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LETTERS TO THE EDITOR

AN UPPER BOUND ON DISPLACEMENTS OF DAMPED LINEAR SYSTEMS

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1. INTRODUCTION

The free vibration of an n-degree-of-freedom linear second order system is represented by

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = \theta_n, \qquad x(0) = x_0, \qquad \dot{x}(0) = \dot{x}_0, \tag{1}$$

for all $t \geqslant 0$. In equation (1),

$$x(t) = [x_1(t) \quad x_2(t) \cdots x_n(t)]^{\mathrm{T}} \in \mathbb{R}^n, \tag{2}$$

denotes the vector of displacements (v^{T} denotes the transpose of a vector v); the mass matrix M and the stiffness matrix K belong to $\mathbb{R}^{n \times n}$ and are symmetric and positive definite; the damping matrix C belongs to $\mathbb{R}^{n \times n}$ and is symmetric and positive semi-definite; $x_0 \in \mathbb{R}^n$ and $\dot{x}_0 \in \mathbb{R}^n$ are the vectors of initial displacements and velocities, respectively; θ_n denotes the zero vector in $\mathbb{R}^n$.

In this note, we plan to derive *a priori* upper bounds on the sizes (norms) of displacements of the system (1) without solving the system (numerically). In recent years, researchers have derived bounds on the sizes of displacements and velocities of free or forced vibratory systems; see, e.g., references [1, 2], [3, p. 136], [4–7], [8, pp. 177–178], [9]. Such bounds can be used in the design and analysis of systems.

Bounds on the sizes of displacements of the system (1) are useful only if (1) they are easily computable; (2) they are tight. If the bounds are not easily computable, then one might as well solve the system (1) (numerically) in order to obtain the exact (very accurate) values for the displacement peaks. If, on the other hand, the bounds are easily computable, but are conservatively large, then they furnish no useful information to be used in the system design and analysis. It appears that the two requirements of ease-of-computation and tightness of the upper bounds oppose each other: the less (more, respectively) computational effort, the more (less) conservative bounds on the sizes of displacements. Despite this fact, one should attempt to derive easy-to-compute and tight bounds.

Most available bounds in the literature are not easily computable, except those in references [1–3] and [7]. In reference [7], upper bounds on the sizes of displacements of the system (1) are computed as follows. Let

$$\|x_i\|_\infty := \max_{t \geqslant 0} |x_i(t)|, \tag{3}$$

denote the L_∞-norm of the displacement $x_i(\cdot)$ for an $i = 1, 2, \ldots, n$. Let

$$E_0 := \tfrac{1}{2} x_0^{\mathrm{T}} K x_0 + \tfrac{1}{2} \dot{x}_0^{\mathrm{T}} M \dot{x}_0, \tag{4}$$

denote the initial energy of the system (1). According to reference [7], the norm $\|x_i\|_\infty$ for an $i = 1, 2, \ldots, n$ satisfies

$$\|x_i\|_\infty \leqslant [2(K^{-1})_{ii} E_0]^{1/2}, \tag{5}$$

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where $(K^{-1})_{ii} > 0$ denotes the ith diagonal element of the matrix K^{-1}. There are two comments to follow:

1) The upper bounds in inequality (5) depend on the matrices M and K, where the dependence on M is through E_0. Computing K^{-1} in order to obtain $(K^{-1})_{ii}$ for an $i = 1, 2, \ldots, n$ requires some computational effort, because K is in general a full matrix. An interesting feature of the bounds in inequality (5) is that they do not depend on the damping matrix C.

2) A careful examination of the bounds in inequality (5) reveals that they can be conservatively large for some $i = 1, 2, \ldots, n$.

