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OPTIMAL SUPPORT POSITIONS FOR A STRUCTURE TO MAXIMIZE ITS FUNDAMENTAL NATURAL FREQUENCY

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A procedure and related theories are developed to find the loci of optimal support positions for a structure to maximize its fundamental eigenvalue by increasing the support stiffness. The concept of limit eigenvalue, which is the upper bound of fundamental eigenvalue achieved by adding supports, is introduced. A condition is derived on which the fundamental eigenvalue can be reached to its limit eigenvalue. A sensitivity formula of eigenvalues with respect to the change of support positions is also derived to set up an optimization problem and to obtain its optimal support positions. It is found that the loci of m supports start from the maximum displacement position of the structure's first eigenfunction and end at certain positions on the nodal line of its $(m + 1)$th eigenfunction if the fundamental eigenvalue can reach its limit eigenvalue. The suggested method is tested to find the loci for a beam and a plate structure.

1. INTRODUCTION

The design concept to increase the fundamental natural frequency or eigenvalue of a structure as high as possible is commonly adopted to make the structure better in dynamic environment. The eigenvalue shift is achieved by changing the size, shape, and boundary conditions of a structure. Adding and/or changing support positions are also frequently used when the size or shape of the structure cannot be altered due to design limitations.

There have been some studies concerning the design issues of supports in a somewhat different aspect. Szelag and Mroz [1] is one of the earlier studies on this topic. They have treated optimal design problems of vibrating beams having unspecified thickness and lengths as well as supports whose positions and stiffnesses are to be determined. They have also shown that bimodal solutions occur in designing the elastic support. Akesson and Olhoff [2] have studied to find optimal support locations of a beam for the maximum fundamental natural frequency by applying Courant's maximum–minium principle. They have also investigated the minimum support stiffness leading to the maximum possible value of fundamental eigenvalue. Son and Kwak [3] have derived sensitivity of eigenvalues with respect to the change of boundary conditions using the material derivative concept, and the results have been applied to find the optimal support locations for the maximum

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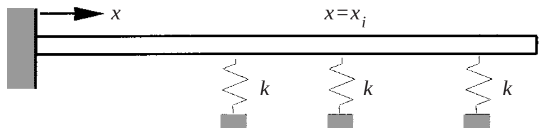

Figure 1. A beam supported by m points of stiffness k at x_i, $i = 1, \ldots, m$.

fundamental natural frequency of a plate structure. Pitarresi and Edwardo [4] have proposed an approach to improve initial guesses of support positions for a vibrating circuit card to increase its fundamental natural frequency. Narita [5] has studied the effects of the point constraint position of cantilever plates on the vibration characteristics. Liew and Lam [6] have studied the effects of the stiffness of elastic support constraints on the vibration response of plates.

From previous works [1, 2], it is apparent that the optimal support positions for the maximum fundamental eigenvalue are dependent on its support stiffness. Thus, it is of importance to study the pattern of optimal support positions by varying the support stiffness.

This paper is concerned with finding the loci of multiple support positions of a structure by varying its support stiffness for the maximum fundamental eigenvalue and developing the related principles. First, after defining a model, the concept of limit eigenvalue is introduced. Then, a cantilever plate having one support is examined to study the typical pattern of locus. From the results, it is shown that there are some characteristic points which govern the starting and ending points of the loci. Multiple sets of optimal support positions can be obtained with some discrete values of stiffness. Thus, the optimal loci can be found by connecting those points. As an example, the loci of a cantilever beam and cantilever rectangular plate are obtained and discussed.

2. MODEL AND LIMIT EIGENVALUE

The models used in this study are a beam and a plate structure having multiple supports with a certain value of stiffness which maximize those fundamental eigenvalues. Figures 1 and 2 show these structures. Two major assumptions are used in this study. First, the support has a translational spring which acts only in the direction of transverse displacement. Second, all the supports have identical stiffnesses.

There is a well-known theorem regarding the limit eigenvalue. The limit eigenvalue is the upper bound of potential fundamental eigenvalue achieved by adding m supports to the original structure. The Courant–Fisher theorem [7] states that all the eigenvalues of a structure increase if m supports with positive stiffness are placed on that structure, and

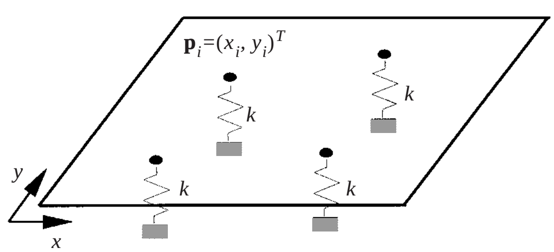

Figure 2. A plate supported by m points of stiffness k at $\mathbf{p}_i = (x_i, y_i)^T$, $i = 1, \ldots, m$.

