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# MODE LOCALIZATION IN SIMPLY SUPPORTED TWO-SPAN BEAMS OF ARBITRARY SPAN LENGTHS

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## 1. INTRODUCTION

Natural frequencies and mode shapes are the dynamic characteristics of structural systems, which are functions of the geometric configuration and the material properties of the structures. The dynamic characteristics may be changed radically by structural changes in some structures when mode localization occurs. The mode localization is a phenomenon in which the magnitude of the specific part of the free vibrational mode is large relative to the rest of the mode. It is well known that weakly coupled periodic structures are sensitive to certain types of periodicity breaking disorder, resulting in mode localization with serious implication for the control problem. For some structures structural damages or manufacturing errors may produce undesirable mode localization. It is, therefore, very important not only to calculate the natural frequencies and the mode shapes but to identify the degrees of localization and the localization sensitivities of the modes.

In solid state physics, the localization phenomenon of electron field in disordered solid was first observed by Anderson [1]. Anderson and Mott [2] shared the 1977 Nobel Prize in physics for their work in this area. The mode localization phenomenon is found to exist in the field of structural dynamics. Many works were concerned with cyclically symmetric structures with weak coupling in order to explain the unpredicted fatigue failure of the mistuned blades of turbomachinery [3–5]. Bendiksen [6] investigated the mode localization in a simple model of a space structure. Hodges [7] was first to recognize that wave localization may occur in disordered periodic structures leading to mode localization. Wave localization is the phenomenon that the vibrational energy imparted to the structure by an external source cannot propagate to arbitrary long distances but is instead substantially confined to a region close to the source. Since his work, there have been several studies on the localization in periodic engineering structures [8–11]. Pierre, Tang and Dowell [9] studied the mode localization of weakly coupled disordered multispan beams using the modified perturbation method and the experimental method. Bouzit and Pierre [10] demonstrated weak and strong localization behaviors and calculated the localization factor for a multispan beam on randomly spaced simple supports, the torsional rigidity of which could be varied. The localization factor is defined by the average exponential rate at which a structural wave decays with respect to the wave propagation distance in a disordered periodic structure and it can be calculated by the transfer matrix method. The mode localization of non-linear systems was studied by Vakakis *et al*. [12, 13], and Zevin [14].

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Previous works on mode localization concerned mainly disordered periodic structures. However, the mode localization phenomenon in non-periodic structures has been passed over. The objective of this study is to show the possibility of drastic occurrences of mode localization in non-periodic structures. Free vibration analysis of simply supported two-span beams of arbitrary span lengths is theoretically investigated. The beam can be periodic or non-periodic. Degrees of mode localization and their sensitivities to system parameters are appraised by considering the characteristic graph and the structural line defined in this study.

## 2. FREE VIBRATION ANALYSIS OF THE TWO-SPAN BEAM

Consider the two-span beam shown in Figure 1. The beam is simply supported at both ends, and is constrained to have zero deflection at $x_1 = l_1$ and/or $x_2 = l_2$. Moreover, a torsional spring of $K_R$ exerts a restoring moment at $x_1 = l_1$ and/or $x_2 = l_2$. The system can be divided into two substructures and for the convenience of simple analysis the coordinates of the substructures are determined as in Figure 1. The eigenvalue problems for free bending vibrations of each substructure can be written as

$$EI_1\, \mathrm{d}^4 y_1/\mathrm{d}x_1^4 - \omega^2 m_1 y_1 = 0, \qquad EI_2\, \mathrm{d}^4 y_2/\mathrm{d}x_2^4 - \omega^2 m_2 y_2 = 0, \tag{1, 2}$$

where $EI_1$ and $EI_2$ are flexural rigidity, $m_1$ and $m_2$ are masses per unit length of each substructure respectively, $\omega$ is natural frequency of the system, and $y_1$ and $y_2$ are the transverse displacements of each substructure. The general solutions of equations (1) and (2) can be written as

$$y_1(x_1) = A_1 \sin \lambda_1 x_1 + B_1 \cos \lambda_1 x_1 + C_1 \sinh \lambda_1 x_1 + D_1 \cosh \lambda_1 x_1 \tag{3}$$

and

$$y_2(x_2) = A_2 \sin \lambda_2 x_2 + B_2 \cos \lambda_2 x_2 + C_2 \sinh \lambda_2 x_2 + D_2 \cosh \lambda_2 x_2, \tag{4}$$

where

$$\lambda_1^4 = \omega^2 m_1/EI_1, \qquad \lambda_2^4 = \omega^2 m_2/EI_2. \tag{5, 6}$$

