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FREQUENCY ANALYSIS OF INFINITE CONTINUOUS BEAM UNDER AXIAL LOADS

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The problem of lateral vibration of an axially loaded infinite uniform Bernoulli–Euler beam is investigated by setting up a model of the infinite beam with a harmonically varying, transverse concentrated force at the centre of the beam. A static axial load either tensile or compressive is applied to the beams. The elastic and stiff discrete supports are treated in the analysis. The influences of the axial load on the vibration modes and resonance frequencies of the beams are studied. Some numerical results such as the varying characteristics of resonance frequencies and the corresponding maximum amplitudes of the central deflection are presented with attention focused on the effect of axial load on vibration of the beam.

1. INTRODUCTION

Lateral vibration of beams under an axial load has been of practical interest in recent years. The influences of axial load on vibration characteristics of one-span or two-span beams have been well investigated, including the effects of various transverse forces and viscous damping [1–5]. Wang *et al*. [6] researched on elastic buckling of the beam by a hybrid method, in which more accurate results can be obtained. The results of the research also have wide application in civil, mechanical and aerospace engineering. For example, it may be possible non-destructively to estimate buckling loads by measuring frequencies at several load levels and extrapolating the results [7, 8]. In contrast, investigation of the effect of axial load on an infinite beam has received little attention. Some engineering problems, such as the buckling and stabling of railway continuous weld rail, have been of great concern. A recent paper by Kukla [9, 10] analysed the free vibration of an axially loaded beam with intermediate elastic supports and concentrated masses. The Green function method has been used in the study. In the case of a beam with n elastic supports, the frequency equation is expressed by means of an nth order determinant. The solution would be, however, complicated with n is large enough.

In the present investigation, the mathematical model of the infinite uniform Bernoulli–Euler beam is established. The discrete elastic and stiff supports are considered, and the steady state response is determined analytically. The transverse concentrated force at the centre of the beam varies harmonically in time. The damping of the transverse deflection, the strain and the discrete supports in the beam are included. The amplitude of the lateral motion at the centre of the beam is computed, and the variations as the functions of the forcing frequency and axial load are also examined. The static axial load may be tensile or compressive. The influences of the parameters such as the damping of the beam and the stiffness of the supports are compared. Then, displacement maximum

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value (over the range of all forcing frequencies) and corresponding forcing frequencies are analysed with attention focused on effects of the axial load in this paper.

2. MATHEMATICAL MODEL

Consider an infinite elastic beam with elastic discrete supports at regular intervals L, and axial co-ordinate $X (0 \leqslant X \leqslant L)$. Let the beam be excited at the fixed point of the centre of one span by vertical force which varies harmonically with time at circular frequency ω. It is shown in Figure 1. The modulus of elasticity is E, moment of inertia I, mass per unit length Θ, viscous damping coefficient of transverse deflection C, viscoelastic damping coefficient of the strain C_s, axial load P (positive if compressive), the transverse motion $W(X, t)$ which is positive if downward, t time. The equation of motion is [11]

$$C_s I \frac{\partial^5 W}{\partial X^4 \partial t} + EI \frac{\partial^4 W}{\partial X^4} + P \frac{\partial^2 W}{\partial X^2} + C \frac{\partial W}{\partial t} + \Theta \frac{\partial^2 W}{\partial t^2} = 0. \tag{1}$$

By introducing the non-dimensional co-ordinates $x = X/L$, $w = W/L$, $\beta = C_s I/EI$, $p = PL^2/EI$, $c = CL^4/EI$, $\rho = \Theta L^4/EI$, equation (1) becomes

$$\beta \frac{\partial^5 w}{\partial x^4 \partial t} + \frac{\partial^4 w}{\partial x^4} + p \frac{\partial^2 w}{\partial x^2} + c \frac{\partial w}{\partial t} + \rho \frac{\partial^2 w}{\partial t^2} = 0 \tag{2}$$

with $0 \leqslant x \leqslant 1$. If the steady state response is written as $w(x, t) = \mathrm{Re}\,\{u(x)\,\mathrm{e}^{i\omega t}\}$, the equation becomes

$$u''''(x) + au''(x) + bu(x) = 0 \tag{3}$$

where $a = P/(i\omega\beta + 1)$, $b = (i\omega c - \omega^2 \rho)/(i\omega\beta + 1)$.