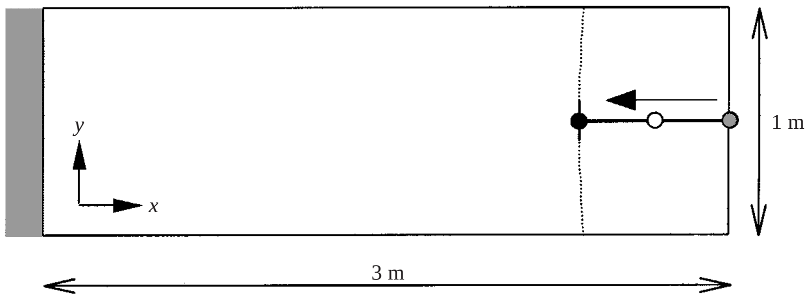

Figure 3. The locus of the optimal support position as support stiffness is increased. —, Nodal line of the second mode of the original structure; ●, $k = 1{\cdot}72 \times 10^4$ N/m; ○, $k = 4 \times 10^4$ N/m; ●, $K = 1{\cdot}74 \times 10^5$ N/m.

the modified eigenvalues are bounded as

$$\lambda_i^0 \leqslant \lambda_i \leqslant \lambda_{i+m}^0 \qquad i = 1, 2, \ldots, \tag{1}$$

where λ_i^0 and λ_i denote the ith original eigenvalue and the ith modified eigenvalue, respectively. The boundness of eigenvalues of the modified structure can also be explained by the eigenvalue separation principle in the case of rigid supports. From equation (1) it is seen that the fundamental eigenvalue of the modified structure cannot exceed λ_{m+1}^0 after adding m supports, and λ_{m+1}^0 is the limit eigenvalue. In addition, the fundamental mode shape of the modified structure, ϕ_1, should be the $(m+1)$th mode shape of the original structure, ϕ_{m+1}^0, in order for λ_1 to be λ_{m+1}^0. This fact means that the necessary condition of $\lambda_1 = \lambda_{m+1}^0$ is that the supports must be placed on the nodal line of ϕ_{m+1}^0.

3. TYPICAL PATTERN OF LOCUS

To clearly investigate the pattern of a locus while varying the support stiffness, a simple example is used. Figure 3 shows a cantilever plate to be constrained by one additional support. The size of the plate is 3×1 m having a thickness of 0·05 m. The elasticity, mass density, and Poisson ratio are 200×10^9 N/m^2, 7800 kg/m^3 and 0·3, respectively. In this case, the problem is to find the locus of support, where its fundamental natural frequency becomes maximum, as its stiffness is varied.

The result is illustrated in Figure 3. The dotted line in Figure 3 shows the nodal line of the second mode of the original structure. When the supporting point stiffness is very low (almost zero), the optimum support position is (3, 0·5). The support position remains fixed until the stiffness is increased to $1{\cdot}72 \times 10^4$ N/m. As the stiffness is increased further, the optimal position moves toward the second nodal line following the horizontal centre line of the plate. Finally, it reaches a point on the second nodal line when the stiffness becomes $1{\cdot}74 \times 10^5$ N/m. That point and the stiffness will be called the separation point and the critical stiffness, k_c. Beyond the critical stiffness, the optimal support position is not determined uniquely but it can be any point within a certain region on the second nodal line of the original structure (ϕ_2^0), and its fundamental eigenvalue is saturated to the second eigenvalue of the original structure (λ_2^0). The result is clearly plotted in Figure 4, showing the y co-ordinate of the support position and the maximized fundamental eigenvalues with an increase in the support stiffness. The region of optimal support position becomes wider as the stiffness is increased above the critical value k_c.

Thus, the pattern of the locus is characterized as follows. (1) It starts from the optimal support position when the support stiffness is almost zero. (2) Then it moves towards the second nodal line of the original structure as the support stiffness is increased. (3) It reaches

