, 14] using the Receptance method. Our results agree with those obtained by Laura *et al*. [15] for isotropic circular plates of linearly varying thickness and are given in Table 11. A comparison of results with those of Pardoen [16] for isotropic uniform circular plates are given in Table 12. They are obtained for $n = 0$, 1 and 2 in the first three modes of vibrations. The above comparison Tables 10–12 clearly show an excellent agreement of the results in particular cases obtained by using different methods such as the Receptance method [13, 14], Polynomial co-ordinate function [15], Finite element method [16] and also with the exact solution [1]. The present method of solution has a faster rate of convergence as compared to the polynomial co-ordinate functions, as can be seen from Table 13.

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