As a first step a single typical unload span extending from $x = 0$ (just to the left of a support), to $x = 1$ (just to the a right of a support) (as shown in Figure 1) is considered. The general solution of the equation (3) is

$$u_0(x) = A_0\,\mathrm{e}^{k_1 x} + B_0\,\mathrm{e}^{k_2 x} + C_0\,\mathrm{e}^{k_3 x} + D_0\,\mathrm{e}^{k_4 x} \tag{4a}$$

and the general solution in the right neighboring span is

$$u_1(x) = A_1\,\mathrm{e}^{k_1 x} + B_1\,\mathrm{e}^{k_2 x} + C_1\,\mathrm{e}^{k_3 x} + D_1\,\mathrm{e}^{k_4 x} \tag{4b}$$

where $A_0 \sim D_0$ and $A_1 \sim D_1$ are unknown constants, and $k_1 \sim k_4$ are the root of the equation (5)

$$r^4 + ar^2 + b = 0. \tag{5}$$

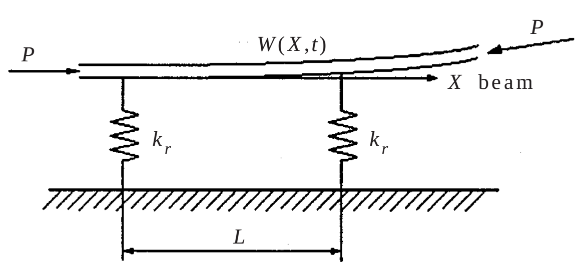

Figure 1. An infinite beam with elastic discrete supports.

The boundary conditions to be satisfied by $u_0(x)$ and $u_1(x)$ at the supporting point B are

$$\begin{cases} u_1(0) = u_0(1) \\ u_1'(0) = u_0(1) \\ u_1''(0) = u_0''(1) \\ u_1'''(0) = u_0'''(1) - k_r/(EI)u_0(1) \end{cases} \tag{6}$$

where k_r is the complex stiffness constant of the elastic supports, k_0 the real stiffness constant of the supports, and η_r the loss factor of the supports.

$$k_r = k_0(1 + i\eta_r). \tag{7}$$

Substituting equations (4a) and (4b) into equation (6) yields the following relation among the unknown constants $A_0 \sim D_0$ and $A_1 \sim D_1$

$$(A_1, B_1, C_1, D_1)^T = \mathbf{T}(A_0, B_0, C_0, D_0)^T \tag{8}$$

where $\mathbf{T}$ is the transfer matrix relating the constants $A_1 \sim D_1$ with the corresponding constants $A_0 \sim D_0$. Clearly

$$(A_n, B_n, C_n, D_n)^T = \mathbf{T}^n(A_0, B_0, C_0, D_0)^T \tag{9}$$

$A_n \sim D_n$ is the unknown constants of the right nth span. The eigenvalue and vectors of $\mathbf{T}$ are of vital importance. Using the numerical method, the eigenvalues r_i $(i = 1, 2, 3, 4)$ and eigenvectors $\mathbf{V}_i$ $(i = 1, 2, 3, 4)$ of $\mathbf{T}$ can be obtained. It has been proved that there are two self-reciprocal pairs of eigenvalues [12], i.e.

$$r_1 \cdot r_3 = 1, \qquad r_2 \cdot r_4 = 1. \tag{10}$$

Because the eigenvalues are self-reciprocal pairs, two of them, say r_1 and r_2, have module less than unity whereas the other two, $r_3 = 1/r_1$ and $r_4 = 1/r_2$, have module greater than unity. Now suppose that the displacement amplitude in some span is expressed as

$$v_0 = (A_0\mathbf{V}_1 + B_0\mathbf{V}_2 + C_0\mathbf{V}_3 + D_0\mathbf{V}_4)(e^{k_1x}, e^{k_2x}, e^{k_3x}, e^{k_4x})' \tag{11}$$

and the displacement amplitude in nth span is

$$v_n = (A_0r_1^n\mathbf{V}_1 + B_0r_2^n\mathbf{V}_2 + C_0r_3^n\mathbf{V}_3 + D_0r_4^n\mathbf{V}_4)(e^{k_1x}, e^{k_2x}, e^{k_3x}, e^{k_4x})'. \tag{12}$$

