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ON THE ANALYSIS OF CERTAIN HIGH DIMENSIONAL SYSTEMS WITH INNER RESONANCES

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In this paper, the normal forms and the related coefficients, of high dimensional inner resonant systems, are explored. Using a recently developed approach, calculations of normal forms (especially the related coefficients) are much easier, compared to the existing methods. A general four dimensional system with two pairs of pure imaginary eigenvalues is used as an example, and normal forms in resonant model $p:q$ are determined. The coefficients of normal forms related to different possible resonant models; namely, 1:2, 1:3, 1:4, 2:1, 3:1 and 4:1, are considered. The theory presented here can be applied to other higher order inner resonant cases as well.

1. INTRODUCTION

Consider the system

$$\dot{z} = Az + F^2(z) + F^3(z) + \text{h.o.t}, \tag{1}$$

where $z \in C^n$; A is a matrix; $F^m \in H_n^m$, and H_n^m is a vector space of homogeneous polynomials of degree m and is described by n variables; "h.o.t" means higher order terms; $m = 2, 3, \ldots, n$; $F^m = (F_{(1)}^m \ F_{(2)}^m \ \cdots \ F_{(n)}^m)^{\mathrm{T}}$ and

$$F_{(p)}^2 = \sum_{s_1+s_2+\cdots+s_n=2} a_{s_1 s_2 \cdots s_n (p)} z_1^{s_1} z_2^{s_2} \cdots z_n^{s_n},$$

$$F_{(p)}^3 = \sum_{s_1+s_2+\cdots+s_n=3} a_{s_1 s_2 \cdots s_n (p)} z_1^{s_1} z_2^{s_2} \cdots z_n^{s_n},$$

$$F_{(p)}^n = \sum_{s_1+s_2+\cdots+s_n=n} a_{s_1 s_2 \cdots s_n (p)} z_1^{s_1} z_2^{s_2} \cdots z_n^{s_n}.$$

Suppose these are $l(2l = n)$ pairs of pure imaginary eigenvalues for the linearized part of equation (1). Let $A = \text{diag}\,(i\omega_1, i\omega_2, \ldots, i\omega_l, i\omega, -i\omega_2, \ldots, -i\omega_l)$, where ω_k are the eigenvalues. Here, it is assumed that A is diagonalizable. Then, normal forms of order k

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are composed of resonant monomials of order k [2]. A system is said to have inner resonance, if it satisfies

$$\sum_{i=1}^{l} m_i^e \omega = 0, \qquad e = 1, 2, \ldots, M; \qquad M < 1, \tag{2}$$

where

$$m_i^e \in Z, \; \sum_{i=1}^{l} |m_i^e| \neq 0.$$

Further, if

$$\sum_{i=1}^{l} |m_i^e| < 4,$$

it is called lower order inner resonance.

Two facts make researches on lower order inner resonance interesting. Firstly, many cases of lower order inner resonances exist in real engineering systems, especially in high dimensional nonlinear systems. Secondly, when a system has lower order inner resonance, both its higher order and lower order normal forms are affected. On the other hand, higher order inner resonance do not change lower order normal forms, which are more important for studying characteristics of the system, such as stability. Thus, when calculating the lower order normal forms, there is no need to consider higher order inner resonance.

It is well accepted that the existing high dimensional normal form theory is too cumbersome to use, especially for the calculations of the coefficients of normal forms. The theory has to employ some modern but very abstract mathematical theories (such as the representation theory and adjoint operator methods [2]) to obtain the normal forms and the related coefficients. Above all, calculation of the coefficients related to the normal forms poses especially serious difficulties. Some high dimensional systems without lower order inner resonance have been studied [1, 2, 9]. Certain particular high dimensional systems with lower order inner resonances have been investigated [3, 4, 7, 8]. In spite of the efforts, there still exists no single general method, and all approaches seem to be too cumbersome to use. In this paper, a general method is proposed. The systems with inner resonance, both lower and higher order inner resonances, are examined through a modified normal form approach. This approach is conveniently applicable to higher dimensional systems with different inner resonances. Using MAPLE, the calculations of both basic terms of normal forms and the associated coefficients are carried out readily, and are easier than any of current approach. Furthermore, it is shown that the results of the existing normal form theory are identical to those of the modified approach.

In the following section, a general theory is presented, which forms the basis of our analysis and can also be applied to higher order inner resonant cases. This theory is based on earlier work [5, 6] and can be considered as an extension of reference [6]. Following the general theory, the effects of lower order inner resonance, where

$$\sum_{i=1}^{l} |m_i^e| < 4,$$

on lower order normal forms are discussed. Then, higher order inner resonance in a degenerate system, where

$$\sum_{i=1}^{l} |m_i^e| < 6,$$

is examined.

In our analysis, a general four dimensional system with two pairs of pure imaginary eigenvalues is studied as an example. In lower order inner resonant cases, where

$$\sum_{i=1}^{l} |m_i^e| < 4,$$

according to equation (2), there are five possible resonant models. They are: (1) $\omega_1 = 2\omega_2$, (2) $\omega_1 = 3\omega_2$, (3) $\omega_2 = 2\omega_1$, (4) $\omega_2 = 3\omega_1$, (5) $\omega_1 = \omega_2$.

In this paper, the first four resonant models are considered in Section 3, and the fifth one can be examined by similar procedure. Using MAPLE, the normal forms and the related coefficients and transformation functions, in different resonant models, are obtained in seconds.

In higher order inner resonant cases, where

$$\sum_{i=1}^{l} |m_i^e| < 6$$

according to equation (2), there are seven possible resonant models. They are: (1) $\omega_1 = 2\omega_2$, (2) $\omega_1 = 3\omega_2$, (3) $\omega_1 = 4\omega_2$, (4) $\omega_2 = 2\omega_1$, (5) $\omega_2 = 3\omega_1$, (6) $\omega_2 = 4\omega_1$, (7) $\omega_1 = \omega_2$.

Again, in this paper, the first six resonant models are discussed in Section 4, and the seventh one can be examined by a similar procedure. Using MAPLE, the normal forms and the related coefficients and transformation functions, in different resonant models, are obtained in seconds.

2. GENERAL THEORY

Consider equation (1). Suppose there are different inner resonances in equation (1) and the resonant conditions are given by equation (2).

Introduce a near identity transformation

$$z = y + P^2(y), \qquad P^2 \in H_n^2, \tag{3}$$

where $P^2(y)$ is an undefined function, which will be determined such that the terms of order 2 in the transformed form will be simplified as resonant polynomial of order 2.

Substituting equation (3) into equation (1) results in

$$\dot{y} = Ay + F_1^2(y) + F_1^3(y) + \text{h.o.t}, \tag{4}$$

where $F_1^2 = F^2 + AP^2 - DP^2Ay$; $F_1^3 = F^3 + DF^2P^2 - DP^2F_1^2$.

Now, introduce another near identity transformation,

$$y = x + P^3(x), \qquad P^3 \in H_n^3, \tag{5}$$

where $P^3(x)$ is an undefined function, which will be determined such that the terms of order 3 in the transformed form will be simplified as a resonant polynomial of order 3.

Substituting equation (5) into equation (4) leads to

$$\dot{x} = Ax + F_1^2(x) + F_2^3(x) + \text{h.o.t}, \tag{6}$$

where $F_2^3 = F_1^3 + AP^3 - DP^3Ax$.

Suppose $F_{k-1}^k(x) = G^k(x)$ $(k = 2, 3, \ldots)$ in equation (6), where $G^k(x)$ are the resonant polynomials of order m. Solving $P^2(y)$ from $F_1^2(y, P^2) = G^2(y)$; substituting $P^2(y)$ into $F_1^3(y, P^2)$ defines F_1^3 as $F_1^3(y)$. Then solving $P^3(x)$ from $F_2^3(x, P^3) = G^3(x)$; the coefficients in $G^3(x)$ can be determined. This is the basic procedure of the existing normal form theory.

In order to determine the normal forms and the related coefficients more conveniently, introduce the transformation

$$x = \mathrm{e}^{At}u \tag{7}$$

into equations (6) to obtain

$$\dot{u} = \mathrm{e}^{-At}[F_1^2(\mathrm{e}^{At}u) + F_2^3(\mathrm{e}^{At}u)] + \text{h.o.t}, \tag{8}$$

where

$$\mathrm{e}^{At}u = \begin{pmatrix} \mathrm{e}^{\lambda_1 t}u_1 \\ \cdots\cdots \\ \mathrm{e}^{\lambda_n t}u_n \end{pmatrix}.$$

Suppose the mth row of

$$F_{k-1}^k(x) \text{ is } F_{k-1(m)}^k(x) = \sum_{\bar{s}=k} a_{s_1 s_2 \cdots s_n(m)}^{k-1} \prod_{i=1}^n x_i^{s_i};$$

then, $\mathrm{e}^{-At}F_{k-1(m)}^k$ $(\mathrm{e}^{At}u)$ can be expressed as

$$\begin{aligned} \mathrm{e}^{-At}F_{k-1(m)}^k(\mathrm{e}^{At}u) &= \sum_{\bar{s}=k} \left(\mathrm{e}^{-\lambda_m t} a_{s_1 s_2 \cdots s_n(m)}^{k-1} \prod_{i=1}^n \mathrm{e}^{s_i \lambda_i t} u_i^{s_i} \right) \\ &= \sum_{\bar{s}=k} \left(a_{s_1 s_2 \cdots s_n(m)}^{k-1} \exp\left(-\lambda_m + \sum_{i=1}^n s_i \lambda_i \right) t \prod_{i=1}^n u_i^{s_i} \right), \end{aligned} \tag{9}$$

where $a_{s_1 s_2 \cdots s_n(m)}^{k-1}$ are the coefficients of transformed functions $F_{k-1(m)}^k(x)$; $\bar{s} = s_1 + s_2 + \cdots + s_n$.

According to the assumption $F_{k-1}^k(x) = G^k(x)$, functions $F_{k-1}^k(x)$ are composed of resonant monomials, in which

$$-\lambda_m + \sum_{i=1}^n s_i \lambda_i = 0, \tag{10}$$

where $m = 1, 2, \ldots, n$.

According to equations (8), (9) and (10), one has

$$\mathrm{e}^{-At}F_{k-1}^k(\mathrm{e}^{At}u) = F_{k-1}^k(u) = \underset{t}{M} \{\mathrm{e}^{-At}F_{k-1}^k(\mathrm{e}^{At}u)\} \tag{11}$$

and

$$\begin{aligned} \dot{u} &= \underset{t}{M} \{\mathrm{e}^{-At}[F_1^2(\mathrm{e}^{At}u) + F_2^3(\mathrm{e}^{At}u)]\} + \text{h.o.t} \\ &= F_1^2(u) + F_2^3(u) + \text{h.o.t}, \end{aligned}$$

where $\underset{t}{M}\{f(u, t)\}$ denotes explicit time averaging of function $f(u, t)$, in which

$$\underset{t}{M}\{f(u, t)\} = \underset{t}{M}\{e^{-At}F_{k-1}^{k}(e^{At}u)\} = \frac{1}{(2\pi)^{l}}\int_{0}^{2\pi}\int_{0}^{2\pi}\cdots\int_{0}^{2\pi} e^{-At}F_{k-1}^{k}(e^{At}u)\, d\phi_1\cdots d\phi_l$$

where $\phi_i = \omega_i t$. Similarly, one has

$$e^{At}F_{k-1}^{k}(e^{-At}u) = F_{k-1}^{k}(u). \tag{12}$$

Thus, equation (8) can be expressed as

$$\dot{u} = G^{2}(u) + G^{3}(u) + \text{h.o.t.} \tag{13}$$

Introducing transformation $u = e^{-At}z$ into equation (13), according to equation (12), one has

$$\dot{z} = Az + G^{2}(z) + G^{3}(z) + \text{h.o.t.} \tag{14}$$

This is a normal form of equation (1).