We present an example to examine the bounds in inequality (5). In this example, we compute the upper bounds on the L_∞-norms of displacements of a system using inequality (5) and compare them to the exact values of the norms obtained from the numerical solution of the system. Consider the system in Figure 1 and let $m_i = 1$, $c_i = 0{\cdot}1$, and $k_i = 1$ for all $i = 1, 2, 3$. The free vibration of this sytem is represented by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \\ \ddot{x}_3(t) \end{bmatrix} + \begin{bmatrix} 0{\cdot}2 & -0{\cdot}1 & 0 \\ -0{\cdot}1 & 0{\cdot}2 & -0{\cdot}1 \\ 0 & -0{\cdot}1 & 0{\cdot}1 \end{bmatrix}\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} + \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \theta_3, \tag{6a}$$

for all $t \geqslant 0$, with the initial conditions

$$x_0 = [1 \quad 0 \quad 0]^{\mathrm{T}}, \qquad \dot{x}_0 = \theta_3. \tag{6b}$$

Identifying the matrices M and K in equation (6a), we obtain

$$E_0 = 1, \qquad K^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}. \tag{7a, b}$$

Therefore, by inequality (5), we obtain

$$\|x_1\|_\infty \leqslant 1{\cdot}4142, \qquad \|x_2\|_\infty \leqslant 2, \qquad \|x_3\|_\infty \leqslant 2{\cdot}4495. \tag{8}$$

By the numerical integration, we obtained responses of the system (6) that are depicted in Figures 2, 3 and 4. From these figures, we obtain

$$\|x_1\|_\infty = 1, \qquad \|x_2\|_\infty = 0{\cdot}43, \qquad \|x_3\|_\infty = 0{\cdot}52. \tag{9}$$

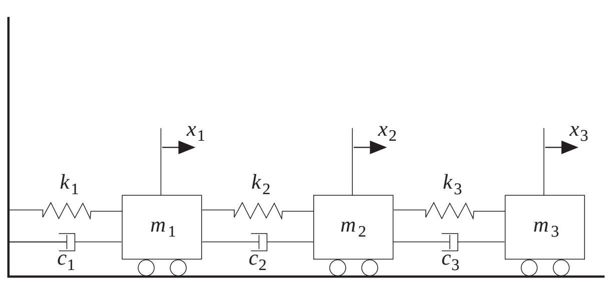

Figure 1. A system with three degrees of freedom, where $m_i = 1$, $c_i = 0{\cdot}1$, and $k_i = 1$ for all $i = 1, 2, 3$.

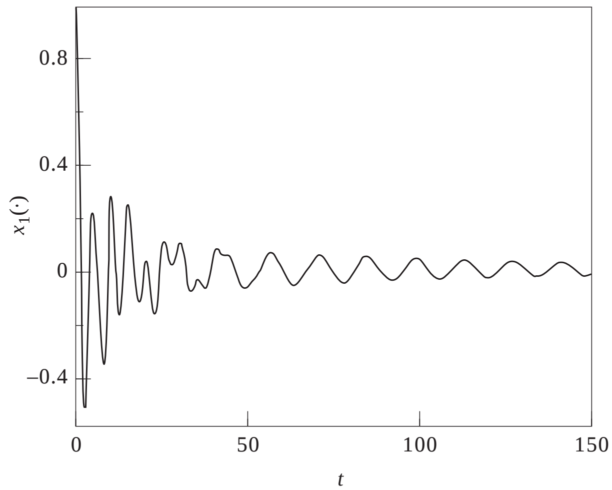

Figure 2. The time history of $x_1(\cdot)$ of the system (6).

Comparing the exact values of $\|x_i\|_\infty$ for all $i = 1, 2, 3$ in equation (9) and their corresponding upper bounds in inequalities (8), we conclude that the bounds are conservatively large for $i = 2, 3$.

Realizing the importance of having bounds on the sizes of responses of systems, we pursue the goal of deriving easy-to-compute and tight bounds on the sizes of displacements of the system (1).