As we move progressively further from the excited span, n increases and the displacement amplitude v_n tends toward zero. Equation (12) shows that this will only happen if C_0 and D_0 are both zero.

$$v_n = (A_0r_1^n\mathbf{V}_1 + B_0r_2^n\mathbf{V}_2)(e^{k_1x}, e^{k_2x}, e^{k_3x}, e^{k_4x})' \qquad (0 \leqslant x \leqslant 1). \tag{13a}$$

In the same way, the displacement amplitude in the left mth span v_m' is

$$v_m' = (A_0'r_1^m\mathbf{V}_1 + B_0'r_2^m\mathbf{V}_2)(e^{k_1x'}, e^{k_2x'}, e^{k_3x'}, e^{k_4x'})' \qquad (0 \leqslant x \leqslant 1). \tag{13b}$$

The next step is to consider the loaded span excited at the centred point O by a force F which varies harmonically with time (positive if downward), as shown as

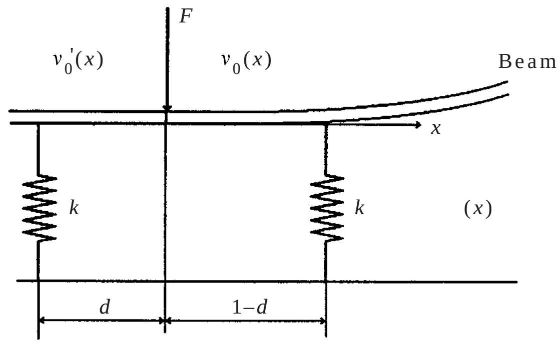

Figure 2. Beam acted on by a point force.

Figure 2. The displacement amplitude on both sides of the point O must satisfy the conditions

$$\begin{cases} u_1(x_c) = u_0(x_c) \\ u_1'(x_c) = u_0(x_c) \\ u_1''(x_c) = u_0''(x_c) \\ u_1'''(x_c) = u_0'''(x_c) - F/(EI). \end{cases} \tag{14}$$

Suppose that the load acts in the zeroth span. The displacement amplitude is

$$v_0 = (A_0\mathbf{V}_1 + B_0\mathbf{V}_2)(e^{k_1 x}, e^{k_2 x}, e^{k_3 x}, e^{k_4 x})' \qquad (d \leqslant x \leqslant 1) \tag{15a}$$

in the region to the right of the exciting point, and

$$v_0' = (A_0'\mathbf{V}_1 + B_0'\mathbf{V}_2)(e^{k_1 x'}, e^{k_2 x'}, e^{k_3 x'}, e^{k_4 x'})' \qquad (0 \leqslant x \leqslant d) \tag{15b}$$

in the region to the left of the exciting point, where $x' = 1 - x$. By applying the boundary condition at the exciting point (14) to v_0 and v_0', A_0, B_0, A_0' and B_0' can be solved.

Substitution of the coefficients A_0, B_0, A_0' and B_0' into equation (13a), (13b), (15a) and (15b) yields the displacement amplitude response corresponding to stationary point O exciting force.

3. NUMERICAL ANALYSIS OF DYNAMIC RESPONSE

Consider an infinite uniform continuous beam with elastic supports stiffness k_0. Transverse force F that varies harmonically with time acts as the center of one span, i.e. $d = 1/2$. The following parameters are kept constant throughout this investigation.

$$\begin{aligned} &\Theta = 10\ \text{kg/m} \qquad C = 0.2 \qquad C_s = 0.2 \\ &E = 2.1 \times 10^{11}\ \text{kg/m}^2 \qquad I = 3.22 \times 10^{-6}\ \text{m}^4 \qquad \eta_r = 0.2 \\ &L = 5\ \text{m}. \end{aligned} \tag{16}$$