To determine the coefficients of the general solutions, one can use eight boundary conditions. By applying boundary conditions that the deflections and the moments at $x_1 = 0$ and $x_2 = 0$ are zeros, equations (3) and (4) yield $B_1 = D_1 = B_2 = D_2 = 0$. Application of boundary conditions that the deflections at $x_1 = l_1$ and $x_2 = l_2$ are zeros yield

$$C_1 = -A_1 \sin \lambda_1 l_1/ \sinh \lambda_1 l_1, \qquad C_2 = -A_2 \sin \lambda_2 l_2/ \sinh \lambda_2 l_2. \tag{7, 8}$$

Equations (7) and (8), and the two continuity conditions such as

$$\frac{\mathrm{d}y_1(l_1)}{\mathrm{d}x_1} = \frac{\mathrm{d}y_2(l_2)}{\mathrm{d}x_2}, \qquad EI_1 \frac{\mathrm{d}^2 y_1(l_1)}{\mathrm{d}x_1^2} + EI_2 \frac{\mathrm{d}^2 y_2(l_2)}{\mathrm{d}x_2^2} = -K_R \frac{\mathrm{d}y_2(l_2)}{\mathrm{d}x_2}, \tag{9, 10}$$

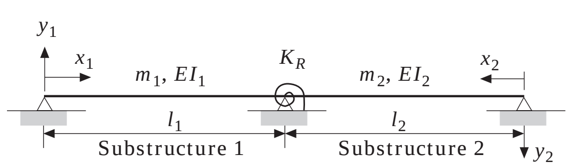


Figure 1. Simply supported two-span beam with rotational stiffness at mid support.

give two algebraic equations for $A_1$ and $A_2$, and that can be written in a matrix form as

$$\begin{bmatrix} \lambda_1\theta_1 & -\lambda_2\theta_2 \\ -2EI_1\lambda_1^2\sin\lambda_1 l_1 & -2EI_2\lambda_2^2\sin\lambda_2 l_2 + K_R\lambda_2\theta_2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix}, \tag{11}$$

where

$$\theta_1(\lambda_1 l_1) = \cos\lambda_1 l_1 - \frac{\cosh\lambda_1 l_1}{\sinh\lambda_1 l_1}\sin\lambda_1 l_1, \quad \theta_2(\lambda_2 l_2) = \cos\lambda_2 l_2 - \frac{\cosh\lambda_2 l_2}{\sinh\lambda_2 l_2}\sin\lambda_2 l_2. \quad (12,\ 13)$$

Non-trival solutions of equation (11) can be obtained if and only if the determinant of the coefficient matrix vanishes. This gives an equation for the determination of natural frequencies, $\omega$, which is called the frequency equation or characteristic equation:

$$K_R\theta_1\theta_2 - 2EI_2\lambda_2\theta_1\sin\lambda_2 l_2 - 2EI_1\lambda_1\theta_2\sin\lambda_1 l_2 = 0. \tag{14}$$

In equation (14), the only unknown is the natural frequency $\omega$. However, it is convenient to use two variables, $\beta_1$ and $\beta_2$, for describing the characteristics of the system. Using $\beta_1$ and $\beta_2$ one can rewrite the characteristic equation as

$$K_R\theta_1\theta_2 - 2(EI_2/l_2)\beta_2\theta_1\sin\lambda_2 l_2 - 2(EI_1/l_1)\beta_1\theta_2\sin\lambda_1 l_1 = 0, \tag{15}$$

where

$$\beta_1 = \lambda_1 l_1 = \omega^{1/2} l_1 (m_1/EI_1)^{1/4}, \qquad \beta_2 = \lambda_2 l_2 = \omega^{1/2} l_2 (m_2/EI_2)^{1/4}. \quad (16,\ 17)$$

From equations (16) and (17),

$$\beta_2 = \alpha\beta_1, \tag{18}$$

where

$$\alpha = l_2/l_1((m_2/EI_2)(EI_1/m_1))^{1/4}. \tag{19}$$

Equation (18) is an equation for a line and $\alpha$ is the slope of the line. In this study, the line and the slope were named structural line and structural slope respectively since they represent the geometry and material properties of the structure. These are very useful to describe the characteristic of the system, and will be used in the next section.