2. AN UPPER BOUND ON DISPLACEMENTS

In this section, we consider the system (1) and *assume* that:

($A1$). The mass matrix $M = \mathrm{diag}\,[m_1, m_2, \ldots, m_n]$, where, without the loss of generality, the diagonal elements are ordered as $m_1 \leqslant m_2 \leqslant \cdots \leqslant m_n$. (This assumption will be relaxed at the end of this section.)

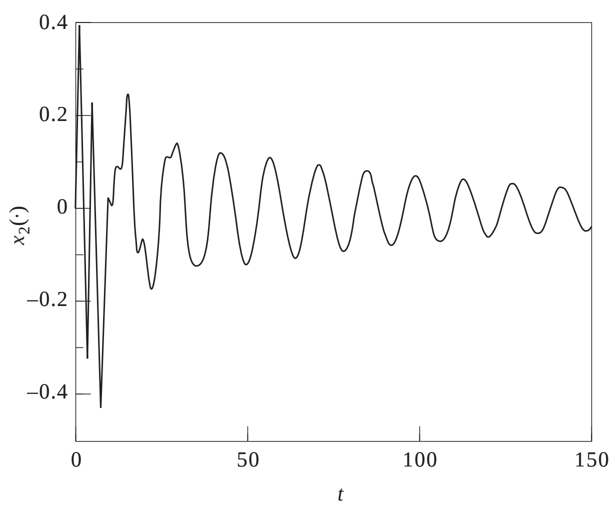

Figure 3. The time history of $x_2(\cdot)$ of the system (6).

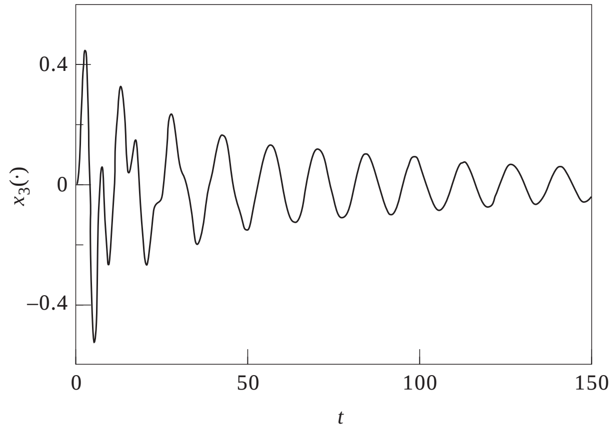

Figure 4. The time history of $x_3(\cdot)$ of the system (6).

($A2$). The matrix C is a classical damping matrix, that is, it satisfies $(M^{-1}C)(M^{-1}K) = (M^{-1}K)(M^{-1}C)$. □

For a proportional (Rayleigh) damping matrix C (see, e.g., references [8, p. 176], [10, pp. 201–202]), which satisfies $C = \alpha M + \beta K$ for some α and $\beta \in \mathbb{R}$, assumption $A2$ holds. It is known that when assumption $A2$ holds, the normalized representation of the system (1) is a set of n decoupled second order linear systems in the normalized co-ordinates (see, e.g., references [3, pp. 144–145], [11]). The normalized representation is obtained via a linear change of co-ordinates applied to the system (1). The change of co-ordinates is

$$x(t) = Uq(t), \tag{10}$$

for all $t \geqslant 0$, where $U \in \mathbb{R}^{n \times n}$ is the (nonsingular) modal matrix corresponding to the system (1) (see, e.g., references [8, pp. 173–175], [10, pp. 178–181]). The columns of the modal matrix are the eigenvectors of the symmetric generalized eigenvalue problem

$$Ku^{(i)} = \omega_i^2 Mu^{(i)}, \tag{11}$$

where $\omega_i^2 > 0$ and $u^{(i)} \in \mathbb{R}^n$ are an eigenvalue (undamped natural frequency squared) and the corresponding eigenvector, respectively. The modal matrix is commonly orthonormalized according to

$$U^{\mathrm{T}}MU = I_n, \tag{12}$$

where U^{T} denotes the transpose of the matrix U and I_n denotes the $n \times n$ identity matrix. Since equation (12) holds, the matrix K satisfies

$$U^{\mathrm{T}}KU = \mathrm{diag}\,[\omega_1^2, \omega_2^2, \ldots, \omega_n^2] =: \Omega^2, \tag{13}$$

where, without the loss of generality, the natural frequencies are ordered as $\omega_1 \leqslant \omega_2 \leqslant, \ldots, \leqslant \omega_n$.