It is evident that the results of the existing normal form theory are identical to those of the new approach. Consider the following relations:

$$e^{-At}(DP^{k}A\, e^{At}u - AP^{k}) = \frac{\partial}{\partial t}[e^{-At}P^{k}(e^{At}u)], \qquad F_{k=1}^{k}F_{k-2}^{k} + AP^{k} - DP^{k}Ax. \tag{15}$$

Then, one has

$$e^{-At}F_{k-1}^{k}(e^{At}u) + \frac{\partial}{\partial t}[e^{-At}P^{k}(e^{At}u)] = e^{-At}F_{k-2}^{k}(e^{At}u). \tag{16}$$

Suppose the mth row of $P^{k}(x)$ is $P_{(m)}^{k}(x)$, then, the arbitrary term in polynomial $e^{-At}P_{(m)}^{k}(e^{At}u)$ can be expressed as

$$e^{-\lambda_m t}b_{s_1 s_2\cdots s_n(m)}^{k}\prod_{\substack{i=1\\ \bar{s}=k}}^{n} e^{s_i\lambda_i t}u_i^{s_i} = b_{s_1 s_2\cdots s_n(m)}^{k}\exp\left(-\lambda_m + \sum_{i=1}^{n} s_i\lambda_i\right)t\prod_{\substack{i=1\\ \bar{s}=k}}^{n} u_i^{s_i},$$

where $b_{s_1 s_2\cdots s_n(m)}^{k}$ are the coefficients of functions $P_{(m)}^{k}(x)$.

The above equation and equation (10) lead to

$$\underset{t}{M}\left\{\frac{\partial}{\partial t}[e^{-At}P^{k}(e^{At}u)]\right\} = 0.$$

According to equation (16), one has

$$\underset{t}{M}\{e^{-At}F_{k-1}^{k}(e^{At}u)\} = \underset{t}{M}\{e^{-At}F_{k-2}^{k}(e^{At}u)\}. \tag{17}$$

From equations (11) and (17), one has

$$G^{k}(u) = \frac{1}{(2\pi)^{l}}\int_{0}^{2\pi}\int_{0}^{2\pi}\cdots\int_{0}^{2\pi} e^{-At}F_{k-2}^{k}(e^{At}u)\, d\phi_1\cdots d\phi_l. \tag{18}$$

According to equation (16), one has

$$P^k(u) = e^{-At} P^k(e^{At}u)|_{t=0}$$

$$= \int e^{-At}(F^k_{k-2}(e^{At}u) - F^k_{k-1}(e^{At}u))\,dt|_{t=0} = \int [e^{-At}(F^k_{k-2}(e^{At}u)) - G^k(u)]\,dt|_{t=0}. \quad (19)$$

According to equations (2) and (10), in the inner resonant case, resonant monomials consist of terms satisfying

$$\pm\omega_j + \sum_{i=1}^{l} (m_i - n_i)\omega_i = 0,$$

$$\sum_{i=1}^{l} m_i^e \omega_i = 0, \qquad e = 1, 2, \ldots, M, \quad M < l. \quad (20)$$

This condition can be written as $D^e = 0$.

Let $G^k(u) = (G^k_{(1)}\ G^k_{(2)}\ \cdots\ G^k_{(n)})^T$; then, $G^k_{(p)}(u)$ can be expressed as

$$G^k_{(p)}(u) = \sum_{e=1}^{M} \sum_{\substack{D^e=0 \\ \bar{m}=k}} a^{k-2}_{m_1\cdots m_k n_1 \cdots n_k(p)} \prod_{q=1}^{l} u_q^{m_q} \bar{u}_q^{n_q}, \quad (21)$$

where

$$\bar{m} = \sum_{i=1}^{l} (m_i + n_i);$$

$a^{k-2}_{m_1\cdots m_k n_1\cdots n_k(p)}$ are the coefficients in function $F^k_{k-2}(u)$.

Solving equation (16) for $P^k(u)$, one has

$$P^k(u) = \begin{pmatrix} \displaystyle\sum_{\substack{\delta-\omega_1\neq 0 \\ \bar{m}=k}} \frac{1}{\delta-\omega_1} a^{k-2}_{m_1\cdots m_k n_1\cdots n_k(1)} \prod_{q=1}^{l} u_q^{m_q}\bar{u}_q^{n_q} \\ \cdots\cdots \\ \displaystyle\sum_{\substack{\delta-\omega_l\neq 0 \\ \bar{m}=k}} \frac{1}{\delta-\omega_l} a^{k-2}_{m_1\cdots m_k n_1\cdots n_k(k)} \prod_{q=1}^{l} u_q^{m_q}\bar{u}_q^{n_q} \\ \displaystyle\sum_{\substack{\delta+\omega_1\neq 0 \\ \bar{m}=k}} \frac{1}{\delta+\omega_1} a^{k-2}_{m_1\cdots m_k n_1\cdots n_k(l+1)} \prod_{q=1}^{l} u_q^{m_q}\bar{u}_q^{n_q} \\ \cdots\cdots \\ \displaystyle\sum_{\substack{\delta+\omega_1\neq 0 \\ \bar{m}=k}} \frac{1}{\delta+\omega_l} a^{k-2}_{m_1\cdots m_k n_l\cdots n_k(2l)} \prod_{q=1}^{l} u_q^{m_q}\bar{u}_q^{n_q} \end{pmatrix}, \quad (22)$$

where

$$\delta = \sum_{i=1}^{l} (m_i - n_i)\omega_i,\ k \geqslant 2.$$

When $l = 2$, suppose there are $p{:}q$ resonances, equation (20) can be expressed as

$$\pm\omega_j + m\omega_1 + n\omega_2 = 0, \qquad \frac{\omega_1}{\omega_2} = \frac{p}{q}, \tag{23}$$

where, $\omega_j = \omega_1, \omega_2$.

Equation (21) leads to

$$G^k(u) = \begin{pmatrix} \sum\limits_{\substack{\bar{m}=k \\ \delta - w_1 = 0}} a^{k-2}_{m_1m_2n_1n_2}\, u_1^{m_1}u_2^{m_2}\bar{u}_1^{n_1}\bar{u}_2^{n_2} \\ \sum\limits_{\substack{\bar{m}=k \\ \delta - w_2 = 0}} b^{k-2}_{m_1m_2n_1n_2}\, u_1^{m_1}u_2^{m_2}\bar{u}_1^{n_1}\bar{u}_2^{n_2} \\ \sum\limits_{\substack{\bar{m}=k \\ \delta + w_1 = 0}} c^{k-2}_{m_1m_2n_1n_2}\, u_1^{m_1}u_2^{m_2}\bar{u}_1^{n_1}\bar{u}_2^{n_2} \\ \sum\limits_{\substack{\bar{m}=k \\ \delta + w_2 = 0}} d^{k-2}_{m_1m_2n_1n_2}\, u_1^{m_1}u_2^{m_2}\bar{u}_1^{n_1}\bar{u}_2^{n_2} \end{pmatrix}, \tag{24}$$

where $a^{k-2}_{m_1m_2n_1n_2}$, $b^{k-2}_{m_1m_2n_1n_2}$, $c^{k-2}_{m_1m_2n_1n_2}$, and $d^{k-2}_{m_1m_2n_1n_2}$ are the coefficients in function $F^k_{k-2}(u)$.

According to equation (23), this result can be expressed in another form. If $\omega_j = \omega_1$, equation (23) becomes $(m-1)p + nq = 0$. The solutions of this equation are: $m = 1$, $n = 0$; $m = 1 + s$, $n = (-(p/q)s)$. The resonant monomials can be expressed as

$$G^k(u) = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix} = \begin{pmatrix} u_1 g_{11}(u_1\bar{u}_1, u_2\bar{u}_2, u_1^q\bar{u}_2^p) + u_2^p\bar{u}_1^{q-1} g_{12}(u_1\bar{u}_1, u_2\bar{u}_2, \bar{u}_1^q u_2^p) \\ u_2 g_{21}(u_1\bar{u}_1, u_2\bar{u}_2, u_1^q\bar{u}_2^p) + u_2^{p+1}\bar{u}_1^q g_{22}(u_1\bar{u}_1, u_2, \bar{u}_2, \bar{u}_1^q u_2^p) \\ \bar{u}_1 g_{31}(u_1\bar{u}_1, u_2\bar{u}_2, u_1^q\bar{u}_2^p) + u_2^p\bar{u}_1^{q+1} g_{32}(u_1\bar{u}_1, u_2, \bar{u}_2, \bar{u}_1^q u_2^p) \\ \bar{u}_2 g_{41}(u_1\bar{u}_1, u_2\bar{u}_2, u_1^q\bar{u}_2^p) + u_2^{p-1}\bar{u}_1^q g_{42}(u_1\bar{u}_1, u_2, \bar{u}_2, \bar{u}_1^q u_2^p) \end{pmatrix}, \tag{24*}$$

where g_{i1} are any functions of $u_1\bar{u}_1$, $u_2\bar{u}_2$ and $u_1^q\bar{u}_2^p$; g_{i2} are any functions of $u_1\bar{u}_1$, $u_2\bar{u}_2$ and $\bar{u}_1^q u_2^p$; the coefficients of term $u_1^{m_1}\bar{u}_1^{m_2}u_2^{n_1}\bar{u}_2^{n_2}$ in functions g_k are $a^{k-2}_{m_1m_2n_1n_2}$, $b^{k-2}_{m_1m_2n_1n_2}$, $c^{k-2}_{m_1m_2n_1n_2}$ and $d^{k-2}_{m_1m_2n_1n_2}$. Equation (22) leads to

$$P^k(u) = \begin{pmatrix} \sum\limits_{\substack{\bar{m}=k \\ \delta - w_1 \neq 0}} \frac{1}{\delta - w_1} a^{k-2}_{m_1m_2n_1n_2}\, u_1^{m_1}u_2^{m_2}\bar{u}_1^{n_1}\bar{u}_2^{n_2} \\ \sum\limits_{\substack{\bar{m}=k \\ \delta - w_2 \neq 0}} \frac{1}{\delta - w_2} b^{k-2}_{m_1m_2n_1n_2}\, u_1^{m_1}u_2^{m_2}\bar{u}_1^{n_1}\bar{u}_2^{n_2} \\ \sum\limits_{\substack{\bar{m}=k \\ \delta + w_1 \neq 0}} \frac{1}{\delta + w_1} c^{k-2}_{m_1m_2n_1n_2}\, u_1^{m_1}u_2^{m_2}\bar{u}_1^{n_1}\bar{u}_2^{n_2} \\ \sum\limits_{\substack{\bar{m}=k \\ \delta + w_2 \neq 0}} \frac{1}{\delta + w_2} d^{k-2}_{m_1m_2n_1n_2}\, u_1^{m_1}u_2^{m_2}\bar{u}_1^{n_1}\bar{u}_2^{n_2} \end{pmatrix}. \tag{25}$$

A major difference between the results of this paper and those concerning non-resonant cases is that normal forms in inner resonant situations have more terms. This difference is attributed to the fact that there are more resonant monomials in inner resonant cases. Using existing normal form theory, however, it is hard to obtain these additional terms and the related coefficients.

The above analysis outlines the underlying theory of using the modified normal form approach to analyse high dimensional systems with inner resonance. More details about the modified approach can be found in reference [6].

In the following sections, certain four dimensional systems with two pairs of pure imaginary eigenvalues and inner resonance are studied as examples.