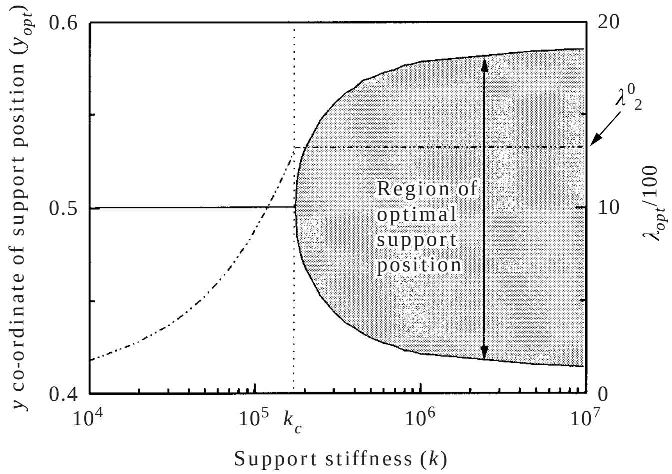

Figure 4. y co-ordinate of the support position and resulting fundamental eigenvalues as the support stiffness is varied. ——, y_{opt}; – · · –, λ_{opt}.

a point on the second nodal line when the stiffness is increased to its critical stiffness (k_c), and the optimal eigenvalue is saturated at its limit eigenvalue, the second eigenvalue of the original structure (λ_2^0). (4) When the support stiffness is further increased above the critical value, the optimal support position is not determined uniquely but can be placed at any point within a certain region on the second nodal line. The region becomes wider as the stiffness goes beyond the critical value, and the optimal eigenvalue is limited to the second eigenvalue of the original structure.

4. PROCEDURES TO FIND LOCUS

In this section, the procedure to calculate the loci of m optimal support positions within a structure is studied.

From the results in the previous section, the procedures for searching the optimal loci can be summarized as follows. (1) Find the optimal positions when the support stiffness is almost zero. (2) Investigate the existence condition where λ_1 can be increased to λ_{m+1}^0. (3) If the existence of condition (2) is satisfied, find the separation points and its critical stiffness. (4) If the condition is not satisfied, find the optimal support positions when the support stiffness is infinite. (5) Find the optimal support positions by increasing the stiffness from zero to the critical stiffness or infinite stiffness. (6) Connect those results to complete the loci.

For this purpose, first the free vibration equation of a structure having multiple supports is constructed and the structural design modification technique is utilized to calculate eigenvalues. Then the condition of procedure (2) is provided and an efficient method to obtain the critical stiffness and separation points is suggested for procedure (3). Also an optimization problem is formulated and eigenvalue sensitivity with respect to the support positions is derived for procedures (4) and (5).

4.1. FREE VIBRATION EQUATION

The free vibration equation can be formulated in both a variational form and in a discrete form. Both forms are used in this work because each has its own advantages depending on the problem in hand.

The variational form of the free vibration equation of a structure having m supports with stiffness k at $\mathbf{p}_1$, $\mathbf{p}_2, \ldots,$ and $\mathbf{p}_m$ is

$$a(u, \bar{u}) - \lambda d(u, \bar{u}) + \sum_{i=1,m} k u(\mathbf{p}_i)\bar{u}(\mathbf{p}_i) = 0. \tag{2}$$

The eigenvalue, λ, and the eigenfunction, u, must satisfy equation (2) for all $\bar{u} \in U_{ad}$, where U_{ad} denotes the set of admissible variations. The kinetic and potential energy bilinear forms for beam and plate, $a(\bullet, \bullet)$ and $d(\bullet, \bullet)$, are found in the book authored by Haug *et al.* [8].

The discrete form of the free vibration equation is

$$(\mathbf{K} - \lambda\mathbf{M})\mathbf{u} + k\mathbf{P}\mathbf{P}^T\mathbf{u} = \mathbf{0}, \tag{3}$$

where $\mathbf{M}$ and $\mathbf{K}$ are mass and stiffness matrices, respectively, obtained from the finite element analysis, $\mathbf{u}$ denotes the displacement vector, and $\mathbf{P}$ is a matrix whose column vector associates the displacement vector with the support position such as

$$\mathbf{P}_i^T\mathbf{u} = u(\mathbf{p}_i), \tag{4}$$

where $\mathbf{P}_i$ is the ith column vector of $\mathbf{P}$.

One additional consideration that should be accounted for is the reanalysis technique, since almost all computing efforts are used in solving equation (3) for lots of expected support positions. The dimension of equation (3) can be greatly reduced by using the reanalysis technique from the structural design modification method which utilizes the existing modal properties of the original structure as the basis vectors to span the displacement vector in a smaller space than the original space [4, 9].

4.2. OPTIMAL SUPPORT POSITIONS FOR $k \cong 0$

From the theory of small modification or sensitivity analysis [8], the fundamental eigenvalue after adding m supports, whose stiffnesses are almost zero, can be expressed as

$$\lambda_1 = \lambda_1^0 + \lambda_1^0\{(\phi_1^0(\mathbf{p}_1))^2 + (\phi_1^0(\mathbf{p}_2))^2 + \cdots + (\phi_1^0(\mathbf{p}_m))^2\}\delta k. \tag{5}$$

Thus, the optimal support positions for the case of almost zero stiffness are the maximum displacement positions of ϕ_1^0. This means that all of the loci of m supports start from that point. It should be noted that equation (5) holds for a structure having only a single fundamental eigenvalue.