The normalized representation of the system (1) obtained via the change of co-ordinates in equation (10) is (see, e.g., references [3, pp. 144–145], [11])

$$I_n\ddot{q}(t) + \tilde{C}\dot{q}(t) + \Omega^2 q(t) = \theta_n, \tag{14a}$$

for all $t \geqslant 0$, with the initial conditions

$$q_0 := q(0) = U^{-1}x_0 = U^{\mathrm{T}}Mx_0, \qquad \dot{q}_0 := \dot{q}(0) = U^{-1}\dot{x}_0 = U^{\mathrm{T}}M\dot{x}_0. \tag{14b}$$

In equation (14),

$$q(t) = [q_1(t) \quad q_2(t) \cdots q_n(t)]^{\mathrm{T}} \in \mathbb{R}^n, \tag{15}$$

denotes the vector of normalized displacements; the matrix $\tilde{C} := U^{\mathrm{T}} C U \in \mathbb{R}^{n \times n}$ is known as the *normalized* damping matrix. By assumption $A2$, the matrix $\tilde{C}$ is diagonal and can be written as

$$\tilde{C} = \mathrm{diag}\,[2\xi_1\omega_1, 2\xi_2\omega_2, \ldots, 2\xi_n\omega_n], \tag{16}$$

where $\xi_i \geqslant 0$ denotes the ith normalized damping ratio for an $i = 1, 2, \ldots, n$. Note that the non-negativeness of ξ_i for all $i = 1, 2, \ldots, n$ follows from the positive semi-definiteness of the damping matrix C. The diagonal matrix $\tilde{C}$ in equation (16) decouples the system (14) to a set of n scalar second order linear systems. The ith element of this set for an $i = 1, 2, \ldots, n$ is

$$\ddot{q}_i(t) + 2\xi_i\omega_i\dot{q}_i(t) + \omega_i^2 q_i(t) = 0, \qquad q_{i_0} := q_i(0), \qquad \dot{q}_{i_0} := \dot{q}_i(0), \tag{17}$$

for all $t \geqslant 0$, where $q_i(t) \in \mathbb{R}$.

The system (17) can be readily solved for $q_i(\cdot)$; see, e.g., references [8, pp. 18–20, pp. 28–33], [10, pp. 106–110] for the exact solutions of the system (17) for all possible values of $\xi_i \geqslant 0$. We, however, are interested in an upper bound on $\|q_i\|_\infty$ for an $i = 1, 2, \ldots, n$, and not the exact time history of $q_i(\cdot)$. An upper bound is given in the following.

Lemma 2.1. The L_∞-norm of the solution of the system (17), $q_i(\cdot)$, for an $i = 1, 2, \ldots, n$, satisfies

$$\|q_i\|_\infty \leqslant [(q_{i_0})^2 + (\dot{q}_{i_0})^2/\omega_i^2]^{1/2}. \tag{18}$$

Proof. The energy of the system (17) is given by

$$E_i(t) = \omega_i^2 q_i^2(t)/2 + \dot{q}_i^2(t)/2, \tag{19}$$

for all $t \geqslant 0$, where at $t = 0$,

$$E_{i_0} := E_i(0) = \omega_i^2 (q_{i_0})^2/2 + (\dot{q}_{i_0})^2/2. \tag{20}$$

We study the evolution of $E_i(\cdot)$ along the solution of the system (17) for two cases.