## 3. MODE LOCALIZATION OF THE TWO-SPAN BEAM

In this section a mode localization factor and a characteristic graph are defined here. The mode localization factor can be used as a measure of the degree of mode localization of each mode. Using the characteristic graph, one can roughly forecast the effects of the changes in the system parameters on the mode localization.

As aforementioned, the mode localization is the vibration energy confinement, so the degree of mode localization can be represented by logarithmic value of the ratio of the mean squared vibrational magnitude of the second substructure to that of the first substructure as

$$\gamma \equiv \log \eta_2/\eta_1, \tag{20}$$

where $\gamma$ is the mode localization factor, and $\eta_1$ and $\eta_2$ are the mean squared free vibrational magnitude of the first and second substructures respectively, which can be expressed as

$$\eta_1 \equiv \frac{1}{l_1}\int_0^{l_1} y_1^2(x_1)\,\mathrm{d}x_1, \qquad \eta_2 \equiv \frac{1}{l_2}\int_0^{l_2} y_2^2(x_2)\,\mathrm{d}x_2. \quad (21,\ 22)$$

For simple analysis, considering $\beta \gg 1$, one can conclude that $C_1 \approx 0$ and $C_2 \approx 0$ from equations (7) and (8), and approximate $\gamma$ as

$$\bar{\gamma} = \log A_2^2/A_1^2. \tag{23}$$

If the vibration is confined at one of the substructures, the absolute value of $\gamma$ becomes large. On the contrary, if the vibrational magnitudes are the same with each other, $|A_1| = |A_2|$, $\bar{\gamma}$ becomes zero. The sign of $\bar{\gamma}$ means the substructure at which the vibration is confined, positive at the second substructure and negative at the first one. Considering equation (11) and the assumption that two substructures have the same material properties, one can rewrite equation (23) as

$$\bar{\gamma} = \log(\cos\beta_1 - (\cosh\beta_1/\sinh\beta_1)\sin\beta_1)^2 - \log(\cos\beta_2 - (\cosh\beta_2/\sinh\beta_2)\sin\beta_2)^2. \tag{24}$$

The mode localization factor $\bar{\gamma}$ is close to zero when $(\beta_1 - \beta_2) = n\pi$ and positive or negative infinite when $(\beta_1 - \beta_2) = (n + 0{\cdot}5)\pi$, where $n$ is zero or integer. If only one of the following conditions is satisfied, the absolute value of $\bar{\gamma}$ is large and the mode is localized.

$$\beta_1 \approx (m + \tfrac{1}{4})\pi, \qquad \beta_2 \approx (m + \tfrac{1}{4})\pi \qquad \text{where } m = 1, 2, 3, \ldots . \tag{25, 26}$$

A mode was considered as localized when the span response ratio was less than 0·1 in reference 9. To be consistent with that terminology, a mode is considered as localized when $|\bar{\gamma}| \geqslant 2$.

Considering the above localization conditions and the results of the previous section, one can draw a characteristic graph as in Figure 2. In Figure 2, the horizontal and vertical axes represent $\beta_1$ and $\beta_2$ respectively. The characteristic curves represent the characteristic equation, equation (15), and the structural line and the structural slope represent equations (18) and (19) respectively. The localization conditions that are vertical and horizontal thin solid lines mean equations (25) and (26) respectively. The crossing points of the characteristic curves and the structural line indicate the eigensolutions of the system. By

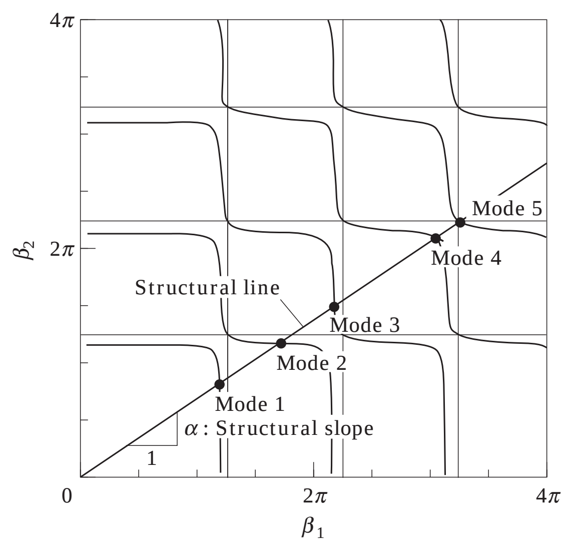


Figure 2. Characteristic graph of the two-span beam. Key: +, localization conditions; ——, characteristic curve; ●, eigensolution point.