3. LOWER ORDER NORMAL FORMS (LOWER ORDER INNER RESONANT CASES)

3.1. THE CASE OF $\omega_1 = 2\omega_2$

Only lower order inner resonance is considered here, which means

$$\sum_{i=1}^{l} |m_i^e| < 4.$$

Suppose $l = 2$; according to the conclusion given in references [2] and [6], if there is no inner resonance, the formal normal form of equation (1) is given by

$$\dot{z}_1 = i\omega_1 z_1 + a_{11} z_1^2 \bar{z}_1 + a_{12} z_1 z_2 \bar{z}_2, \qquad \dot{z}_2 = i\omega_2 z_2 + a_{21} z_2^2 \bar{z}_2 + a_{22} z_2 z_1 \bar{z}_1. \tag{26}$$

It contains the first kind of resonant monomials (without inner resonance). To be distinguished from the above resonant monomials, a second kind is defined. Resonant monomials fall to the category "second kind of resonant monomials", if their eigenvalues satisfy

$$-\lambda_k + m\omega_1 + n\omega_2 = 0, \qquad \omega_1 = 2\omega_2,$$

where $\lambda_k = \pm\omega_1, \ \pm\omega_2$.

Suppose $\lambda_k = \omega_1$. Above equation can be expressed as

$$2(m-1) + n = 0.$$

Thus, the solution of this equation is $m = 0$, $n = 2$, in which $m + n < 4$.

So, the resonant monomial is z_2^2. Similarly,

if $\lambda_k = -\omega_1$, there is a second kind of resonant monomial $\bar{z}_2^2$;

if $\lambda_k = \omega_2$, there is a second kind of resonant monomial $z_1\bar{z}_2$;

if $\lambda_k = -\omega_2$, there is a second kind of resonant monomial $\bar{z}_1 z_2$.

Therefore, in complex coordinates, the formal normal form of equation (1) is given by

$$\dot{z}_1 = i\omega_1 z_1 + a_{11} z_1^2 \bar{z}_1 + a_{12} z_1 z_2 \bar{z}_2 + a_{13} z_2^2,$$

$$\dot{z}_2 = i\omega_2 z_2 + a_{21} z_2^2 \bar{z}_2 + a_{22} z_1 z_2 \bar{z}_1 + a_{23} z_1 \bar{z}_2. \tag{27}$$

Note that the corresponding conjugate equations are not given.

It is evident that the formal normal form (27) can be obtained directly from equation (24*). Substituting $z_j = r_j\,e^{i\theta_j}$, $j = 1, 2$ into equation (27) and transforming to polar coordinates, one has

$$\dot{r}_1 + ir_1(\dot{\theta}_1 - \omega_1) = a_{11}r_1^3 + a_{12}r_1r_2^2 + a_{13}r_2^2\,e^{i(2\psi_2 - \psi_1)},$$
$$\dot{r}_2 + ir_2(\dot{\theta}_2 - \omega_2) = a_{21}r_2^3 + a_{22}r_2r_1^2 + a_{23}r_1r_2\,e^{-i(2\psi_2 - \psi_1)},$$

where $\theta_j = \omega_j t + \psi_j$. So,

$$\dot{r}_1 = \mathrm{Re}\,(a_{11})r_1^3 + \mathrm{Re}\,(a_{12})r_1r_2^2 - \mathrm{Im}\,(a_{13})r_2^2\sin\alpha_1 + \mathrm{Re}\,(a_{13})r_2^2\cos\alpha_1,$$
$$\dot{r}_2 = \mathrm{Re}\,(a_{21})r_2^3 + \mathrm{Re}\,(a_{22})r_2r_1^2 + \mathrm{Im}\,(a_{23})r_1r_2\sin\alpha_1 + \mathrm{Re}\,(a_{23})r_1r_2\cos\alpha_1,$$
$$\dot{\theta}_1 = \omega_1 + \mathrm{Im}\,(a_{11})r_1^2 + \mathrm{Re}\,(a_{12})r_2^2 + \mathrm{Re}\,(a_{13})\frac{r_2^2}{r_1}\sin\alpha_1 + \mathrm{Im}\,(a_{13})\frac{r_2^2}{r_1}\cos\alpha_1,$$
$$\dot{\theta}_2 = \omega_2 + \mathrm{Im}\,(a_{21})r_2^2 + \mathrm{Re}\,(a_{22})r_1^2 - \mathrm{Re}\,(a_{23})r_1\sin\alpha_1 + \mathrm{Im}\,(a_{23})r_1\cos\alpha_1,$$
$$\dot{\alpha}_1 - 2\dot{\theta}_2 - \dot{\theta}_1, \qquad (28)$$

where $\alpha_1 = 2\theta_2 - \theta_1$.

According to equation (24), the coefficients related to normal forms can be calculated by

$$G^2 = \begin{bmatrix} a_{0020}z_2^2 \\ b_{1001}z_1\bar{z}_2 \\ c_{0002}\bar{z}_2^2 \\ d_{0110}\bar{z}_1z_2 \end{bmatrix}, \qquad G^3 = \begin{bmatrix} a^1_{2010}z_1^2\bar{z}_1 + a^1_{1101}z_1z_2\bar{z}_2 \\ b^1_{0201}z_2^2\bar{z}_2 + b^1_{1110}z_1\bar{z}_1z_2 \\ c^1_{1020}\bar{z}_1^2z_1 + c^1_{0111}\bar{z}_1z_2\bar{z}_2 \\ d^1_{0102}z_2\bar{z}_2^2 + d^1_{1011}z_1\bar{z}_1\bar{z}_2 \end{bmatrix},$$

where $a^1_{m_1m_2n_1n_2}$, $b^1_{m_1m_2n_1n_2}$, $c^1_{m_1m_2n_1n_2}$ and $d^1_{m_1m_2n_1n_2}$ are the coefficients in function F_1^3; $a_{m_1m_2n_1n_2}$, $b_{m_1m_2n_1n_2}$, $c_{m_1m_2n_1n_2}$ and $d_{m_1m_2n_1n_2}$ are the coefficients in function F^2. $P^2 = (P^2_{(1)}\ P^2_{(2)}\ P^2_{(3)}\ P^2_{(4)})^T$ can be determined readily through equation (25) as follows:

$$P^2_{(1)} = \frac{1}{\omega_2 i}\left(\tfrac{1}{2}a_{2000}y_1^2 - \tfrac{1}{2}a_{1100}y_1y_2 - \tfrac{1}{6}a_{0200}y_2^2 + a_{1010}y_1y_3 - \tfrac{1}{3}a_{0110}y_2y_3 - a_{1001}y_1y_4\right.$$
$$\left. - \tfrac{1}{5}a_{0101}y_2y_4 - \tfrac{1}{2}a_{0011}y_3y_4 - \tfrac{1}{4}a_{0002}y_4^2\right),$$
$$P^2_{(2)} = \frac{1}{i\omega_2}\left(\tfrac{1}{6}b_{2000}y_1^2 + \tfrac{1}{2}b_{1100}y_1y_2 - \tfrac{1}{2}b_{0200}y_2^2 + \tfrac{1}{5}b_{1010}y_1y_3 + b_{0110}y_2y_3 + \tfrac{1}{4}b_{0020}y_3^2\right.$$
$$\left. + \tfrac{1}{3}b_{1001}y_1y_4 - b_{0101}y_2y_4 + \tfrac{1}{2}b_{0011}y_3y_4\right),$$
$$P^2_{(3)} = \frac{1}{i\omega_2}\left(\tfrac{1}{3}c_{2000}y_1^2 - c_{1100}y_1y_2 - \tfrac{1}{5}c_{0200}y_2^2 + \tfrac{1}{2}c_{1010}y_1y_3 - \tfrac{1}{2}c_{0110}y_2y_3 + c_{0020}y_3^2\right.$$
$$\left. - \tfrac{1}{4}c_{0101}y_2y_4 - c_{0011}y_3y_4 - \tfrac{1}{3}c_{0002}y_4^2\right),$$
$$P^2_{(4)} = \frac{1}{i\omega_2}\left(\tfrac{1}{5}d_{2000}y_1^2 + d_{1100}y_1y_2 - \tfrac{1}{3}d_{0200}y_2^2 + \tfrac{1}{4}d_{1010}y_1y_3 + \tfrac{1}{3}d_{0020}y_3^2 + \tfrac{1}{2}d_{1001}y_1y_4\right.$$
$$\left. - \tfrac{1}{2}d_{0101}y_2y_4 + d_{0011}y_3y_4 - d_{0002}y_4^2\right).$$

where $y_1 = z_1$, $y_2 = z_2$, $y_3 = \bar{z}_1$, $y_4 = \bar{z}_2$.

After simple iteration, $F_1^2 = F^2 + AP^2 - DP^2Ay$; $F_1^3 = F^3 + DF^2P^2 - DP^2F_1^2$, the coefficients $a^1_{m_1m_2n_1n_2}$, $b^1_{m_1m_2n_1n_2}$, $c^1_{m_1m_2n_1n_2}$ and $d^1_{m_1m_2n_1n_2}$ in F_1^3 are determined. Then, the coefficients in G^2 and G^3 are determined. Using MAPLE, the formal normal forms and the related coefficients and the transformation functions P^k are obtained in seconds. To save space, the related coefficients are not given here.

It is evident that the normal forms of the systems with inner resonance are quite different from those of the systems with no inner resonance. In the latter cases, the normal forms are in the form of $\dot{r} = r(r)$, while in the former cases, normal forms are in the form of $\dot{r} = r(r, \alpha_1)$. Therefore, their dynamical behaviours also vary.

When $\omega_1 = 2\omega_2$, there are terms of order 2 in the normal form. Compared to normal forms of systems with no inner resonance, this is another significant difference. As a result, for example, *the stability of Hopf bifurcation is determined by the terms of order* 2.

3.2. THE CASE OF $\omega_1 = 3\omega_2$

If there exists no inner resonance, the formal normal form of equation (1) is given by equation (26). In the resonant model $\omega_1 = 3\omega_2$, following a similar analysis as above, the formal normal form is given by

$$\dot{z}_1 = i\omega_1 z_1 + a_{11} z_1^2 \bar{z}_1 + a_{12} z_1 z_2 \bar{z}_2 + a_{13}^2 z_2^3,$$
$$\dot{z}_2 = i\omega_2 z_2 + a_{21} z_2^2 \bar{z}_2 + a_{22} z_2 z_1 \bar{z}_1 + a_{23}^2 z_1 \bar{z}_2^2. \tag{29}$$

Note that the corresponding conjugate equations are not given. Substituting $z_j = r_j\, e^{i\theta_j}$, $j = 1, 2$ into equation (29) and transforming to polar coordinates results in

$$\dot{r}_1 + i r_1(\dot{\theta}_1 - \omega_1) = a_{11} r_1^3 + a_{12} r_1 r_2^2 + a_{13}^2 r_2^3\, e^{i(3\psi_2 - \psi_1)},$$
$$\dot{r}_2 + i r_2(\dot{\theta}_2 - \omega_2) = a_{21} r_2^3 + a_{22} r_2 r_1^2 + a_{23}^2 r_1 r_2^2\, e^{-i(3\psi_2 - \psi_1)},$$

where $\theta_j = \omega_j t + \psi_j$. Then, one has

$$\dot{r}_1 = \mathrm{Re}\,(a_{11}) r_1^3 + \mathrm{Re}\,(a_{12}) r_1 r_2^2 - \mathrm{Im}\,(a_{13}) r_2^3 \sin\alpha_2 + \mathrm{Re}\,(a_{13}) r_2^3 \cos\alpha_2,$$
$$\dot{r}_2 = \mathrm{Re}\,(a_{21}) r_2^3 + \mathrm{Re}\,(a_{22}) r_2 r_1^2 + \mathrm{Im}\,(a_{23}) r_1 r_2^2 \sin\alpha_2 + \mathrm{Re}\,(a_{23}) r_1 r_2^2 \cos\alpha_2,$$
$$\dot{\theta}_1 = \omega_1 + \mathrm{Im}\,(a_{11}) r_1^2 + \mathrm{Im}\,(a_{12}) r_2^2 + \mathrm{Re}\,(a_{13}) \frac{r_2^3}{r_1} \sin\alpha_2 + \mathrm{Im}\,(a_{13}) \frac{r_2^3}{r_1} \cos\alpha_2,$$
$$\dot{\theta}_2 = \omega_2 + \mathrm{Im}\,(a_{21}) r_2^2 + \mathrm{Im}\,(a_{22}) r_1^2 - \mathrm{Re}\,(a_{23}) r_1 r_2 \sin\alpha_2 + \mathrm{Im}\,(a_{23}) r_2 r_1 \cos\alpha_2,$$
$$\dot{\alpha}_2 = 3\dot{\theta}_2 - \dot{\theta}_1, \tag{30}$$

where $\alpha_2 = 3\theta_2 - \theta_1$.