Case 1. $\xi_i = 0$: The derivative of $E_i(\cdot)$ along the solution of the system (17) is $\dot{E}_i(t) = 0$ for all $t \geqslant 0$. Therefore, the energy of the system is conserved. That is, $E_i(t) = E_{i_0}$ for all $t \geqslant 0$, which can be written as

$$q_i^2(t) + \dot{q}_i^2(t)/\omega_i^2 = (q_{i_0})^2 + (\dot{q}_{i_0})^2/\omega_i^2. \tag{21}$$

Therefore, the trajectory corresponding to the solution of the system (17) in the phase plane $(q_i, \dot{q}_i)$ is the ellipse in equation (21), which is depicted in Figure 5. From equation (21) or Figure 5, it is clear that equation (18) holds with the equality sign.

Case 2. $\xi_i > 0$: The derivative of $E_i(\cdot)$ along the solution of the system (17) is $\dot{E}_i(t) = -\xi_i\dot{q}_i^2(t) \leqslant 0$ for all $t \geqslant 0$. Therefore, the energy of the system is non-increasing. That is, $E_i(t) \leqslant E_{i_0}$ for all $t \geqslant 0$, which can be written as

$$q_i^2(t) + \dot{q}_i^2(t)/\omega_i^2 \leqslant (q_{i_0})^2 + (\dot{q}_{i_0})^2/\omega_i^2. \tag{22}$$

Therefore, the trajectory corresponding to the solution of the system (17) in the phase plane $(q_i, \dot{q}_i)$ is inside the ellipse in equation (21). Hence, inequality (18) holds. □

Remark. It is straightforward to apply the LaSalle invariance principle (see, e.g., references [12, pp. 115–116], [13, pp. 178–179]) to show that when $\xi_i > 0$, the trajectories corresponding to the solutions of the system (17) are not only inside the ellipse in equation

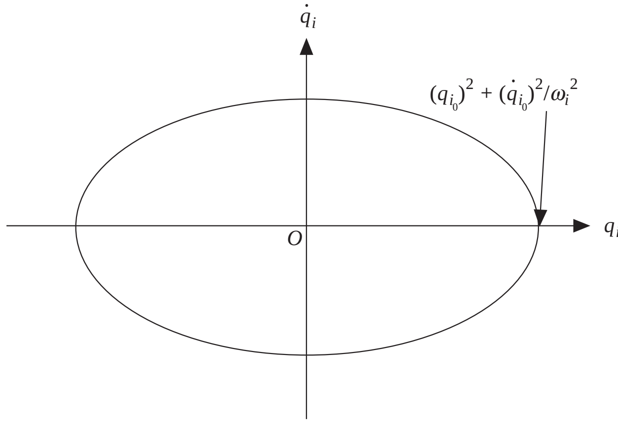

Figure 5. The ellipse $q_i^2(t) + \dot{q}_i^2(t)/\omega_i^2 = (q_{i_0})^2 + (\dot{q}_{i_0})^2/\omega_i^2$ in the phase plane $(q_i, \dot{q}_i)$.

(21), but they converge to the origin of the phase plane as $t \to \infty$. However, in the proof of *Lemma* 2.1, all that matters is the fact that the trajectories do not leave the ellipse in equation (21). □

Having the upper bounds in inequality (18), we next derive an upper bound on $\|x_i\|_\infty$ of the system (1) for all $i = 1, 2, \ldots, n$. First, we establish some preliminary results. In the following, by $\lambda(A)$ we denote an eigenvalue of a matrix A.

Lemma 2.2. The modal matrix U satisfies

$$\lambda_{max}(U^{-\mathrm{T}}\, U^{-1}) = m_n, \qquad \lambda_{max}(U^{\mathrm{T}}\, U) = m_1^{-1}, \tag{23a, b}$$

where m_1 and m_n denote the smallest and the largest elements of the diagonal mass matrix M, respectively.