4.3. CRITICAL STIFFNESS AND SEPARATION POINTS

It is not always possible to increase λ_1 of a structure to λ_{m+1}^0 after placing m supports on the nodal line of ϕ_{m+1}^0. This condition can be easily checked by solving an optimization problem as

$$\min\ (\lambda_1 - \lambda_{m+1}^0((\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m))^2 \tag{6}$$

$$\text{subject to } \mathbf{p}_1, \mathbf{p}_2, \ldots, \quad \text{and} \quad \mathbf{p}_m \in \Omega_{\mathrm{NL}},$$

and $k = \infty$, where Ω_{NL} is the nodal line of ϕ_{m+1}^0. The λ_1 can be increased to λ_{m+1}^0 when the optimal value of equation (6) becomes zero.

To derive critical stiffness and separation points, another condition is developed for $\lambda_1 = \lambda_{m+1}^0$ by placing m supports on the nodal line of ϕ_{m+1}^0.

[Existence condition of $\lambda_1 = \lambda^0_{m+1}$.] Let $\mathbf{p}_1, \mathbf{p}_2, \ldots$, and $\mathbf{p}_m$ be the positions of m supports on the nodal line of ϕ^0_{m+1}. If all non-zero solutions, k, of

$$\text{Det}\,(\mathbf{K} - \lambda^0_{m+1}\,\mathbf{M} + k\mathbf{P}^T\mathbf{P}) = 0 \tag{7}$$

are positive, then λ_1 can be increased to λ^0_{m+1} by the supports. Moreover, the minimum stiffness for $\lambda_1 = \lambda^0_{m+1}$ is max (k).

The derivation of the existence condition of $\lambda_1 = \lambda^0_{m+1}$ is based on the property of the inertia of a matrix [7]. First, it is assumed that λ^0_{m+1} is one of the eigenvalues of the structure after placing m supports to the nodal line of ϕ^0_{m+1}. Then, the characteristic equation becomes

$$\text{Det}\,(\mathbf{K} - \lambda^0_{m+1}\,\mathbf{M} + k\mathbf{P}\mathbf{P}^T) = 0. \tag{8}$$

The characteristic equation is only a function of the support stiffness, k, and there are m non-zero k satisfying that equation, since the rank of $\mathbf{P}\mathbf{P}^T$ is m. Let us define the number of negative eigenvalues of the matrix of equation (8) as

$$\pi(k) \equiv \pi(\mathbf{K} - \lambda^0_{m+1}\,\mathbf{M} + k\mathbf{P}\mathbf{P}^T). \tag{9}$$

It is apparent that

$$\pi(0) = m, \tag{10}$$

since there are m original eigenvalues less than λ^0_{m+1}. The eigenvalues of the matrix of equation (8) always rise as k is increased because of the positive semi-definiteness of $k\mathbf{P}\mathbf{P}^T$. To increase λ_1 to λ^0_{m+1}, the following condition applies to k_M:

$$\pi(k) = 0 \quad \text{for } k \geqslant k_M. \tag{11}$$

Let k_i^+ be the ith positive k satisfying equation (8), then equation (9) becomes

$$\pi(k_i^+) = m - i. \tag{12}$$

Thus, there must be m positive k to satisfy equation (11). Furthermore, the minimum stiffness, k_M, is the maximum k satisfying equation (8) in this case.

The direct use of equation (7) in obtaining k is inefficient because the dimension of the matrix is very large. There is an efficient method using the modal properties of the original structure. Equation (7) can be transformed as [10]

$$\text{Det}\,[\mathbf{I} + k\mathbf{H}_m\,(\lambda^0_{m+1})] = 0. \tag{13}$$

$\mathbf{H}_m$ is defined as

$$\mathbf{H}_m = \mathbf{P}^T[\mathbf{\Phi}^T(\mathbf{K} - \lambda\mathbf{M})\mathbf{\Phi}]^{-1}\mathbf{P} = \mathbf{P}^T(\mathbf{\Lambda} - \lambda\mathbf{I})^{-1}\mathbf{P}, \tag{14}$$

where $\mathbf{\Lambda}$ is the diagonal matrix composed of the eigenvalues of the original structure and $\mathbf{\Phi}$ is the eigenmatrix. Since all supports are placed on the nodal line of ϕ^0_{m+1}, $\mathbf{P}$ and ϕ^0_{m+1} are orthogonal. Thus, the $(m+1)$th modal property is excluded in deriving equation (14). The dimension of equation (13) is only m.