using $\beta_1$ or $\beta_2$ of the eigensolution points and equation (16) or (17), one can calculate the natural frequencies of the system. The more an eigensolution is close to one of the thin solid lines and is far from the other ones, the more the mode is strongly localized. However, if an eigensolution is close to the vertical and the horizontal localization condition lines concurrent or on their crossing point, the mode is not to be localized since the points satisfy $(\beta_1 - \beta_2) = n\pi$.

For example, in Figure 2, the first, second and third modes are localized while the fourth and fifth modes are not. Because the first three modes are close to one of the localization conditions, but the fourth mode is close to two localization conditions and the fifth one is at a crossing point. The first and the third modes especially are close to the vertical localization conditions. That means that the $\beta_1$'s of them satisfy equation (25), so it is certain that their mode localization factors have negative large values and the vibration is confined at the first substructure of the system. And for the second mode, it is close to the horizontal localization condition, so the vibration is localized at the second substructure.

What makes the best use of the characteristic graph is that one can roughly predict the mode localization phenomenon incurred by any disturbances introduced into a two-span beam. Small changes in system parameters produce the changes in the structural slope and the characteristic curves, but the variation in characteristic curves are small. The change in structural slope shifts the eigensolution points and the degrees of mode localization of each mode are varied also. Since the structural line starts at the origin, the shifting caused by the change in slope is increasingly steeper with the increased mode number. For such reason, the shifting of higher modes can be very serious and so is the variation in degree of mode localization. Considering the system depicted in Figure 2, for instance, one can say that if the structural slope is increased by some disturbances, all the eigensolution points are shifted. And as a result of that the degrees of mode localization of the first and second modes are decreased while the others are increased.

## 4. EXAMPLES

### 4.1. *The system considered*

In this section the foregoing results are confirmed by some examples. In this example, the mode localization phenomenon caused by the small changes in system parameters in the strongly or weakly coupled periodic and non-periodic systems are discussed by using characteristic graphs.

The example structures are continuous beam structures resting on the three simple supports, which are constrained by torsional spring at mid support as shown schematically in Figure 1. The torsional spring plays the role of a decoupler. As $K_R \to \infty$, the spans are fully decoupled from each other because no moment can be transmitted from one substructure to the other. For $K_R = 0$, the substructures are strongly coupled since no restoring moment is exerted. The effects of the coupling and the periodicity on the mode localization are studied by considering four cases. In the first two cases the effect of the coupling on mode localization in periodic structure is studied, and in the next two cases the non-periodic structure is considered. All the example structures have the same material properties and each of the cases is classified by the strength of the torsional spring and the span length of the second substructure. However, one can study the effects of any other material properties using the same procedure presented in this paper, and will get similar results because the non-dimensional parameters $\beta_1$ and $\beta_2$ are mainly used in the procedure.

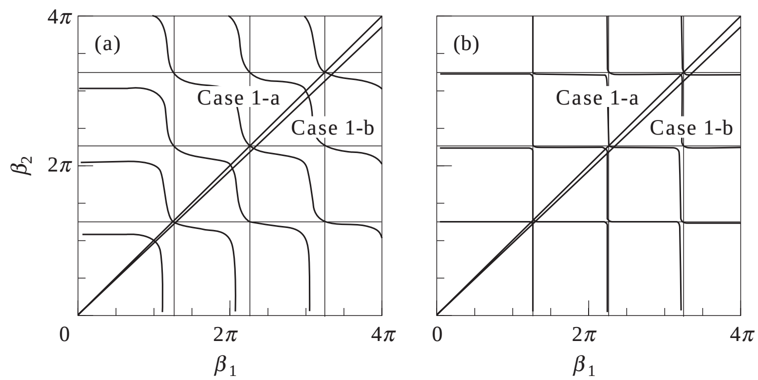


Figure 3. Characteristic graph of case 1 and 2. (a) Case 1-a, $\alpha = 1{\cdot}0$; case 1-b, $\alpha = 0{\cdot}95$; (b) Case 2-a, $\alpha = 1{\cdot}0$; case 2-b, $\alpha = 0{\cdot}95$.