Proof. From the orthonormalization condition (12), we obtain $U^{-\mathrm{T}}\, U^{-1} = M$. Thus, equation (23a) follows from the fact that $\lambda_{max}(M) = m_n$. Furthermore, from equation (12), we obtain $U^{\mathrm{T}}U = U^{-1}M^{-1}U$. Therefore, the eigenvalue problem corresponding to the matrix $U^{\mathrm{T}}U$ reads as

$$U^{\mathrm{T}}Uv = U^{-1}M^{-1}Uv = \lambda(U^{\mathrm{T}}U)v, \tag{24}$$

where $\lambda(U^{\mathrm{T}}\, U)$ and $v \in \mathbb{R}^n$ are an eigenvalue and an eigenvector of the matrix $U^{\mathrm{T}}U$, respectively. We can rewrite the last indentity in equation (24) as $M^{-1}w = \lambda(U^{\mathrm{T}}\, U)w$, where $w := Uv \in \mathbb{R}^n$. Thus, $\lambda_{max}(U^{\mathrm{T}}\, U) = \lambda_{max}(M^{-1}) = \lambda_{min}(M)$, and equation (23b) follows. □

Next, we obtain relations between the initial conditions in the normalized and physical co-ordinates.

Lemma 2.3. The vectors of initial conditions in the normalized co-ordinates, q_0 and $\dot{q}_0$, and those in the physical co-ordinates, x_0 and $\dot{x}_0$, satisfy

$$q_0^{\mathrm{T}} q_0 \leqslant m_n x_0^{\mathrm{T}} x_0, \qquad \dot{q}_0^{\mathrm{T}} \Omega^{-2} \dot{q}_0 \leqslant (m_n/\omega_1^2)\dot{x}_0^{\mathrm{T}} \dot{x}_0, \tag{25a, b}$$

where ω_1 is the lowest undamped natural frequency of the system (1).

Proof. Using equation (14b), we can write

$$q_0^{\mathrm{T}} q_0 = x_0^{\mathrm{T}} U^{-\mathrm{T}} U^{-1} x_0 \leqslant \lambda_{max}(U^{-\mathrm{T}}\, U^{-1}) x_0^{\mathrm{T}} x_0, \tag{26}$$

where the inequality follows from the definition of Rayleigh's quotient (see, e.g., references [10, pp. 237–243], [14, pp. 176–181]). Using equation (23a) in inequality (26), we obtain inequality (25a).

Using equation (14b), we next write

$$\dot{q}_0^{\mathrm{T}}\Omega^{-2}\dot{q}_0 = \dot{x}_0^{\mathrm{T}}U^{-\mathrm{T}}\Omega^{-2}U^{-1}\dot{x}_0 \leqslant \lambda_{max}(\Omega^{-2})\dot{x}_0^{\mathrm{T}}U^{-\mathrm{T}}U^{-1}\dot{x}_0 \leqslant \lambda_{max}(\Omega^{-2})\lambda_{max}(U^{-\mathrm{T}}\,U^{-1})\dot{x}_0^{\mathrm{T}}\dot{x}_0, \tag{27}$$

where the last two inequalities follow from the definition of Rayleigh's quotient. Using the identity $\lambda_{max}(\Omega^{-2}) = 1/\omega_1^2$ and equation (23a) in inequality (27), we obtain inequality (25b). □

Now, we can present a single upper bound on $\|x_i\|_\infty$ for all $i = 1, 2, \ldots, n$.

Theorem 2.4. Consider the system (1) and let assumptions $A1$ and $A2$ hold. The L_∞-norm of the displacement $x_i(\cdot)$ satisfies

$$\|x_i\|_\infty \leqslant [(m_n/m_1)(x_0^{\mathrm{T}}x_0 + \dot{x}_0^{\mathrm{T}}\dot{x}_0/\omega_1^2)]^{1/2}, \tag{28}$$

for all $i = 1, 2, \ldots, n$, where m_n *and* m_1 are the largest and smallest elements of the mass matrix M, respectively, ω_1 is the lowest undamped natural frequency of the system, and x_0 and $\dot{x}_0$ are the vectors of initial displacements and velocities, respectively.