According to equation (24), the coefficients related to normal form can be calculated by

$$G^2 = 0, \quad G^3 = \begin{Bmatrix} a_{2010}^1 z_1^2 \bar{z}_1 + a_{1101}^1 z_1 z_2 \bar{z}_2 + a_{0300}^1 z_2^3 \\ b_{0201}^1 z_2^2 \bar{z}_2 + b_{1110}^1 z_1 \bar{z}_1 z_2 + b_{1002}^1 z_1 \bar{z}_2^2 \\ c_{1020}^1 \bar{z}_1^2 z_1 + c_{0111}^1 \bar{z}_1 z_2 \bar{z}_2 + c_{0003}^1 \bar{z}_2^3 \\ d_{0102}^1 z_2 \bar{z}_2^2 + d_{1011}^1 z_1 \bar{z}_1 \bar{z}_2 + d_{0210}^1 \bar{z}_1 z_2^2 \end{Bmatrix},$$

where $a_{m_1 m_2 n_1 n_2}^1$, $b_{m_1 m_2 n_1 n_2}^1$, $c_{m_1 m_2 n_1 n_2}^1$ and $d_{m_1 m_2 n_1 n_2}^1$ are the coefficients in function F_1^3. $P^2 = (P_{(1)}^2\ P_{(2)}^2\ P_{(3)}^2\ P_{(4)}^2)^T$ can be determined through equation (25) as before.

3.3. OTHER CASES

For resonant models $2\omega_1 = \omega_2$ and $3\omega_1 = \omega_2$, the procedures of the analysis are similar to those in Sections 3.1 and 3.2. To save space, only the final results are given here and the details are omitted. For the resonant model $2\omega_1 = \omega_2$, the formal normal form can be obtained in complex coordinates as

$$\dot{z}_1 = i\omega_1 z_1 + a_{2010}^1 z_1^2 \bar{z}_1 + a_{1101}^1 z_1 z_2 \bar{z}_2 + a_{0110} \bar{z}_1 z_2.$$
$$\dot{z}_2 = i\omega_2 z_2 + b_{0201}^1 z_2^2 \bar{z}_2 + b_{1110}^1 z_2 z_1 \bar{z}_1 + b_{2000} z_1^2. \tag{31}$$

In polar coordinates, one has

$$\dot r_1 = \mathrm{Re}\,(a^1_{2010})r_1^3 + \mathrm{Re}\,(a^1_{1101})r_1r_2^2 + \mathrm{Im}\,(a_{0110})r_1r_2\sin\alpha_3 + \mathrm{Re}\,(a_{0110})r_1r_2\cos\alpha_3,$$

$$\dot r_2 = \mathrm{Re}\,(b^1_{0201})r_2^3 + \mathrm{Re}\,(b^1_{1110})r_2r_1^2 - \mathrm{Im}\,(b_{2000})r_1^2\sin\alpha_3 + \mathrm{Re}\,(b_{2000})r_1^2\cos\alpha_3,$$

$$\dot\theta_1 = \omega_1 + \mathrm{Im}\,(a^1_{2010})r_1^2 + \mathrm{Re}\,(a^1_{1101})r_2^2 - \mathrm{Re}\,(a_{0110})r_2\sin\alpha_3 + \mathrm{Im}\,(a_{0110})r_2\cos\alpha_3,$$

$$\dot\theta_2 = \omega_2 + \mathrm{Im}\,(b^1_{0201})r_2^2 + \mathrm{Re}\,(b^1_{1110})r_1^2 + \mathrm{Re}\,(b_{2000})\frac{r_1^2}{r_2}\sin\alpha_3 + \mathrm{Im}\,(b_{2000})\frac{r_1^2}{r_2}\cos\alpha_3,$$

$$\dot\alpha_3 = 2\dot\theta_1 - \dot\theta_2, \tag{32}$$

where $\alpha_3 = 2\theta_1 - \theta_2$.

For the resonant model $3\omega_1 = \omega_2$, the formal normal form can be obtained in complex coordinates as

$$\dot z_1 = i\omega_1 z_1 + a^1_{2010}z_1^2\bar z_1 + a^1_{1101}z_1z_2\bar z_2 + a^1_{0120}\bar z_1^2 z_2,$$

$$\dot z_2 = i\omega_2 z_2 + b^1_{0201}z_2^2\bar z_2 + b^1_{1110}z_2z_1\bar z_1 + b^1_{3000}z_1^3. \tag{33}$$

In polar coordinates, one has

$$\dot r_1 = \mathrm{Re}\,(a^1_{2010})r_1^3 + \mathrm{Re}\,(a^1_{1101})r_1r_2^2 + \mathrm{Im}\,(a^1_{0120})r_2r_1^2\sin\alpha_4 + \mathrm{Re}\,(a^1_{0120})r_2r_1^2\cos\alpha_4,$$

$$\dot r_2 = \mathrm{Re}\,(b^1_{0201})r_2^3 + \mathrm{Re}\,(b^1_{1110})r_2r_1^2 - \mathrm{Im}\,(b^1_{3000})r_1^3\sin\alpha_4 + \mathrm{Re}\,(b^1_{3000})r_1^3\cos\alpha_4,$$

$$\dot\theta_1 = \omega_1 + \mathrm{Im}\,(a^1_{2010})r_1^2 + \mathrm{Re}\,(a^1_{1101})r_2^2 - \mathrm{Re}\,(a^1_{0120})r_1r_2\sin\alpha_4 + \mathrm{Im}\,(a^1_{0120})r_1r_2\cos\alpha_4,$$

$$\dot\theta_2 = \omega_2 + \mathrm{Im}\,(b^1_{0201})r_2^2 + \mathrm{Re}\,(b^1_{1110})r_1^2 + \mathrm{Re}\,(b^1_{3000})\frac{r_1^3}{r_2}\sin\alpha_4 + \mathrm{Im}\,(b^1_{3000})\frac{r_1^3}{r_2}\cos\alpha_4,$$

$$\dot\alpha_4 = 3\dot\theta_1 - \dot\theta_2.$$

where $\alpha_4 = 3\theta_1 - \theta_2$.

The transformation functions P^2 can be calculated through equation (25) as before.

4. HIGHER ORDER INNER RESONANCE IN DEGENERATE SYSTEMS

When the coefficients of lower order normal forms are equal to zero, higher order normal forms have to be examined. In this case, if

$$\sum_{i=1}^{l} |m_i^e| > 4,$$

higher order inner resonance must be taken into account as well. Suppose

$$\sum_{i=1}^{l} |m_i^e| < 6.$$

Different resonant cases are considered in the following sections.

4.1. THE CASE OF $\omega_1 = 2\omega_2$

If there exists no inner resonance, in degenerate cases, the formal normal form of equation (1) is given by

$$\dot z_1 = i\omega_1 z_1 + a_{11}z_1^2\bar z_1 + a_{12}z_1^3\bar z_1^2 + a_{13}z_1z_2\bar z_2 + a_{14}z_1\bar z_2^2 z_2^2 + a_{15}z_1^2z_2\bar z_1\bar z_2,$$

$$\dot z_2 = i\omega_2 z_2 + a_{21}z_2^2\bar z_2 + a_{22}z_2^3\bar z_2^2 + a_{23}z_2z_1\bar z_1 + a_{24}z_1^2\bar z_1^2 z_2 + a_{25}z_2^2z_1\bar z_1\bar z_2. \tag{35}$$

In the resonant model $\omega_1 = 2\omega_2$, following a similar analysis as in section 3.1, the formal normal form of equation (1) is given by

$$\begin{aligned} \dot{z}_1 &= i\omega_1 z_1 + a_{11} z_1^2 \bar{z}_1 + a_{12} z_1^3 \bar{z}_1^2 + a_{13} z_1 z_2 \bar{z}_2 + a_{14} z_1 \bar{z}_2^2 z_2^2 + a_{15} z_1^2 z_2 \bar{z}_1 \bar{z}_2 \\ &\quad + b_{11} z_1 \bar{z}_1 z_2^2 + b_{12} z_2^2 + b_{13} \bar{z}_1 z_2^4 + b_{14} z_1^2 \bar{z}_2^2 + b_{15} \bar{z}_2 z_2^3, \\ \dot{z}_2 &= i\omega_2 z_2 + a_{21} z_2^2 \bar{z}_2 + a_{22} z_2^3 \bar{z}_2^2 + a_{23} z_2 z_1 \bar{z}_1 + a_{24} z_1^2 \bar{z}_1^2 z_2 + a_{25} z_2^2 z_1 \bar{z}_2 \\ &\quad + b_{21} z_1 \bar{z}_2 + b_{22} z_1 z_2 \bar{z}_2^2 + b_{23} \bar{z}_1 z_1^2 \bar{z}_2 + b_{24} z_1^2 \bar{z}_2^3 + b_{25} \bar{z}_1 z_2^3. \end{aligned} \tag{36}$$