### 4.2. *The periodic structures*; *cases* 1 *and* 2

The mode localization phenomena in the ordered and the disordered periodic two-span beams are examined for two coupling conditions. In the disordered cases, the ratio of the span length of the second substructure to that of the first one is 0·95 while the ratio is unity in the ordered cases. The masses per unit length of each substructure are $m_1 = m_2 = 25{\cdot}0$ kg/m, the flexural rigidities $EI_1 = EI_2 = 2 \times 10^7$ Nm$^2$, and the span length $l_1 = l_2 = 1{\cdot}0$ m in ordered cases and $l_2 = 0{\cdot}95$ m in disordered ones. The torsional spring constants are $K_R = 0{\cdot}00$ Nm and $K_R = 2 \times 10^8$ Nm in cases 1 and 2 respectively. The substructures of case 1 are strongly coupled with each other since they have no torsional spring, and the next case is a weakly coupled system. The results of the ordered and the disordered cases are discussed in pairs because this example is performed on the assumption that some errors such as manufacturing errors or structural damages make an ordered system into disordered one. Figure 3 shows the characteristic graph of the cases 1 and 2. Subcases *a* and *b* imply an ordered system and a disordered one respectively. Selected mode shapes of each case are shown in Figure 4. The lowest five natural frequencies and the degrees of mode localization of each case and its differences are given in Tables 1 and 2.

All the eigensolution points of the case 1-a and case 2-a satisfy $(\beta_1 - \beta_2) = n\pi$ where $n = 0$ as shown in Figure 3, and it is clear that all the modes are not localized as shown

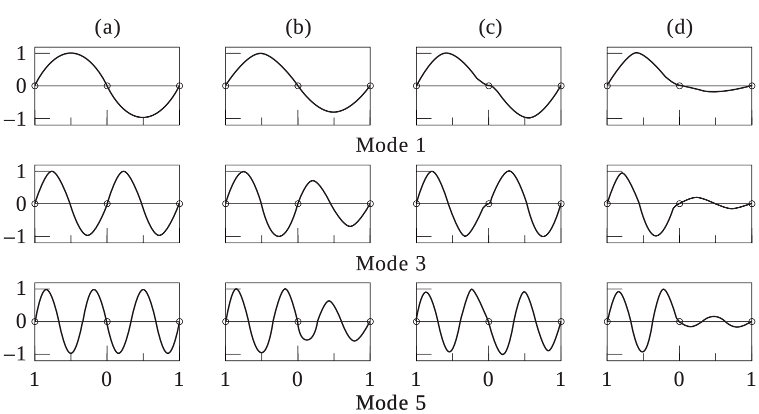


Figure 4. Mode shapes of case 1 and 2. (a) Case 1-a; (b) case 1-b; (c) case 2-a; (d) case 2-b.

TABLE 1

*Natural frequencies and degrees of mode localization: case 1*

| | Case 1-a | | Case 1-b | | Differences | |
|---|---|---|---|---|---|---|
| Mode | $f$ (Hz) | $\bar{\gamma}$ | $f$ (Hz) | $\bar{\gamma}$ | $\delta f$ (%) | $\delta\bar{\gamma}$ |
| 1 | 444·28 | 0·000 | 465·46 | −0·1384 | 4·77 | −0·1384 |
| 2 | 694·06 | 0·000 | 733·86 | 0·1731 | 5·73 | 0·1731 |
| 3 | 1777·2 | 0·000 | 1854·7 | −0·2695 | 4·36 | −0·2695 |
| 4 | 2249·2 | 0·000 | 2387·3 | 0·3032 | 6·14 | 0·3032 |
| 5 | 3998·6 | 0·000 | 4158·4 | −0·3880 | 4·00 | −0·3880 |