The vertical displacement amplitude at the exciting point of transverse force is calculated with all range of the forcing frequency. The maximum amplitude value of the vibration modes and the corresponding force frequency at the different axial loads (tensile or compressive load) are discussed. All the solutions are calculated as complex displacements

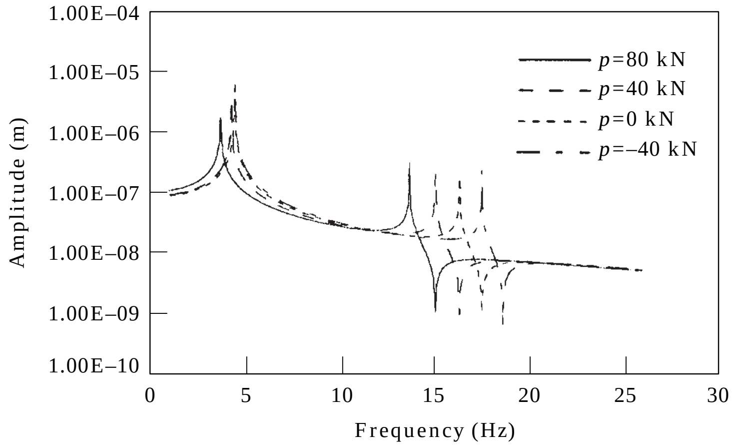

Figure 3. Amplitude versus frequency for the different axial loads.

but, to simplify the discussion, the module is presented here. Note that real displacement is most commonly analyzed in problems of technical vibrations. That is

$$A = |v_0|. \tag{17}$$

In the example of investigation, the elastic and stiff supports are considered. The forcing frequency is from 0 to 25 Hz in which the main vibration frequency would be included.

3.1. ELASTIC SUPPORTS

Figure 3 shows the results of amplitude vs. the corresponding frequency for the different axial loads. The stiffness of support k_0 is 5×10^6 N/m in the example. It can be clearly seen from the figure that there are two vibration modes in all range of the forcing frequency. They are denoted as P_1 and P_2. The first peak P_1 that corresponds to the first vibration mode does not vary obviously in frequency as the axial load P increases. Another one, associated with the ''higher'' frequency, decreases in frequency as P increases (Figure 3).

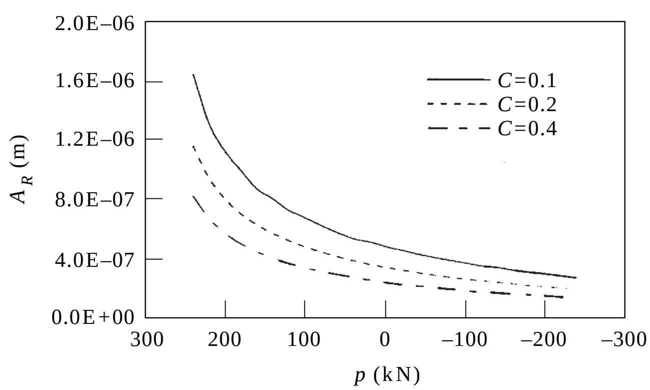

Figure 4. Resonant amplitude of P_2 versus axial load for different damping.

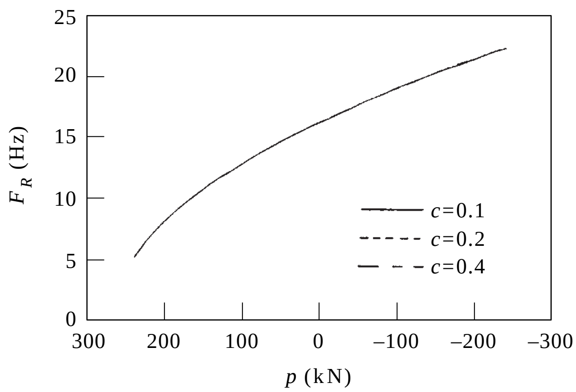

Figure 5. Resonant frequency of P_2 versus axial load for different damping.