Again, the corresponding conjugate equations are not given here. Substituting $z_j = r_j \, e^{i\theta_j}$, $j = 1, 2$, into equation (36) and transforming into polar coordinates, one has

$$\begin{aligned} \dot{r}_1 + ir_1(\dot{\theta}_1 - \omega_1) &= a_{11} r_1^3 + a_{12} r_1^5 + a_{13} r_1 r_2^2 + a_{14} r_1 r_2^4 + a_{15} r_1^3 r_2^2 \\ &\quad + b_{11} r_1^2 r_2^2 \, e^{i\alpha_1} + b_{12} r_2^2 \, e^{i\alpha_1} + b_{13} r_1 r_2^4 \, e^{2i\alpha_1} + b_{14} r_1^2 r_2^2 \, e^{-i\alpha_1} + b_{15} r_2^4 \, e^{i\alpha_1}, \\ \dot{r}_2 + ir_2(\dot{\theta}_2 - \omega_2) &= a_{21} r_2^3 + a_{22} r_2^5 + a_{23} r_2 r_1^2 + a_{24} r_2 r + a_{25} r_2^3 r_1^2 + b_{21} r_1 r_2 \, e^{-i\alpha_1} \\ &\quad + b_{22} r_1 r_2^3 \, e^{-i\alpha_1} + b_{23} r_1^3 r_2 \, e^{-i\alpha_1} + b_{24} r_1^2 r_2^3 \, e^{-2i\alpha_1} + b_{25} r_1 r_2^3 \, e^{i\alpha_1}, \end{aligned}$$

where $\theta_j = \omega_j t + \psi_j$ and $\alpha_1 = 2\psi_2 - \psi_1$. So,

$$\begin{aligned} \dot{r}_1 &= \mathrm{Re}\,(a_{11}) r_1^3 + \mathrm{Re}\,(a_{12}) r_1^5 + \mathrm{Re}\,(a_{13}) r_1 r_2^2 + \mathrm{Re}\,(a_{14}) r_1 r_2^4 + \mathrm{Re}\,(a_{15}) r_1^3 r_2^2 \\ &\quad - \mathrm{Im}\,(b_{11}) r_2^2 r_1^2 \sin\alpha_1 - \mathrm{Im}\,(b_{12}) r_2^2 \sin\alpha_1 - \mathrm{Im}\,(b_{13}) r_1 r_2^4 \sin 2\alpha_1 + \mathrm{Im}\,(b_{14}) r_2^2 r_1^2 \sin\alpha_1 \\ &\quad - \mathrm{Im}\,(b_{15}) r_2^4 \sin\alpha_1 + \mathrm{Re}\,(b_{11}) r_2^2 r_1^2 \cos\alpha_1 + \mathrm{Re}\,(b_{12}) r_2^2 \cos\alpha_1 + \mathrm{Re}\,(b_{13}) r_1 r_2^4 \cos 2\alpha_1 \\ &\quad + \mathrm{Re}\,(b_{14}) r_2^2 r_1^2 \cos\alpha_1 + \mathrm{Re}\,(b_{15}) r_2^4 \cos\alpha_1, \\ \dot{r}_2 &= \mathrm{Re}\,(a_{21}) r_2^3 + \mathrm{Re}\,(a_{22}) r_2^5 + \mathrm{Re}\,(a_{23}) r_2 r_1^2 + \mathrm{Re}\,(a_{24}) r_2 r_1^4 + \mathrm{Re}\,(a_{25}) r_1^2 r_2^3 \\ &\quad + \mathrm{Im}\,(b_{21}) r_2 r_1 \sin\alpha_1 + \mathrm{Im}\,(b_{22}) r_1 r_2^3 \sin\alpha_1 + \mathrm{Im}\,(b_{23}) r_1^3 r_2 \sin\alpha_1 + \mathrm{Im}\,(b_{24}) r_2^3 r_1^2 \sin 2\alpha_1 \\ &\quad - \mathrm{Im}\,(b_{25}) r_1 r_2^3 \sin\alpha_1 + \mathrm{Re}\,(b_{21}) r_2 r_1 \cos\alpha_1 + \mathrm{Re}\,(b_{22}) r_1 r_2^3 \cos\alpha_1 + \mathrm{Re}\,(b_{23}) r_1^3 r_2 \cos\alpha_1 \\ &\quad + \mathrm{Re}\,(b_{24}) r_2^3 r_1^2 \cos 2\alpha_1 + \mathrm{Re}\,(b_{25}) r_1 r_2^3 \cos\alpha_1, \\ \dot{\theta}_1 &= \omega_1 + \mathrm{Im}\,(a_{11}) r_1^2 + \mathrm{Im}\,(a_{12}) r_1^4 + \mathrm{Im}\,(a_{13}) r_2^2 + \mathrm{Im}\,(a_{14}) r_2^4 + \mathrm{Im}\,(a_{15}) r_1^2 r_2^2 \\ &\quad + \mathrm{Im}\,(b_{11}) r_1 r_2^2 \cos\alpha_1 + \mathrm{Im}\,(b_{12}) \frac{r_2^2}{r_1} \cos\alpha_1 + \mathrm{Im}\,(b_{13}) r_2^4 \cos 2\alpha_1 + \mathrm{Im}\,(b_{14}) r_1 r_2^2 \cos\alpha_1 \\ &\quad + \mathrm{Im}\,(b_{15}) \frac{r_2^4}{r_1} \cos\alpha_1 + \mathrm{Re}\,(b_{11}) r_1 r_2^2 \sin\alpha_1 + \mathrm{Re}\,(b_{12}) \frac{r_2^2}{r_1} \sin\alpha_1 + \mathrm{Re}\,(b_{13}) r_2^4 \sin 2\alpha_1 \\ &\quad - \mathrm{Re}\,(b_{14}) r_1 r_2^2 \sin\alpha_1 + \mathrm{Re}\,(b_{15}) \frac{r_2^4}{r_1} \sin\alpha_1, \\ \dot{\theta}_2 &= \omega_2 + \mathrm{Im}\,(a_{21}) r_2^2 + \mathrm{Im}\,(a_{22}) r_2^4 + \mathrm{Im}\,(a_{23}) r_1^2 + \mathrm{Im}\,(a_{24}) r_1^4 + \mathrm{Im}\,(a_{25}) r_1^2 r_2^2 \\ &\quad + \mathrm{Im}\,(b_{21}) r_1 \cos\alpha_1 + \mathrm{Im}\,(b_{22}) r_1 r_2^2 \cos\alpha_1 + \mathrm{Im}\,(b_{23}) r_1^3 \cos\alpha_1 + \mathrm{Im}\,(b_{24}) r_1^2 r_2^2 \cos 2\alpha_1 \\ &\quad + \mathrm{Im}\,(b_{25}) r_1 r_2^2 \cos\alpha_1 - \mathrm{Re}\,(b_{21}) r_1 \sin\alpha_1 - \mathrm{Re}\,(b_{22}) r_1 r_2^2 \sin\alpha_1 - \mathrm{Re}\,(b_{23}) r_1^3 \sin\alpha_1 \\ &\quad - \mathrm{Re}\,(b_{24}) r_1^2 r_2^2 \sin 2\alpha_1 + \mathrm{Re}\,(b_{25}) r_1 r_2^2 \sin\alpha_1, \end{aligned} \tag{37}$$

$$\dot{\alpha}_1 = 2\dot{\theta}_2 - \dot{\theta}_1.$$

According to equation (24), the coefficients related to normal form can be calculated by

$$G^2 = \begin{bmatrix} a_{0020} z_2^2 \\ b_{1001} z_1 \bar{z}_2 \\ c_{0002} \bar{z}_2^2 \\ d_{0110} \bar{z}_1 z_2 \end{bmatrix}, \qquad G^3 = \begin{bmatrix} a^1_{2010} z_1^2 \bar{z}_1 + a^1_{1101} z_1 z_2 \bar{z}_2 \\ b^1_{0201} z_2^2 \bar{z}_2 + b^1_{1110} z_1 \bar{z}_1 z_2 \\ c^1_{1020} \bar{z}_1^2 z_1 + c^1_{0111} \bar{z}_1 z_2 \bar{z}_2 \\ d^1_{0102} z_2 \bar{z}_2^2 + d^1_{1011} z \bar{z}_1 \bar{z}_2 \end{bmatrix},$$

$$G^4 = \begin{bmatrix} a^2_{1210} z_1 \bar{z}_1 z_2^2 + a^2_{0301} z_2^3 \bar{z}_2 + a^2_{2001} z_1^2 \bar{z}_2^2 \\ b^2_{0310} z_2^3 \bar{z}_1 + b^2_{1102} z_1 z_2 \bar{z}_2^2 + b^2_{2011} z_1^2 \bar{z}_1 \bar{z}_2 \\ c^2_{1012} z_1 \bar{z}_1 \bar{z}_2^2 + c^2_{0103} z_2 \bar{z}_2^3 + c^2_{0220} z_2^2 \bar{z}_1^2 \\ d^2_{1003} z_1 \bar{z}_2^3 + d^2_{0211} z_2^2 \bar{z}_1 \bar{z}_2 + d^2_{1120} z_1 z_2 \bar{z}_1^2 \end{bmatrix},$$

$$G^5 = \begin{bmatrix} a^3_{3020} z_1^3 \bar{z}_1^2 + a^3_{1202} z_1 z_2^2 \bar{z}_2^2 + a^3_{0410} z_2^4 \bar{z}_1 + a^3_{2111} z_1^2 z_2 \bar{z}_1 \bar{z}_2 \\ b^3_{0302} z_2^3 \bar{z}_2^2 + b^3_{2120} z_1^2 z_2 \bar{z}_1^2 + b^3_{2003} z_1^2 \bar{z}_2^3 + b^3_{1211} z_2^2 z_1 \bar{z}_2 \bar{z}_1 \\ c^3_{2030} z_1^2 \bar{z}_1^3 + c^3_{0212} \bar{z}_1 z_2^2 \bar{z}_2^2 + c^3_{1004} z_1 \bar{z}_2^4 + b^3_{1121} \bar{z}_1^2 z_1 z_2 \\ d^3_{0203} z_2^2 \bar{z}_2^3 + d^3_{2021} z_1^2 \bar{z}_2 \bar{z}_1^2 + d^3_{0320} z_2^3 \bar{z}_1^2 + b^3_{1112} \bar{z}_2^2 \bar{z}_1 z_2 z_1 \end{bmatrix},$$

where $a^2_{m_1m_2n_1n_2}$, $b^2_{m_1m_2n_1n_2}$, $c^2_{m_1m_2n_1n_2}$ and $d^2_{m_1m_2n_1n_2}$ are the coefficients in function F_2^4; $a^3_{m_1m_2n_1n_2}$, $b^3_{m_1m_2n_1n_2}$, $c^3_{m_1m_2n_1n_2}$ and $d^3_{m_1m_2n_1n_2}$ are the coefficients in function F_2^5; P^2, P^3 and P^4 can be determined readily through equation (25) as before.

4.2. THE CASE OF $\omega_1 = 3\omega_2$

If there exists no inner resonance, the normal form of equation (1) is given by equation (35). In the resonant model $\omega_1 = 3\omega_2$, following a similar analysis, the formal normal form in complex coordinates is

$$\begin{aligned} \dot{z}_1 &= i\omega_1 z_1 + a_{11} z_1^2 \bar{z}_1 + a_{12} z_1^3 \bar{z}_1^2 + a_{13} z_1 z_2 \bar{z}_2 + a_{14} z_1 \bar{z}_2^2 z_2^2 + a_{15} z_1^2 z_2 \bar{z}_1 \bar{z}_2 \\ &\quad + b_{11} z_1 \bar{z}_1 z_2^3 + b_{12} z_2^3 + b_{13} z_1^2 \bar{z}_2^3 + b_{14} z_2^4 \bar{z}_2, \\ \dot{z}_2 &= i\omega_2 z_2 + a_{21} z_2^2 \bar{z}_2 + a_{22} z_2^3 \bar{z}_2^2 + a_{23} z_2 z_1 \bar{z}_1 + a_{24} z_1^2 \bar{z}_1^2 z_2 + a_{25} z_2^2 z_1 \bar{z}_1 \bar{z}_2 \\ &\quad + b_{21} z_1 \bar{z}_2^2 + b_{22} z_1^2 \bar{z}_1 \bar{z}_2^2 + b_{23} \bar{z}_1 z_2^4 + b_{24} z_1 z_2 \bar{z}_2^3. \end{aligned} \tag{38}$$