in Figure 4. However, for the disordered cases such as case 1-b and case 2-b, $(\beta_1 - \beta_2) = n\pi$ is not satisfied and all the modes are localized to some degree as shown in Figure 4. The eigensolution points are shifted by the length disturbance and the shift steeper in the higher modes. That is, the mode localization occurs in all modes simultaneously, and the degree of mode localization increases with the mode number. As shown in Figure 3(b), the characteristic curves of the weakly coupled systems are close to the localization conditions. This means that if the coupling is very weak, many eigensolution points of the disordered system satisfy the localization condition, equation (25) or (26), and the localization occurs to a large extent in those modes as shown in Figure 4 and Tables 1 and 2. The degrees of mode localization are more severe in the weakly coupled systems than in strongly coupled ones. The effect of the coupling strength is more serious in lowest modes. The extents of the differences of each mode increase with the strength of the torsional spring, and the rate of the increase in the lowest mode is much higher than that in the higher modes as shown in Tables 1 and 2. Considering the characteristic equation, equation (15), one concludes that the first term of that equation can be neglected since $\beta_1$ and $\beta_2$ are very large in higher modes, and for this reason the mode localizations of the higher modes are rarely affected by the coupling strength.

4.3. *The non-periodic structures; cases* 3 *and* 4

The normal modes of the non-periodic two-span beams are examined for two coupling conditions. In the cases of the non-periodic systems, the words of *ordered* and *disordered* do not have meanings any longer because the undisturbed initial structures have no regularity now. So in this study the structures of case 3-a and case 4-a are called the *initial* system, and the others the *disturbed* system. The material properties, mass per unit length and flexural rigidity, of these cases, are equal to those of the cases 1 and 2. The span

TABLE 2

*Natural frequencies and degrees of mode localization: case 2*

| | Case 2-a | | Case 2-b | | Differences | |
|---|---|---|---|---|---|---|
| Mode | $f$ (Hz) | $\bar{\gamma}$ | $f$ (Hz) | $\bar{\gamma}$ | $\delta f$ (%) | $\delta\bar{\gamma}$ |
| 1 | 669·01 | 0·00 | 679·16 | −1·487 | 1·52 | −1·487 |
| 2 | 694·06 | 0·00 | 756·90 | 1·485 | 9·05 | 1·485 |
| 3 | 2172·6 | 0·00 | 2204·2 | −1·521 | 1·45 | −1·521 |
| 4 | 2249·2 | 0·00 | 2454·5 | 1·521 | 9·13 | 1·521 |
| 5 | 4541·7 | 0·00 | 4605·4 | −1·546 | 1,40 | −1·546 |

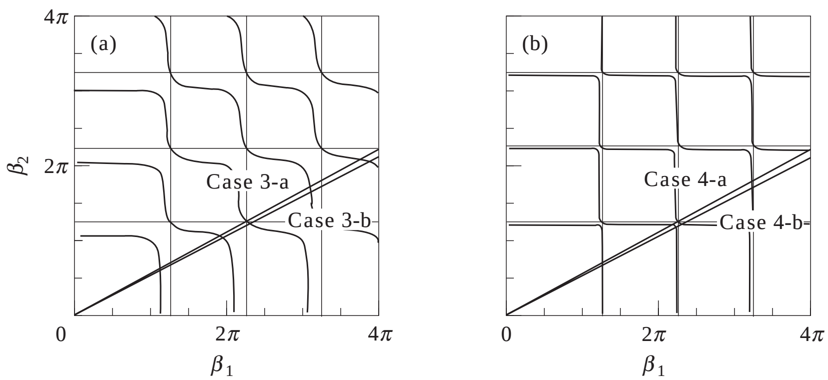


Figure 5. Characteristic graph of case 3 and 4. (a) Case 3-a, $\alpha = 0{\cdot}5556$; case 3-b, $\alpha = 0{\cdot}5278$. case 4-a, $\alpha = 0{\cdot}5556$; case 4-b, $\alpha = 0{\cdot}5278$.