If there are support positions satisfying the existence condition of $\lambda_1 = \lambda^0_{m+1}$, then the separation points and the critical stiffness can be obtained from

$$\min\max\,(k(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m)) \tag{15}$$

$$\text{subject to } \forall(k) > 0$$

$$\mathbf{p}_1, \mathbf{p}_2, \ldots, \quad \text{and } \mathbf{p}_m \in \Omega_{NL},$$

where k represents the non-zero solutions of $\text{Det}\,(\mathbf{K} - \lambda^0_{m+1}\,\mathbf{M} + k\mathbf{P}^T\mathbf{P}) = 0$.

4.4. OPTIMAL SUPPORT POSITIONS FOR $0 < k < k_c$ OR $0 < k < \infty$

The loci are built by connecting the optimal support positions obtained at each discrete value of the stiffness in the range of $0 < k < k_c$ or $0 < k < \infty$. The optimal support positions with k are obtained by solving the optimization problem as

$$\max \min (\lambda(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m)) \tag{16}$$

$$\text{subject to } a(u, \bar{u}) - \lambda d(u, \bar{u}) + \sum_{i=1,m} ku(\mathbf{p}_i)\bar{u}(\mathbf{p}_i) = 0$$

$$\mathbf{p}_1, \mathbf{p}_2, \ldots, \quad \text{and} \quad \mathbf{p}_m \in \Omega,$$

where Ω is the domain of the structure. To solve the optimization problem effectively, it is essential to know the eigenvalue sensitivity with respect to the support positions.

Several authors have derived eigenvalue sensitivity for a beam structure with respect to its support positions. They have used a continuum approach with material derivative concept, normal mode method, or variational principle of Rayleigh's quotient with the Lagrange multiplier [11–13]. The derived sensitivity equation shows that it is proportional to its reaction forces and slopes of the original mode shape at the supports.

In this study, previous work is extended for general structures such as a plate. The key idea is the use of the variational form of the free vibration equation which augments the support stiffness term explicitly as in equation (2).

Figure 5 shows a plate having multiple supports, and each support moves in the direction of $\mathbf{v}_i$. For the sake of simplicity in deriving the sensitivity without loss of generality, it is initially pretended that the plate has one support with stiffness k at $\mathbf{p}$. After substituting $\bar{u} = u$ into equation (2), the free vibration equation becomes

$$a(u, u) - \lambda d(u, u) + ku^2(\mathbf{p}) = 0. \tag{17}$$

The variation of $\mathbf{p}$ can be expressed as

$$\delta\mathbf{p} = \mathbf{v}\delta z, \tag{18}$$

where $\mathbf{v}$ and δz are the direction and the magnitude of the variation, respectively. After taking variation of equation (17), the eigenvalue sensitivity becomes

$$\delta\lambda = 2[a(u, \delta u) - \lambda d(u, \delta u) + ku(\mathbf{p})\delta(u(\mathbf{p}))]. \tag{19}$$

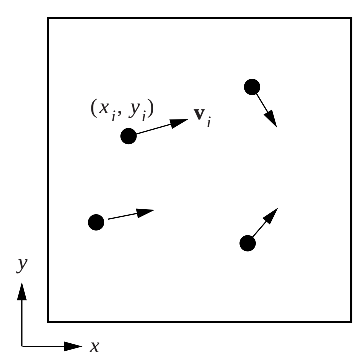

Figure 5. A plate having m supports when the ith support moves in $\mathbf{v}_i$ direction.