Substituting $z_j = r_j\,\mathrm{e}^{i\theta_j}$, $j = 1, 2$ into equation (38) and transforming to polar coordinates, one has

$$\begin{aligned} \dot{r}_1 + ir_1(\dot{\theta}_1 - \omega_1) &= a_{11} r_1^3 + a_{12} r_1^5 + a_{13} r_1 r_2^2 + a_{14} r_1 r_2^4 + a_{15} r_1^3 r_2^2 + b_{11} r_1^2 r_2^3\,\mathrm{e}^{i\alpha_2} \\ &\quad + b_{12} r_2^3\,\mathrm{e}^{i\alpha_2} + b_{13} r_1^2 r_2^3\,\mathrm{e}^{-i\alpha_2} + b_{14} r_2^5\,\mathrm{e}^{i\alpha_2}, \\ \dot{r}_2 + ir_2(\dot{\theta}_2 - \omega_2) &= a_{21} r_2^3 + a_{22} r_2^5 + a_{23} r_2 r_1^2 + a_{24} r_2 r_1^4 + a_{25} r_2^3 r_1^2 \\ &\quad + b_{21} r_1 r_2^2\,\mathrm{e}^{-i\alpha_2} + b_{22} r_1^3 r_2^2\,\mathrm{e}^{-i\alpha_2} + b_{23} r_1 r_2^4\,\mathrm{e}^{i\alpha_2} + b_{24} r_1 r_2^4\,\mathrm{e}^{-i\alpha_2}, \end{aligned}$$

where $\theta_j = \omega_j t + \psi_j$ and $\alpha_2 = 3\psi_2 - \psi_1$;

$$\begin{aligned} \dot{r}_1 &= \mathrm{Re}\,(a_{11}) r_1^3 + \mathrm{Re}\,(a_{12}) r_1^5 + \mathrm{Re}\,(a_{13}) r_1 r_2^2 + \mathrm{Re}\,(a_{14}) r_1 r_2^4 + \mathrm{Re}\,(a_{15}) r_1^3 r_2^2 \\ &\quad - \mathrm{Im}\,(b_{11}) r_2^3 r_1^2 \sin\alpha_2 - \mathrm{Im}\,(b_{12}) r_2^3 \sin\alpha_2 + \mathrm{Im}\,(b_{13}) r_1^2 r_2^3 \sin\alpha_2 - \mathrm{Im}\,(b_{14}) r_2^5 \sin\alpha_2 \\ &\quad + \mathrm{Re}\,(b_{11}) r_2^3 r_1^2 \cos\alpha_2 + \mathrm{Re}\,(b_{12}) r_2^3 \cos\alpha_2 + \mathrm{Re}\,(b_{13}) r_1^2 r_2^3 \cos\alpha_2 + \mathrm{Re}\,(b_{14}) r_2^5 \cos\alpha_2, \end{aligned}$$

$$
\begin{aligned}
\dot{r}_2 = {}& \mathrm{Re}\,(a_{21})r_2^3 + \mathrm{Re}\,(a_{22})r_2^5 + \mathrm{Re}\,(a_{23})r_2 r_1^2 + \mathrm{Re}\,(a_{24})r_2 r_1^4 + \mathrm{Re}\,(a_{25})r_2^3 r_1^2 \\
& + \mathrm{Re}\,(b_{21})r_2^2 r_1 \cos\alpha_2 + \mathrm{Re}\,(b_{22})r_1^3 r_2^2 \cos\alpha_2 + \mathrm{Re}\,(b_{23})r_1 r_2^4 \cos\alpha_2 + \mathrm{Re}\,(b_{24})r_1 r_2^4 \cos\alpha_2 \\
& + \mathrm{Im}\,(b_{21})r_2^2 r_1 \sin\alpha_2 + \mathrm{Im}\,(b_{22})r_1^3 r_2^2 \sin\alpha_2 - \mathrm{Im}\,(b_{23})r_1 r_2^4 \sin\alpha_2 + \mathrm{Im}\,(b_{24})r_1 r_2^4 \sin\alpha_2, \\
\dot{\theta}_1 = {}& \omega_1 + \mathrm{Im}\,(a_{11})r_1^2 + \mathrm{Im}\,(a_{12})r_1^4 + \mathrm{Im}\,(a_{13})r_2^2 + \mathrm{Im}\,(a_{14})r_2^4 + \mathrm{Im}\,(a_{15})r_1^2 r_2^2 \\
& + \mathrm{Im}\,(b_{11})r_1 r_2^3 \cos\alpha_2 + \mathrm{Im}\,(b_{12})\frac{r_2^3}{r_1}\cos\alpha_2 + \mathrm{Im}\,(b_{13})r_1 r_2^3 \cos\alpha_2 + \mathrm{Im}\,(b_{14})\frac{r_2^5}{r_1}\cos\alpha_2 \\
& + \mathrm{Re}\,(b_{11})r_1 r_2^3 \sin\alpha_2 + \mathrm{Re}\,(b_{12})\frac{r_2^3}{r_1}\sin\alpha_2 - \mathrm{Re}\,(b_{13})r_1 r_2^3 \sin\alpha_2 + \mathrm{Re}\,(b_{14})\frac{r_2^5}{r_1}\sin\alpha_2, \\
\dot{\theta}_2 = {}& \omega_2 + \mathrm{Im}\,(a_{21})r_2^2 + \mathrm{Im}\,(a_{22})r_2^4 + \mathrm{Im}\,(a_{23})r_1^2 + \mathrm{Im}\,(a_{24})r_1^4 + \mathrm{Im}\,(a_{25})r_1^2 r_2^2 \\
& + \mathrm{Im}\,(b_{21})r_1 r_2 \cos\alpha_2 + \mathrm{Im}\,(b_{22})r_1^3 r_2 \cos\alpha_2 + \mathrm{Im}\,(b_{23})r_1 r_2^3 \cos\alpha_2 + \mathrm{Im}\,(b_{24})r_1 r_2^3 \cos\alpha_2 \\
& - \mathrm{Re}\,(b_{21})r_1 r_2 \sin\alpha_2 - \mathrm{Re}\,(b_{22})r_1^3 r_2 \sin\alpha_2 + \mathrm{Re}\,(b_{23})r_1 r_2^3 \sin\alpha_2 - \mathrm{Re}\,(b_{24})r_1 r_2^3 \sin\alpha_2, \\
\dot{\alpha}_2 = {}& 3\dot{\theta} - \dot{\theta}_1.
\end{aligned} \tag{39}
$$

According to equation (24), the coefficients related to the normal form can be calculated by

$$
G^2 = G^4 = 0, \qquad G^3 = \begin{pmatrix} a^1_{2010} z_1^2 \bar{z}_1 + a^1_{1101} z_1 z_2 \bar{z}_2 + a^1_{0300} z_2^3 \\ b^1_{0201} z_2^2 \bar{z}_2 + b^1_{1110} z_1 \bar{z}_1 z_2 + b^1_{1002} z_1 \bar{z}_2^2 \\ c^1_{1020} \bar{z}_1^2 z_1 + c^1_{0111} \bar{z}_1 z_2 \bar{z}_2 + c^1_{0003} \bar{z}_2^3 \\ d^1_{0102} z_2 \bar{z}_2^2 + d^1_{1011} z_1 \bar{z}_1 \bar{z}_2 + d^1_{0210} \bar{z}_1 z_2^2 \end{pmatrix},
$$

$$
G^5 = \begin{pmatrix} a^2_{3020} z_1^3 \bar{z}_1^2 + a^2_{1202} z_1 z_2^2 \bar{z}_2^2 + a^2_{1310} z_1 z_2^3 \bar{z}_1 + a^2_{2003} z_1^2 \bar{z}_2^3 + a^2_{0401} z_2^4 \bar{z}_2 + a^2_{2111} z_1^2 z_2 \bar{z}_1 \bar{z}_2 \\ b^2_{0302} z_2^3 \bar{z}_2^2 + b^2_{2120} z_1^2 z_2 \bar{z}_1^2 + b^2_{2012} z_1^2 \bar{z}_1 \bar{z}_2^2 + b^2_{0410} z_2^4 \bar{z}_1 + b^2_{1103} z_1 z_2 \bar{z}_2^3 + b^2_{1211} z_2^2 z_1 \bar{z}_1 \bar{z}_2 \\ c^2_{2030} z_1^2 \bar{z}_1^3 + c^2_{0212} \bar{z}_1 z_2^2 \bar{z}_2^2 + c^2_{1013} z_1 \bar{z}_1 \bar{z}_2^3 + c^2_{0320} z_2^3 \bar{z}_1^2 + c^2_{0104} z_2 \bar{z}_2^4 + c^2_{1121} \bar{z}_1^2 \bar{z}_2 z_1 z_2 \\ d^2_{0203} z_2^2 \bar{z}_2^3 + d^2_{2021} z_1^2 \bar{z}_2 \bar{z}_1^2 + d^2_{1220} z_1 z_2^2 \bar{z}_1^2 + d^2_{1004} z_1 \bar{z}_2^4 + d^2_{0311} z_2^3 \bar{z}_1 \bar{z}_2 + d^2_{1112} \bar{z}_2^2 z_1 z_2 \end{pmatrix},
$$

where $a^2_{m_1 m_2 n_1 n_2}$, $b^2_{m_1 m_2 n_1 n_2}$, $c^2_{m_1 m_2 n_1 n_2}$ and $d^2_{m_1 m_2 n_1 n_2}$ are the coefficients in function F_2^5; P^2 and P^3 can be determined through equation (25) as before.

In this case, since $F_3^5 = F_2^5 + DF_1^2 P^4 - DP^4 F_1^2$ and $F_1^2 = G2 = 0$, $F_3^5 = F_2^5$, one can determine G^5 directly from function F_2^5.

4.3. THE CASE OF $\omega_1 = 4\omega_2$

If there exists no inner resonance, the normal form of equation (1) is given by equation (35). In the resonant model $\omega_1 = 4\omega_2$, following a similar procedure, the formal normal form in complex coordinates is given by

$$
\begin{aligned}
\dot{z}_1 &= i\omega_1 z_1 + a_{11} z_1^2 \bar{z}_1 + a_{12} z_1^3 \bar{z}_1^2 + a_{13} z_1 z_2 \bar{z}_2 + a_{14} z_1 \bar{z}_2^2 z_2^2 + a_{15} z_1^2 z_2 \bar{z}_1 \bar{z}_2 + b_{11} z_2^4, \\
\dot{z}_2 &= i\omega_2 z_2 + a_{21} z_2^2 \overline{z_2} + a_{22} z_2^3 \bar{z}_2^2 + a_{23} z_2 z_1 \bar{z}_1 + a_{24} z_1^2 \bar{z}_1^2 z_2 + a_{25} z_2^2 z_1 \bar{z}_1 \bar{z}_2 + b_{21} z_1 \bar{z}_2^3.
\end{aligned} \tag{40}
$$