(a) *Effect of axial load on* P_2

Let A_R denote the ‘‘resonant amplitude’’ of P_2, and let F_R denote the corresponding ‘‘resonant frequency’’. The amplitude of resonant peak P_2 and the corresponding frequency are plotted as functions of the axial load in Figures 4 and 5, respectively, for $C = 0.1$, 0.2 and 0.4. As the beam is induced by the tensile or the small compressive axial load P, resonant peak P_2 does not vary obviously in size as the P, as shown in Figure 4. As the compressive load P in the beam continues to increase, the amplitude of resonant peak P_2 begins to increase more quickly with increasing P, which can be expressed roughly by the factor.

$$A_R \sim b_{12}/(P - b_2) \qquad (b_1 \text{ and } b_2 \text{ are constants}) \tag{18}$$

This shows that there is a strong influence of the axial load for the displacement amplitude of resonant peak P_2 when the axial load is large. The greater axial load has the greater displacement amplitude until the buckling occurs in the beam.

The resonant frequency decreases with increasing P. But the tendency of change is being weakened with a reducing axial load (see Figure 5). The resonant frequency of P_2 does not decrease to zero, when the buckling of the beam is produced. It is equal to the resonant frequency of P_1 when the buckling takes place (the frequency is about 5 Hz in the example). There is a linear relationship between the square of F_R^2 and P (when $F_R > 0$) with a slope.

In Figure 4, one should notice that an increase in the damping c has a strong effect on the amplitude, but not on the resonant frequency.

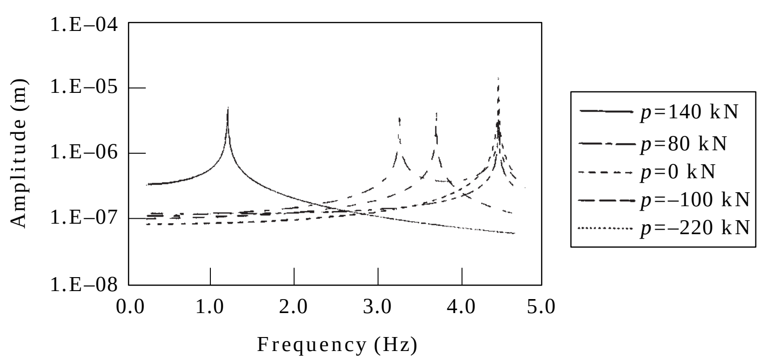

Figure 6. Amplitude of resonant mode P_1 versus frequency for different axial load.

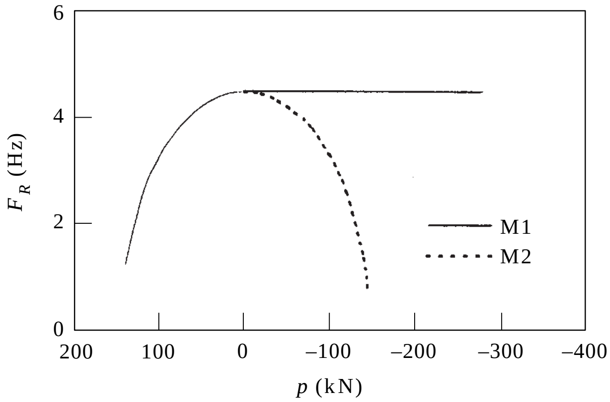

Figure 7. Resonant frequency of P_1 versus axial load.

(b) *Effect of axial load on* P_1

The numerical results for the amplitude of resonant mode P_1 and the corresponding axial load are shown in the Figure 6. One finds some more characteristics in the figure. The axial load can be divided into three regions as follows.

In the large tensile axial load region (beyond -145 kN in the example), the resonant frequency does not increase or decrease with the axial loads. It is equal to about 4.48 Hz (see Figure 7). The resonant amplitude becomes lower with a decreasing axial load, but the variation is small as shown as Figure 8.