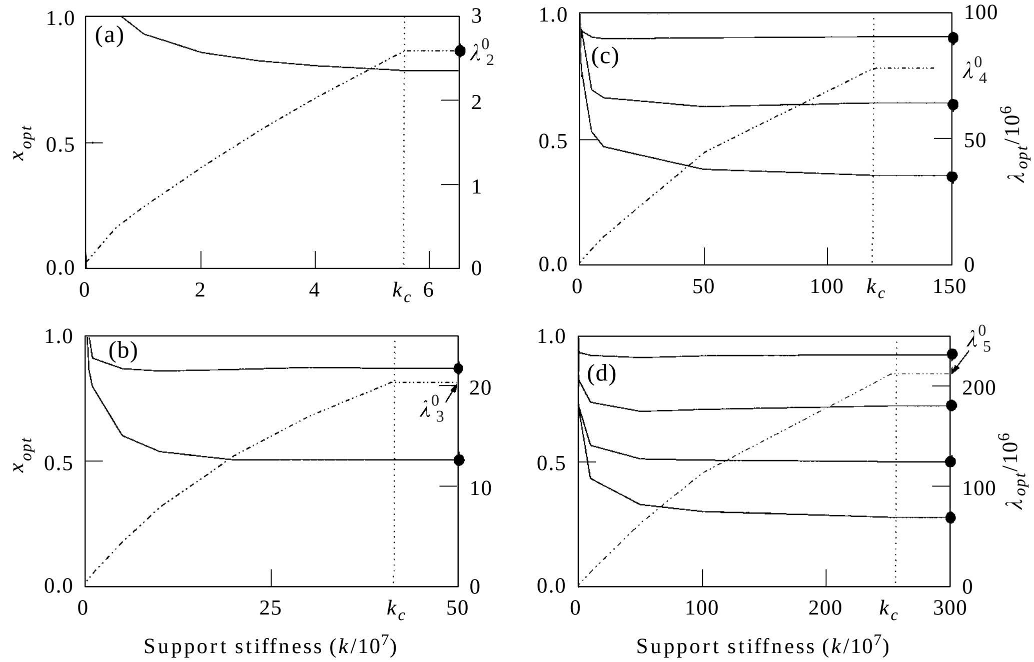

Figure 6. Loci of m optimal supports as k is increased and the corresponding optimal eigenvalues of a cantilever beam. ——, Loci, x_{opt}; $-\cdot\cdot\cdot-$, optimal eigenvalue, λ_{opt}; ●, nodal points of ϕ^0_{m+1}. (a) $m=1$; (b) $m=2$; (c) $m=3$; (d) $m=4$.

The mass normalizing condition is used in the derivation. By definition, the variation of eigenfunction at $\mathbf{p}$ can be expressed as

$$\delta(u(\mathbf{p})) = u(\mathbf{p} + \mathbf{v}\delta z) + \delta u(\mathbf{p} + \mathbf{v}\delta z) - u(\mathbf{p}). \tag{20}$$

Equation (20) is further simplified by expanding the eigenfunction at $\mathbf{p}$ and neglecting higher order variations as

$$\delta(u(\mathbf{p})) = \nabla u(\mathbf{p})^T \delta z + \delta u(\mathbf{p}), \tag{21}$$

where ∇ is the gradient operator. Since δu is also a member of admissible function, it can be substituted into equation (2) for $\bar{u}$ to obtain

$$a(u, \delta u) - \lambda d(u, \delta u) + ku(\mathbf{p})\delta u(\mathbf{p}) = 0. \tag{22}$$

By combining equations (19), (21) and (22), the eigenvalue sensitivity becomes

$$\delta\lambda = 2ku(\mathbf{p})\nabla u(\mathbf{p})^T \mathbf{v}\delta z. \tag{23}$$

Finally, if there are m supports and each of them is moving in $\mathbf{v}_i$ direction with δz_i, then the sensitivity becomes

$$\delta\lambda = 2k \sum_{i=1,m} u(\mathbf{p}_i)\nabla u(\mathbf{p}_i)^T \mathbf{v}_i \, \delta z_i. \tag{24}$$

Similarly, the sensitivity with infinite stiffness can be obtained as

$$\delta\lambda = -2 \sum_{i=1,m} f_i \, \nabla u(\mathbf{p}_i)^T \mathbf{v}_i \, \delta z_i, \tag{25}$$