Substituting $z_j = r_j\,\mathrm{e}^{i\theta_j}$, $j = 1, 2$, into equation (40) and transforming to polar coordinates, one has

$$
\begin{aligned}
\dot{r}_1 + ir_1(\dot{\theta}_1 - \omega_1) &= a_{11} r_1^3 + a_{12} r_1^5 + a_{13} r_1 r_2^2 + a_{14} r_1 r_2^4 + a_{15} r_1^3 r_2^2 + b_{11} r_2^4\,\mathrm{e}^{i\alpha_3}, \\
\dot{r}_2 + ir_2(\dot{\theta}_2 - \omega_2) &= a_{21} r_2^3 + a_{22} r_2^5 + a_{23} r_2 r_1^2 + a_{24} r_2 r_1^4 + a_{25} r_2^3 r_1^2 + b_{21} r_1 r_2^3\,\mathrm{e}^{-i\alpha_3},
\end{aligned}
$$

where $\theta_j = \omega_j t + \psi_j$ and $\alpha_5 = 4\psi_2 - \psi_1$;

$$\begin{aligned}
\dot{r}_1 &= \mathrm{Re}\,(a_{11})r_1^3 + \mathrm{Re}\,(a_{12})r_1^5 + \mathrm{Re}\,(a_{13})r_1 r_2^2 + \mathrm{Re}\,(a_{14})r_1 r_2^4 + \mathrm{Re}\,(a_{15})r_1^3 r_2^2 \\
&\quad - \mathrm{Im}\,(b_{11})r_2^4 \sin\alpha_5 + \mathrm{Re}\,(b_{11})r_2^4 \cos\alpha_5, \\
\dot{r}_2 &= \mathrm{Re}\,(a_{21})r_2^3 + \mathrm{Re}\,(a_{22})r_2^5 + \mathrm{Re}\,(a_{23})r_2 r_1^2 + \mathrm{Re}\,(a_{24})r_2 r_1^4 + \mathrm{Re}\,(a_{25})r_2^3 r_1^2 \\
&\quad + \mathrm{Re}\,(b_{21})r_1 r_2^3 \cos\alpha_5 + \mathrm{Im}\,(b_{21})r_1 r_2^3 \sin\alpha_5, \\
\dot{\theta}_1 &= \omega_1 + \mathrm{Im}\,(a_{11})r_1^2 + \mathrm{Im}\,(a_{12})r_1^4 + \mathrm{Im}\,(a_{13})r_2^2 + \mathrm{Im}\,(a_{14})r_2^4 + \mathrm{Im}\,(a_{15})r_1^2 r_2^2 \\
&\quad + \mathrm{Im}\,(b_{11})\frac{r_2^4}{r_1}\cos\alpha_5 + \mathrm{Re}\,(b_{11})\frac{r_2^4}{r_1}\sin\alpha_5, \\
\dot{\theta}_2 &= \omega_2 + \mathrm{Im}\,(a_{21})r_2^2 + \mathrm{Im}\,(a_{22})r_2^4 + \mathrm{Im}\,(a_{23})r_1^2 + \mathrm{Im}\,(a_{24})r_1^4 + \mathrm{Im}\,(a_{25})r_1^2 r_2^2 \\
&\quad + \mathrm{Im}\,(b_{21})r_1 r_2^2 \cos\alpha_5 - \mathrm{Re}\,(b_{21})r_1 r_2^2 \sin\alpha_5, \\
\dot{\alpha}_5 &= 4\dot{\theta}_2 - \dot{\theta}_1.
\end{aligned} \tag{41}$$

According to equation (24), the coefficients related to normal form can be calculated by $G^2 = 0$,

$$G^3 = \begin{bmatrix} a^1_{2010} z_1^2 \bar{z}_1 + a^1_{1101} z_1 z_2 \bar{z}_2 \\ b^1_{0201} z_2^2 \bar{z}_2 + b^1_{1110} z_1 \bar{z}_1 z_2 \\ c^1_{1020} \bar{z}_1^2 z_1 + c^1_{0111} \bar{z}_1 z_2 \bar{z}_2 \\ d^1_{0102} z_2 \bar{z}_2^2 + d^1_{1011} z_1 \bar{z}_1 \bar{z}_2 \end{bmatrix}, \qquad G^4 = \begin{bmatrix} a^1_{0400} z_2^4 \\ b^1_{1003} z_1 \bar{z}_2^3 \\ c^1_{0004} \bar{z}_2^4 \\ d^1_{0310} z_2^3 \bar{z}_1 \end{bmatrix}$$

$$G^5 = \begin{bmatrix} a^2_{3020} z_1^3 \bar{z}_1^2 + a^2_{1202} z_1 z_2^2 \bar{z}_2^2 + a^2_{2111} z_1^2 z_2 \bar{z}_1 \bar{z}_2 \\ b^2_{0302} z_2^3 \bar{z}_2^2 + b^2_{2120} z_1^2 z_2 \bar{z}_1^2 + b^2_{1211} z_2^2 z_1 \bar{z}_1 \bar{z}_2 \\ c^2_{2030} z_1^2 \bar{z}_1^3 + c^2_{0212} \bar{z}_1 z_2^2 \bar{z}_2^2 + c^2_{1121} \bar{z}_1^2 \bar{z}_2 z_1 z_2 \\ d^2_{0203} z_2^2 \bar{z}_2^3 + d^2_{2021} z_1^2 \bar{z}_2 \bar{z}_1^2 + d^2_{1112} \bar{z}_2^2 \bar{z}_1 z_1 z_2 \end{bmatrix},$$

where $a^2_{m_1 m_2 n_1 n_2}$, $b^2_{m_1 m_2 n_1 n_2}$, $c^2_{m_1 m_2 n_1 n_2}$ and $d^2_{m_1 m_2 n_1 n_2}$ are the coefficients in functions F_2^4 and F_2^5; P^2 and P^3 can be determined through equation (25) as before.

4.4. OTHER RESONANT MODELS

For the other three resonant models, $2\omega_1 = \omega_2$, $3\omega_1 = \omega_2$ and $4\omega_1 = \omega_2$, the procedure is similar to those in sections 4.1, 4.2 and 4.3. To save space, only the final results are given in this section. For the resonant model $2\omega_1 = \omega_2$, the formal normal form in complex coordinates is given by

$$\begin{aligned}
\dot{z}_1 &= i\omega_1 z_1 + a^1_{2010} z_1^2 \bar{z}_1 + a^3_{3020} z_1^3 \bar{z}_1^2 + a^1_{1101} z_1 z_2 \bar{z}_2 + a^3_{1202} z_1 \bar{z}_2^2 z_2^2 + a^3_{2111} z_1^2 z_2 \bar{z}_1 \bar{z}_2 \\
&\quad + a^2_{3001} \bar{z}_2 z_1^3 + a_{0110} \bar{z}_1 z_2 + a^2_{1120} z_1 \bar{z}_1^2 z_2 + a^2_{0211} \bar{z}_1 z_2^2 \bar{z}_2 + a^3_{0230} \bar{z}_1^3 z_2^2, \\
\dot{z}_2 &= i\omega_2 z_2 + b^1_{0201} z_2^2 \bar{z}_2 + b^3_{0302} z_2^3 \bar{z}_2^2 + b^1_{1110} z_2 z_1 \bar{z}_1 + b^3_{2120} z_1^2 \bar{z}_1^2 z_2 + b^3_{1211} z_2^2 z_1 \bar{z}_1 \bar{z}_2 \\
&\quad + b_{2000} z_1^2 + b^2_{2101} z_1^2 \bar{z}_2 z_2 + b^2_{3010} \bar{z}_1 z_1^3 + b^3_{4001} z_1^4 \bar{z}_2 + b^2_{0220} \bar{z}_1^2 z_2^2.
\end{aligned} \tag{42}$$

The normal form in polar coordinates is

$$
\begin{aligned}
\dot{r}_1 = {} & \mathrm{Re}\,(a^1_{2010})r_1^3 + \mathrm{Re}\,(a^3_{3020})r_1^5 + \mathrm{Re}\,(a^1_{1101})r_1 r_2^2 + \mathrm{Re}\,(a^3_{1202})r_1 r_2^4 + \mathrm{Re}\,(a^3_{2111})r_1^3 r_2^2 \\
& + \mathrm{Im}\,(a^2_{3001})r_2 r_1^3 \sin\alpha_3 + \mathrm{Im}\,(a_{0110})r_1 r_2 \sin\alpha_3 + \mathrm{Im}\,(a^2_{1120})r_1^3 r_2 \sin\alpha_3 \\
& + \mathrm{Im}\,(a^2_{0211})r_2^3 r_1 \sin\alpha_3 + \mathrm{Im}\,(a^3_{0230})r_1^3 r_2^2 \sin 2\alpha_3 + \mathrm{Re}\,(a^2_{3001})r_2 r_1^3 \cos\alpha_3 \\
& + \mathrm{Re}\,(a_{0110})r_1 r_2 \cos\alpha_3 + \mathrm{Re}\,(a^2_{1120})r_1^3 r_2 \cos\alpha_3 + \mathrm{Re}\,(a^2_{0211})r_1 r_2^3 \cos\alpha_3 \\
& + \mathrm{Re}\,(a^3_{0230})r_1^3 r_2 \cos 2\alpha_3, \\
\dot{r}_2 = {} & \mathrm{Re}\,(b^1_{0201})r_2^3 + \mathrm{Re}\,(b^3_{0302})r_2^5 + \mathrm{Re}\,(b^1_{1110})r_2 r_1^2 + \mathrm{Re}\,(b^3_{2120})r_2 r_1^4 + \mathrm{Re}\,(b^3_{1211})r_2^3 r_1^2 \\
& - \mathrm{Im}\,(b_{2000})r_1^2 \sin\alpha_3 - \mathrm{Im}\,(b^2_{2101})r_1^2 r_2^2 \sin\alpha_3 - \mathrm{Im}\,(b^2_{3010})r_1^4 \sin\alpha_3 \\
& - \mathrm{Im}\,(b^3_{4001})r_1^4 r_2 \sin 2\alpha_3 + \mathrm{Im}\,(b^2_{0220})r_1^2 r_2^2 \sin\alpha_3 + \mathrm{Re}\,(b_{2000})r_1^2 \cos\alpha_3 \\
& + \mathrm{Re}\,(b^2_{2101})r_1^2 r_2^2 \cos\alpha_3 + \mathrm{Re}\,(b^3_{3101})r_1^4 \cos\alpha_3 + \mathrm{Re}\,(b^3_{4001})r_2 r_1^4 \cos 2\alpha_3 \\
& + \mathrm{Re}\,(b^2_{0220})r_1^2 r_2^2 \cos\alpha_3, \\
\dot{\theta}_1 = {} & \omega_1 + \mathrm{Im}\,(a^1_{3010})r_1^2 + \mathrm{Im}\,(a^3_{3020})r_1^4 + \mathrm{Im}\,(a^1_{1101})r_2^2 + \mathrm{Im}\,(a^3_{1202})r_2^4 + \mathrm{Im}\,(a^3_{2111})r_1^2 r_2^2 \\
& + \mathrm{Im}\,(a^2_{3001})r_2 r_1^2 \cos\alpha_3 + \mathrm{Im}\,(a_{0110})r_2 \cos\alpha_3 + \mathrm{Im}\,(a^2_{1120})r_1^2 r_2 \cos\alpha_3 \\
& + \mathrm{Im}\,(a^2_{0211})r_2^3 \cos\alpha_3 + \mathrm{Im}\,(a^3_{0230})r_1^2 r_2^2 \cos 2\alpha_3 + \mathrm{Re}\,(a^2_{3001})r_2 r_1^2 \sin\alpha_3 \\
& - \mathrm{Re}\,(a_{0110})r_2 \sin\alpha_3 - \mathrm{Re}\,(a^2_{1120})r_2 r_1^2 \sin\alpha_3 - \mathrm{Re}\,(a^2_{0211})r_2^3 \sin\alpha_3 \\
& - \mathrm{Re}\,(a^2_{0230})r_1^2 r_2^2 \sin 2\alpha_3, \\
\dot{\theta}_2 = {} & \omega_2 + \mathrm{Im}\,(b^1_{0201})r_2^2 + \mathrm{Im}\,(b^3_{0302})r_2^4 + \mathrm{Im}\,(b^1_{1110})r_1^2 + \mathrm{Im}\,(b^3_{2120})r_1^4 + \mathrm{Im}\,(b^3_{1211})r_1^2 r_2^2 \\
& + \mathrm{Im}\,(b_{2000})\frac{r_1^2}{r_2}\cos\alpha_3 + \mathrm{Im}\,(b^2_{2101})r_2 r_1^2 \cos\alpha_3 + \mathrm{Im}\,(b^2_{3010})\frac{r_1^4}{r_2}\cos\alpha_3 \\
& + \mathrm{Im}\,(b^3_{4001})r_1^4 \cos 2\alpha_3 + \mathrm{Im}\,(b^2_{0220})r_2 r_1^2 \cos\alpha_3 + \mathrm{Re}\,(b_{2000})\frac{r_1^2}{r_2}\sin\alpha_3 \\
& + \mathrm{Re}\,(b^2_{2101})r_2 r_1^2 \sin\alpha_3 + \mathrm{Re}\,(b^2_{3010})\frac{r_1^4}{r_2}\sin\alpha_3 + \mathrm{Re}\,(b^3_{4001})r_1^4 \sin 2\alpha_3 \\
& - \mathrm{Re}\,(b^2_{0220})r_2 r_1^2 \sin\alpha_3, \\
\dot{\alpha}_3 = {} & 2\dot{\theta} - \dot{\theta}_2,
\end{aligned} \tag{43}
$$

where $\theta_j = \omega_j t + \psi_j$ and $\alpha_3 = 2\psi_1 - \psi_2$.