Proof: We can write

$$\|x_i\|_\infty = \max_{t\geqslant 0}|x_i(t)| \leqslant \max_{t\geqslant 0}[x^{\mathrm{T}}(t)x(t)]^{1/2} = \max_{t\geqslant 0}[q^{\mathrm{T}}(t)U^{\mathrm{T}}Uq(t)]^{1/2}, \tag{29}$$

for all $i = 1, 2, \ldots, n$, where the last identity follows from equation (10). Using the definition of Rayleigh's quotient, we can write

$$\|x_i\|_\infty \leqslant [\lambda_{max}(U^{\mathrm{T}}\,U)\max_{t\geqslant 0}q^{\mathrm{T}}(t)q(t)]^{1/2} = \left[m_1^{-1}\max_{t\geqslant 0}\sum_{i=1}^{n}q_i^2(t)\right]^{1/2}, \tag{30}$$

where the last identity is obtained by using equation (23b). We can further write

$$\|x_i\|_\infty \leqslant \left[m_1^{-1}\sum_{i=1}^{n}\max_{t\geqslant 0}q_i^2(t)\right]^{1/2} \leqslant \left[m_1^{-1}\sum_{i=1}^{n}[(q_{i_0})^2 + (\dot{q}_{i_0})^2/\omega_i^2]\right]^{1/2}, \tag{31}$$

where the last inequality is obtained by using inequality (18). We rewrite inequality (31) in the compact form

$$\|x_i\|_\infty \leqslant [m_1^{-1}(q_0^{\mathrm{T}}q_0 + \dot{q}_0^{\mathrm{T}}\Omega^{-2}\dot{q}_0)]^{1/2}. \tag{32}$$

Using inequalities (25) in inequality (32), we obtain inequality (28). □

Remarks (1) The bound in inequality (28) is a single upper bound on the norms of all displacements of the system (1). Therefore, it is computed only once. Recall that there are n upper bounds in inequality (5), and so are there n times of computation. Computing the bound in inequality (28) is an easy task. In computing this bound some computational effort is required to compute the square of the lowest undamped natural frequency, ω_1^2. It is straightforward to compute ω_1^2, since there are certain numerical methods by which ω_1^2 is readily computed. Some of such numerical methods are power method, Given's method, QR method, inverse iteration method, and Rayleigh's quotient iteration method, which primarily compute the smallest eigenvalue of an eigenvalue problem (see, e.g., references [10, Chapter 6], [15]). It should be pointed out that the lowest natural frequency, ω_1, is an important piece of information for vibratory systems. Therefore, computing ω_1—to be used either in inequality (28) or in the system design and analysis—is a well worthed effort. Note that there is no need to compute ω_1^2, when the initial velocities are zero.

(2) If the mass matrix M is not diagonal, that is, assumption $A1$ does not hold, then the upper bound in inequality (28) is replaced by

$$\|x_i\|_\infty \leqslant [[\lambda_{max}(M)/\lambda_{min}(M)](x_0^{\mathrm{T}} x_0 + \dot{x}_0^{\mathrm{T}} \dot{x}_0/\omega_1^2)]^{1/2}, \tag{33}$$

for all $i = 1, 2, \ldots, n$, where $\lambda_{max}(M)$ and $\lambda_{min}(M)$ are the largest and smallest eigenvalues of the mass matrix M, respectively. The upper bound in inequality (33) can be easily verified by using $\lambda_{max}(U^{-\mathrm{T}} U^{-1}) = \lambda_{max}(M)$ and $\lambda_{max}(U^{\mathrm{T}} U) = \lambda_{max}(M^{-1}) = \lambda_{min}(M)$ in the steps that led to inequality (28). In computing the bound in inequality (33), some computational effort is required to compute $\lambda_{max}(M)$ and $\lambda_{min}(M)$. □