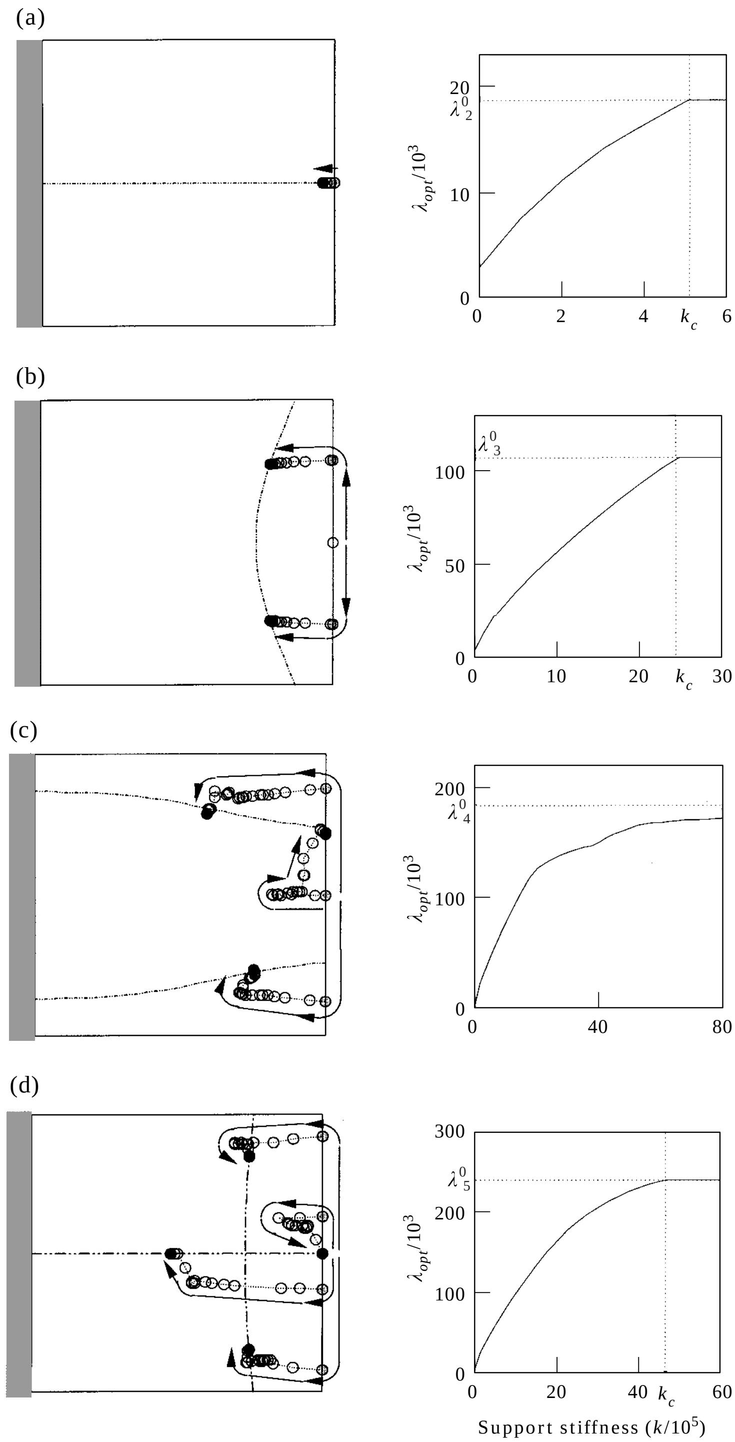

Figure 7. Loci of m optimal supports as k is increased and the corresponding optimal eigenvalues of a cantilever plate. – · · –, Nodal line of ϕ^0_{m+1}; ●, separation points for $m = 1, 2, 4$, and end points for $m = 3$; ●, departing points from the edge; $\Delta k = 2{\cdot}5 \times 10^5$, increment of k. (a) $m = 1$; (b) $m = 2$; (c) $m = 3$; (d) $m = 4$.

where f_i is the support reaction force. Thus, the eigenvalue sensitivity with respect to the support positions is proportional to reaction forces and slopes of eigenfunction along the moving direction at supports.

The optimization problem of equation (16) is carried out using a steepest decent approach.

4.5. OPTIMAL SUPPORT POSITIONS FOR $k_c < k < \infty$

In the case of $k_c < k < \infty$, the optimal support positions are not determined uniquely but they can be placed at any point within a certain region of the nodal line of ϕ^0_{m+1}. Thus, if a designer wants to determine unique support positions, the object of the optimization problem should be modified, such as shifting the second eigenvalue upwards as high as possible while λ_1 remains at λ^0_{m+1}.

5. EXAMPLES

The proposed method is tested to find the locus of optimal support positions for a cantilever beam and a cantilevered rectangular plate.

5.1. BEAM EXAMPLE

The loci of m optimal support positions of a cantilever beam, as shown in Figure 1, are obtained using the suggested method. The length, width, and thickness of the beam are 1, 0·1, and 0·05 m, respectively. The mass density is 7800 kg/m^3 and the elasticity is 200×10^9 N/m^2. Fifty elements are used in meshing the finite elements and 30 original modal properties are used in reanalysis. Figure 6 shows the loci with respect to support stiffness and the corresponding optimal eigenvalues when $m = 1, 2, 3, 4$. In all cases, the fundamental eigenvalue can be increased to λ^0_{m+1} by adding m supports. It is interesting to note that there are m nodal points and these become separation points. All of the loci start from $x = 1$, since this point is the maximum displacement point of ϕ^0_i. The loci move to the nodal points of ϕ^0_{m+1} as the stiffness increases. Finally, they reach the nodal points when the stiffness is increased to its critical stiffness as shown in Figure 6. After reaching the critical stiffness, the optimal support positions remain fixed on the nodal points and the optimal eigenvalue is also unchanged as λ^0_{m+1}.