For the resonant model $3\omega_1 = \omega_2$, the formal normal form in complex coordinates is given by

$$
\begin{aligned}
\dot{z}_1 = {} & i\omega_1 z_1 + a^1_{2010} z_1^2 \bar{z}_1 + a^3_{3020} z_1^3 \bar{z}_1^2 + a^1_{1101} z_1 z_2 \bar{z}_2 + a^3_{1202} z_1 \bar{z}_2^2 z_2^2 + a^3_{2111} z_1^2 \bar{z}_1 \bar{z}_2 \\
& + a^3_{4001} \bar{z}_2 z_1^4 + a^1_{0120} \bar{z}_1^2 z_2 + a^3_{0221} \bar{z}_1^2 z_2^2 \bar{z}_2 + a^3_{1130} z_1 \bar{z}_1^3 z_2, \\
\dot{z}_2 = {} & i\omega_2 z_2 + b^1_{0201} z_2^2 \bar{z}_2 + b^3_{0302} z_2^3 \bar{z}_2^2 + b^1_{1110} z_2 z_1 \bar{z}_1 + b^3_{2120} z_1^2 \bar{z}_1^2 z_2 + b^3_{1211} z_2^2 z_1 \bar{z}_1 \bar{z}_2 \\
& + b^1_{3000} z_1^3 + b^1_{0120} z_1^3 \bar{z}_2 z_2 + b^3_{0230} \bar{z}_1^3 z_2^2 + b^3_{4010} z_1^4 \bar{z}_1.
\end{aligned} \tag{44}
$$

The normal form in polar coordinates is given by

$$
\begin{aligned}
\dot r_1 &= \mathrm{Re}\,(a^1_{2010})r_1^3 + \mathrm{Re}\,(a^3_{3020})r_1^5 + \mathrm{Re}\,(a^1_{1101})r_1 r_2^2 + \mathrm{Re}\,(a^3_{1202})r_1 r_2^4 + \mathrm{Re}\,(a^3_{2111})r_1^3 r_2^2\\
&\quad + \mathrm{Im}\,(a^3_{4001})r_2 r_1^4 \sin\alpha_4 + \mathrm{Im}\,(a^1_{0120})r_1^2 r_2 \sin\alpha_4 + \mathrm{Im}\,(a^3_{0221})r_1^2 r_2^3 \sin\alpha_4\\
&\quad + \mathrm{Im}\,(a^3_{1130})r_1^4 r_2 \sin\alpha_4 + \mathrm{Re}\,(a^3_{4001})r_2 r_1^4 \cos\alpha_4 + \mathrm{Re}\,(a^1_{0120})r_1^2 r_2 \cos\alpha_4\\
&\quad + \mathrm{Re}\,(a^3_{0221})r_1^2 r_2^3 \cos\alpha_4 + \mathrm{Re}\,(a^3_{1130})r_1^4 r_2 \cos\alpha_4,\\
\dot r_2 &= \mathrm{Re}\,(b^1_{0201})r_2^3 + \mathrm{Re}\,(b^3_{0302})r_2^5 + \mathrm{Re}\,(b^1_{1110})r_2 r_1^2 + \mathrm{Re}\,(b^3_{2120})r_2 r_1^4 + \mathrm{Re}\,(b^3_{1211})r_2^3 r_1^2\\
&\quad + \mathrm{Re}\,(b^1_{3000})r_1^3 \cos\alpha_4 + \mathrm{Re}\,(b^3_{3101})r_1^3 r_2^2 \cos\alpha_4 + \mathrm{Re}\,(b^3_{0230})r_1^3 r_2^2 \cos\alpha_4\\
&\quad + \mathrm{Re}\,(b^3_{4010})r_1^5 \cos\alpha_4 - \mathrm{Im}\,(b^1_{3000})r_1^3 \sin\alpha_4 - \mathrm{Im}\,(b^3_{3101})r_1^3 r_2^2 \sin\alpha_4\\
&\quad + \mathrm{Im}\,(b^3_{0230})r_1^3 r_2^2 \sin\alpha_4 - \mathrm{Im}\,(b^3_{4010})r_1^5 \sin\alpha_4,\\
\dot\theta_1 &= \omega_1 + \mathrm{Im}\,(a^1_{3010})r_1^2 + \mathrm{Im}\,(a^3_{3020})r_1^4 + \mathrm{Im}\,(a^1_{1101})r_2^2 + \mathrm{Im}\,(a^3_{1202})r_2^4 + \mathrm{Im}\,(a^3_{2111})r_1^2 r_2^2\\
&\quad + \mathrm{Im}\,(a^3_{4001})r_2 r_1^3 \cos\alpha_4 + \mathrm{Im}\,(a^1_{0120})r_1 r_2 \cos\alpha_4 + \mathrm{Im}\,(a^3_{0221})r_1 r_2^3 \cos\alpha_4\\
&\quad + \mathrm{Im}\,(a^3_{1130})r_1^3 r_2 \cos\alpha_4 + \mathrm{Re}\,(a^3_{4001})r_2 r_1^3 \sin\alpha_4 - \mathrm{Re}\,(a^1_{0120})r_1 r_2 \sin\alpha_4\\
&\quad - \mathrm{Re}\,(a^3_{0221})r_1 r_2^3 \sin\alpha_4 - \mathrm{Re}\,(a^3_{1130})r_1^3 r_2 \sin\alpha_4,\\
\dot\theta_2 &= \omega_2 + \mathrm{Im}\,(b^1_{0201})r_2^2 + \mathrm{Im}\,(b^3_{0302})r_2^4 + \mathrm{Im}\,(b^1_{1110})r_1^2 + \mathrm{Im}\,(b^3_{2120})r_1^4 + \mathrm{Im}\,(b^3_{1211})r_1^2 r_2^2\\
&\quad + \mathrm{Im}\,(b^1_{3000})\frac{r_1^3}{r_2}\cos\alpha_4 + \mathrm{Im}\,(b^3_{3101})r_1^3 r_2 \cos\alpha_4 + \mathrm{Im}\,(b^3_{0230})r_1^3 r_2 \cos\alpha_4\\
&\quad + \mathrm{Im}\,(b^3_{4010})\frac{r_1^5}{r_2}\cos\alpha_4 + \mathrm{Re}\,(b^1_{3000})\frac{r_1^3}{r_2}\sin\alpha_4 + \mathrm{Re}\,(b^3_{3101})r_1^3 r_2 \sin\alpha_4\\
&\quad - \mathrm{Re}\,(b^3_{0230})r_2 r_1^3 \sin\alpha_4 + \mathrm{Re}\,(b^3_{4010})\frac{r_1^5}{r_2}\sin\alpha_4,\\
\dot\alpha_4 &= 3\dot\theta_1 - \dot\theta_2,
\end{aligned}
\tag{45}
$$

where $\theta_j = \omega_j t + \psi_j$ and $\alpha_4 = 3\psi_1 - \psi_2$.

Similarly, for the resonant model $4\omega_1 = \omega_2$, the formal normal form in complex coordinates is given by

$$
\begin{aligned}
\dot z_1 &= i\omega_1 z_1 + a^1_{2010}z_1^2\bar z_1 + a^3_{3020}z_1^3\bar z_1^2 + a^1_{1101}z_1 z_2\bar z_2 + a^3_{1202}z_1\bar z_2^2 z_2^2 + a^3_{2111}z_1^2 z_2\bar z_1\bar z_2 + a^2_{0130}z_2\bar z_1^3,\\
\dot z_2 &= i\omega_2 z_2 + b^1_{0201}z_2^2\bar z_2 + b^3_{0302}z_2^3\bar z_2^2 + b^1_{1110}z_2 z_1\bar z_1 + b^3_{2120}z_1^2\bar z_1^2 z_2 + b^3_{1211}z_2^2 z_1\bar z_1\bar z_2 + b^2_{4000}z_1^4.
\end{aligned}
\tag{46}
$$

The normal form in polar coordinates is

$$
\begin{aligned}
\dot r_1 &= \mathrm{Re}\,(a^1_{2010})r_1^3 + \mathrm{Re}\,(a^3_{3020})r_1^5 + \mathrm{Re}\,(a^1_{1101})r_1 r_2^2 + \mathrm{Re}\,(a^3_{1202})r_1 r_2^4 + \mathrm{Re}\,(a^3_{2111})r_1^3 r_2^2\\
&\quad + \mathrm{Im}\,(a^2_{0130})r_1^3 r_2 \sin\alpha_6 + \mathrm{Re}\,(a^2_{0130})r_1^3 r_2 \cos\alpha_6,\\
\dot r_2 &= \mathrm{Re}\,(b^1_{0201})r_2^3 + \mathrm{Re}\,(b^3_{0302})r_2^5 + \mathrm{Re}\,(b^1_{1110})r_2 r_1^2 + \mathrm{Re}\,(b^3_{2120})r_2 r_1^4 + \mathrm{Re}\,(b^3_{1211})r_2^3 r_1^2\\
&\quad + \mathrm{Re}\,(b^2_{4000})r_1^4 \cos\alpha_6 - \mathrm{Im}\,(b^2_{4000})r_1^4 \sin\alpha_6,\\
\dot\theta_1 &= \omega_1 + \mathrm{Im}\,(a^1_{3010})r_1^2 + \mathrm{Im}\,(a^3_{3020})r_1^4 + \mathrm{Im}\,(a^1_{1101})r_2^2 + \mathrm{Im}\,(a^3_{1202})r_2^4 + \mathrm{Im}\,(a^3_{2111})r_1^2 r_2^2\\
&\quad + \mathrm{Im}\,(a^2_{0130})r_1^2 r_2 \cos\alpha_6 - \mathrm{Re}\,(a^2_{0130})r_1^2 r_2 \sin\alpha_6,
\end{aligned}
$$

$$\dot{\theta}_2 = \omega_2 + \text{Im}\,(b^1_{0201})r_2^2 + \text{Im}\,(b^3_{0302})r_2^4 + \text{Im}\,(b^1_{1110})r_1^2 + \text{Im}\,(b^3_{2120})r_1^4 + \text{Im}\,(b^3_{1211})r_1^2 r_2^2$$
$$+ \text{Im}\,(b^2_{4000})\frac{r_1^4}{r_2}\cos\alpha_6 + \text{Re}\,(b^2_{4000})\frac{r_1^4}{r_2}\sin\alpha_6,$$
$$\dot{\alpha}_6 = 4\dot{\theta}_1 - \dot{\theta}_2, \tag{47}$$

where $\theta_j = \omega_j t + \psi_j$ and $\alpha_6 = 4\psi_1 - \psi_2$.

The transformation functions P^2, P^3 and P^4 can be calculated through equation (25) as before.

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