In the intermediate axial load region (axial load $0 \sim -145$ kN in this example), unlike the first region, there are two vibration peaks in the frequency range of $0 \sim 5$ Hz. One of them, denoted as M1, has a resonant frequncy which does not change with the increasing P. It is 4.48 Hz. Its corresponding resonant amplitude is reduced with a decreasing axial load. The variation is larger when P tends to zero and it reaches a maximum value as P is 0 kN (Figure 8). However, the resonant frequency of another one (denoted as M2) is reduced with a decreasing P. It is equal to 4·48 Hz when P is 0 kN, and it reaches zero when P is about -145 kN. But its resonant amplitude shows some different phenomena. It decreases with an increasing P at first and reaches a minimum value as the axial load is about -100 kN. Then, it begins to increase with a decreasing P. The amplitude varies more rapidly when the axial load decreases to zero and it reaches the greatest value as P

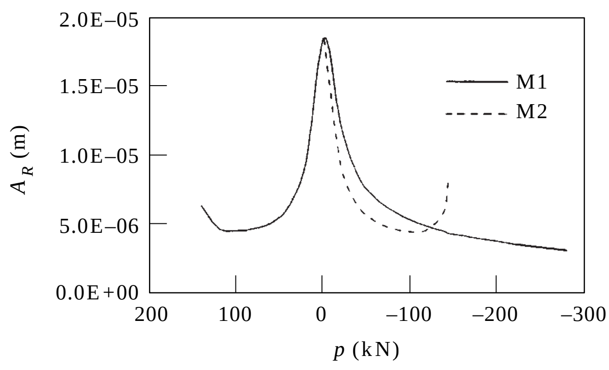

Figure 8. Resonant amplitude of P_1 versus axial load.

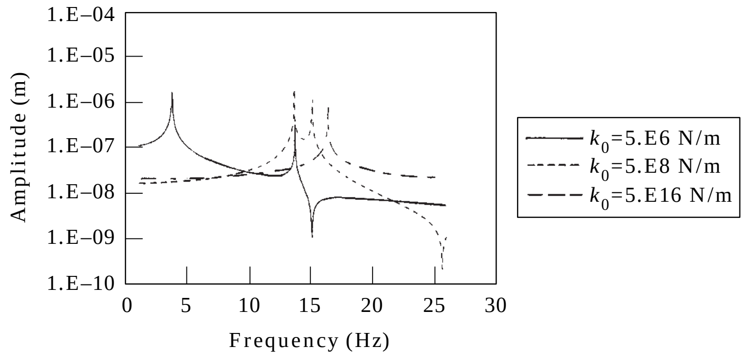

Figure 9. Amplitude versus frequency for different stiffness of the support.

is zero. Also, from Figure 8 it is seen that an increase in P has some similar effect on M1 and M2 on this region.

In the compressive axial load region, the resonant frequency starts to decrease with an increasing P, until the frequency reaches zero. The amplitude decreases with an increasing P at first (see Figures 7 and 8), but after P is greater than 100 kN, this varied tendency is changed. The amplitudes begin to increase with an increasing P. This phenomenon is symmetrical in the intermediate region of M2.

3.2. STIFF SUPPORT

It would be considered that supports of the beam are stiff when stiffness of the support k_0 is very large ($k_0 = 5 \times 10^{16}$ N/m in the investigation).

In Figure 9, the amplitudes of the lateral motion are plotted as a function of the corresponding frequencies for some stiffness of the support k_0. It is shown that, vibration mode P_1 increases with an increase k_0, and it would occur at higher frequencies (beyond the range of the forcing frequency of the example) when k_0 is large enough. But in P_1, the resonant frequency does not vary greatly with different k_0.

The maximum value of lateral displacement and the corresponding forcing frequency in stiff supports are plotted as functions of the axial load P in Figures 10 and 11.

They all have the same tendency of change as that in elastic supports (cf. Figures 4 and 5). That is, the resonant peak of P_2 increases with augmenting P and the corresponding resonant frequency decreases with an increasing P. The changes would be rapid with an

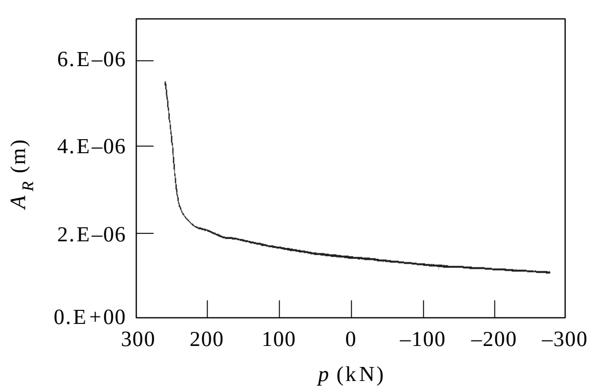

Figure 10. Maximum displacement versus axial load in stiff supports.