5.2. PLATE EXAMPLE

The loci of m optimal support positions of a cantilevered plate, as shown in Figure 2, are obtained using the proposed method. The plate dimensions are 1×1 m, with a thickness of 0·05 m. The mass density, elasticity and Poisson ratio are 7800 kg/m^3, 200×10^9 N/m^2 and 0·3, respectively. Four-hundred elements are used in meshing the finite elements and 100 original modal properties are used in reanalysis. Figure 7 shows the loci and optimal eigenvalues as the stiffness is increased with $m = 1, 2, 3, 4$. All the loci start from (1, 0·5) because this point is the maximum displacement point of ϕ^0_1. It is found that the fundamental eigenvalue can be increased to λ^0_{m+1} in all cases except when $m = 3$.

For $m = 1$, the locus departs from (1, 0·5) and approaches the separation point, (0·96, 0·5), which is on the nodal line of ϕ^0_2. For $m = 2$, the loci start from (1, 0·5) and move along the right edge and depart from that edge as the stiffness is increased. After that they move to the separation points, (0·78, 0·23) and (0·78, 0·77) which are on the nodal line of ϕ^0_3. The loci move in a symmetrical manner about the horizontal centre line of the plate.

In the case of $m = 3$, no points on the nodal line of ϕ^0_4 satisfy the existence condition of critical stiffness. Thus, it is anticipated that the optimal eigenvalue cannot

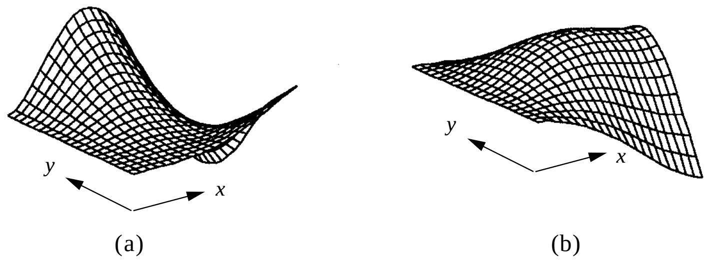

Figure 8. The first two mode shapes of the optimized structure for $m = 3$. (a) First mode shape ($\lambda_1 = 0{\cdot}958\lambda_4^0$); (b) second mode shape ($\lambda_2 = \lambda_4^0$).

reach its limit eigenvalue. To verify this fact, the optimal points of support with infinite stiffness are found on the nodal line of ϕ_4^0. These points are (0·59, 0·21), (1, 0·28) and (0·75, 0·76). The optimal eigenvalue becomes 95·8% of its limit eigenvalue (λ_4^0), and the second eigenvalue becomes its limit eigenvalue in this case. The first two mode shapes of the optimized structure are shown in Figure 8. From the results, it is clear that the optimal mode cannot be increased to ϕ_4^0 but it becomes the mode shown in Figure 8(a). To determine the end points of loci, the optimal supports for $k = \infty$ are found on the whole domain of the structure. These points are (0·60, 0·19), (0·98, 0·26) and (0·75, 0·77). The optimal eigenvalue becomes 98·0% of its limit eigenvalue, and the second eigenvalue becomes the same as the first eigenvalue. The bimodal phenomenon is well known in the area of finding optimal support positions [1–3]. The loci start from (1, 0·5) and move to the optimal points for infinite stiffness in a somewhat complex manner as shown in Figure 7(c).

For $m = 4$, the loci start from (1, 0·5) and move along the edge in a symmetrical manner as the stiffness is increased, as shown in Figure 7(d). After that, the loci depart from the edge and approach the separation points, (0·48, 0·5), (1, 0·5), (0·75, 0·15), and (0·75, 0·85) which are all on the nodal line of ϕ_5^0.

6. CONCLUSIONS

A procedure is proposed to find the loci of optimal support positions of a structure, while varying its support stiffness, in which the fundamental eigenvalue of the structure is maximized. To increase the fundamental eigenvalue to its limit eigenvalue by adding supports, the supports should be placed on the $(m + 1)$th nodal line of the original structure and at the same time a certain existence condition must be satisfied. The loci start from the maximum displacement position of the first eigenfunction of the original structure. The end positions of the loci are located on the nodal line of the $(m + 1)$th eigenfunction if the fundamental eigenvalue of the modified structure can be increased to the $(m + 1)$th eigenvalue of the original structure. Or, these positions are the optimal support positions in the case of infinite support stiffness. The optimal support positions are within a certain region if the end positions are found on the nodal line of the $(m + 1)$th eigenfunction.

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