It can be easily verified that assumptions $A1$ and $A2$ hold for the system (6). Thus, we can use inequality (28) to compute upper bounds on the norms of displacements of the system (6). We obtain

$$\|x_1\|_\infty \leqslant 1, \qquad \|x_2\|_\infty \leqslant 1, \qquad \|x_3\|_\infty \leqslant 1. \tag{34}$$

Clearly, the bounds in inequalities (34) are tighter than those in inequalities (8). We, however, point out that the bounds in inequality (28) are not always tighter than those in inequality (5). There can be systems and initial conditions for which the bounds obtained by inequality (28) are more conservative than the corresponding bounds obtained by inequality (5). Note that the bounds in inequality (28) can be tighter when the ratio $\lambda_{max}(M)/\lambda_{min}(M)$ (m_n/m_1 for a diagonal M) is not much larger than one.

3. CONCLUSIONS

In this note, the free vibratin of n-degree-of-freedom linear second order systems has been considered. A single and easy-to-compute upper bound on the norms of all displacements of such systems, which is given in inequality (28), has been derived. The upper bound depends on the ratio of the largest to the smallest eigenvalues of the mass matrix of the system, the lowest undamped natural frequency of the system, and the vectors of initial displacements and velocities. The upper bound is independent of the lowest natural frequency when the initial velocities are zero.

In the derivation of the upper bound, the damping matrix was assumed to be a classical damping matrix. This assumption led to the decoupling of the system in the normalized co-ordinates. Work to relax the assumption of the classically damped system is in progress, and will be reported at a later time.

REFERENCES

1. B. Hu and W. Schiehlen 1996 *Archive of Applied Mechanics* **66**, 357–368. Amplitude bounds of linear forced vibrations.
2. B. Hu and W. Schiehlen 1996 *European Journal of Mechanics*, *A/Solids* **15**, 617–646. Eigenvalues, frequency response and variance bounds of linear damped systems.
3. P. C. Müller and W. O. Schiehlen 1985 *Linear Vibrations*. Dordrecht, The Netherlands: Martinus Nijhoff.
4. D. W. Nicholson 1980 *Mechanics Research Communications* **7**, 305–308. Bounds on the forced response of damped linear systems.
5. D. W. Nicholson 1982 *IEEE Transactions on Automatic Control* **27**, 704–706. Response bounds for asymptotically stable time-invariant linear dynamical systems.
6. D. W. Nicholson 1987 *ASME Journal of Applied Mechanics* **54**, 430–433. Response bounds for nonclassically damped mechanical systems under transient loads.
7. W. Schiehlen and B. Hu 1995 *ASME Journal of Applied Mechanics* **62**, 231–233. Amplitude bounds of linear free vibrations.

8. W. T. Thomson 1988 *Theory of Vibration with Applications*. Englewood Cliffs, New Jersey: Prentice Hall; third edition.
9. K. H. Yae and D. J. Inman 1987 *ASME Journal of Applied Mechanics* **54**, 419–423. Response bounds for linear underdamped systems.
10. L. Meirovitch 1997 *Principles and Techniques of Vibrations*. Upper Saddle River, New Jersey: Prentice Hall.
11. T. K. Caughey and M. E. J. O'Kelly 1965 *ASME Journal of Applied Mechanics* **32**, 583–588. Classical normal modes in damped linear dynamic systems.
12. H. K. Khalil 1996 *Nonlinear Systems*. Upper Saddle River, New Jersey: Prentice-Hall; second edition.
13. M. Vidyasagar 1993 *Nonlinear Systems Analysis*. Englewood Cliffs, New Jersey: Prentice-Hall; second edition.
14. R. A. Horn and C. R. Johnson 1985 *Matrix Analysis*. Cambridge: Cambridge University Press.
15. G. H. Golub and C. F. Van Loan 1996 *Matrix Computations*. Baltimore, Maryland: Johns Hopkins University Press; third edition.