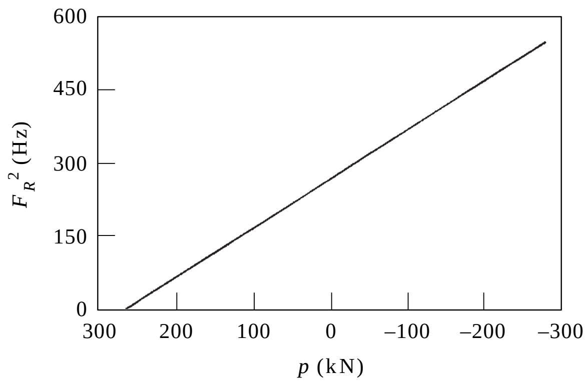

Figure 11. Square of resonant frequency versus axial load with stiff supports.

increasing P when P is large. The buckling load of the beam in stiff support is greater than that in elastic support.

Figure 11 shows a linear relationship between the square of the resonant frequency F_R^2 and axial load. This is the same as that of the one-span.

4. CONCLUSION

A method for solving problems of lateral vibration of axially loaded infinite uniform Bernoulli–Euler beams has been presented. Elastic and stiff supports were considered, and the beams were subjected to a harmonically transverse concentrated force at the centre of one span. The amplitudes of lateral motion and the corresponding forcing frequency vs. the axial load were investigated. The viscous and viscoelastic damping of the beam, and complex stiffness of support, were included in the analysis. The main conclusions are as follows.

In the elastic support, there are two vibration modes for the range of the forcing frequencies ($0 \sim 25$ Hz in the example). It is denoted as P_1 and P_2. In the vibration mode P_1, the amplitude and the corresponding forcing frequency can be divided into three regions according to different axial loads. In the large tensile axial load region, the resonant frequency does not change with different axial loads, and the corresponding amplitude is reduced with a decreasing axial load. In the intermediate axial load region, two resonant peaks are formed on the range of forcing frequency of P_1. One resonant frequency does not change with the increasing axial load and its corresponding amplitude attenuates with a decreasing P; another resonant frequency decreases with reduction of the axial load, and its amplitude increases with a decreasing P initially, then, decreases with a decreasing axial load. In the compressive axial load region, the resonant frequency decreases with an increasing P, and the corresponding amplitudes at first attenuate, with an increasing P, then increases with an augmenting P. The resonant amplitude increases with enlarging P, and the corresponding frequency declines with an increasing P for the vibration mode P_2.

In the stiff support, the natural frequency of the vibration mode P_1 is high enough to be out of all range of the forcing frequency, and the characteristic of vibration mode P_2 resembles that in the elastic support.

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APPENDIX: NOTATION

A	the module of beam amplitude
C	viscous damping coefficient of transverse deflection
c	dimensionless parameter, CL^4/EI
C_s	viscoelastic damping coefficient of strain
E	modulus of elasticity
F	transverse force
I	beam moment of inertia
k_0	real stiffness constant of the supports
k_r	complex stiffness constant of the elastic supports
L	beam length of one-span
P	axial load
p	dimensionless axial load, PL^2/EI
r_i	eigenvalues of $\mathbf{T}$
X	beam axial co-ordinate
x	dimensionless parameter, X/L
t	time
$W(X, t)$	transverse deflection of beam
w	dimensionless parameter, W/L
$\mathbf{T}$	transfer matrix of beam
$\mathbf{V}_i$	eigenvectors of $\mathbf{T}$
β	dimensionless parameter, C_sI/EI
η_r	loss factor of the supports
Θ	beam mass per unit length
ρ	dimensionless parameter, $\Theta L^4/EI$
ω	vibration frequency of beam