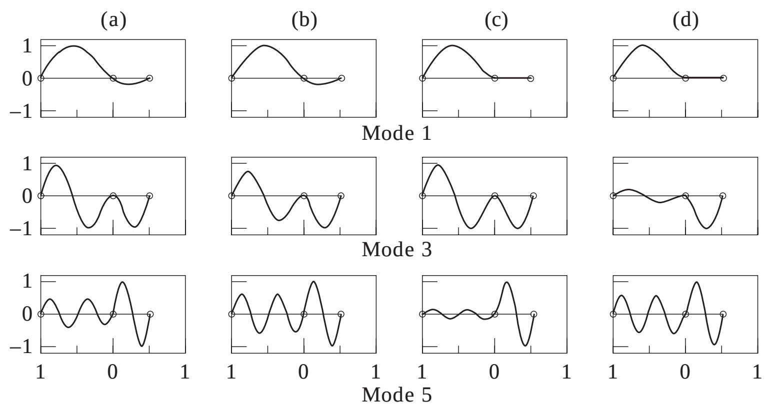


Figure 6. Mode shapes of the case 3 and 4. (a) case 3-a; (b) case 3-b; (c) case 4-a; (d) case 4-b.

lengths, however, are determined by considering the characteristic graph so that the mode localization may not occur in the third mode satisfying $|\beta_1 - \beta_2| = n\pi$ where $n = 1$. Figure 5 shows the characteristic graph of the cases 3 and 4. Selected mode shapes of each case are shown in Figure 6. The lowest five natural frequencies and the degrees of mode localization of each case and its differences are given in Tables 3 and 4.

TABLE 3

*Natural frequencies and degrees of mode localization*: *case* 3

| | Case 3-a | | Case 3-b | | Differences | |
|---|---|---|---|---|---|---|
| Mode | $f$ (Hz) | $\bar{\gamma}$ | $f$ (Hz) | $\bar{\gamma}$ | $\delta f$ (%) | $\delta\bar{\gamma}$ |
| 1 | 560·52 | −0·7785 | 564·94 | −0·7778 | 0·78 | 0·0007 |
| 2 | 1615·1 | 0·3878 | 1703·1 | 0·2031 | 5·45 | −0·1847 |
| 3 | 2249·0 | 0·0000 | 2346·4 | 0·2273 | 4·33 | 0·2273 |
| 4 | 4289·1 | −0·7223 | 3428·1 | −0·7637 | 9·09 | −0·0414 |
| 5 | 6331·6 | 0·6684 | 6771·0 | 0·3960 | 6·94 | −0·2724 |

TABLE 4

*Natural frequencies and degrees of mode localization*: *case* 4

| Mode | Case 4-a | | Case 4-b | | Differences | |
|---|---|---|---|---|---|---|
| | $f$ (Hz) | $\bar{\gamma}$ | $f$ (Hz) | $\bar{\gamma}$ | $\delta f$ (%) | $\delta\bar{\gamma}$ |
| 1 | 681·42 | −2·889 | 681·48 | −2·873 | 0·009 | 0·016 |
| 2 | 2142·8 | 0·511 | 2197·5 | −0·925 | 2·55 | −1·437 |
| 3 | 2249·0 | −0·000 | 2426·1 | 1·428 | 7·87 | 1·428 |
| 4 | 4615·1 | −2·136 | 4614·9 | −2·153 | 0·039 | −0·017 |
| 5 | 7026·7 | 1·617 | 7703·7 | 0·425 | 9·63 | −1·192 |

The characteristic graphs depicted in Figure 5 show that all the modes except the third mode of the initial system are already localized in some degree, and Figure 6 and Tables 3 and 4 confirm that. The length disturbance introduced into the second span yields the changes in the natural frequencies and degrees of mode localization as predicted by the characteristic graphs, and Tables 3 and 4 certify that. Observe that in some modes the degrees of mode localization are increased and in the others decreased by the disturbance. That is not in accordance with the result of the periodic system in which the trends of variations are consistent with the mode number in all modes. It is obvious, however, that the weak coupling makes the system sensitive to the disturbance independently of the periodicity of the system.

## 5. CONCLUSIONS

The present study is concerned with the occurrence and the variation of the mode localization in both periodic beams and non-periodic ones. It investigates the effects of the coupling strength on mode localization in those systems. The main findings of the work are summarized as follows.

(1) A mode satisfying $(\beta_1 - \beta_2) = n\pi$ makes possible the drastic occurrence of mode localization independent of the periodicity of the two-span beams when the coupling is weak.
(2) Mode localization of a two-span beam with equal span lengths occurs simultaneously in all modes.
(3) Mode localization of a two-span beam with arbitrary span lengths occurs in some (not all) modes.
(4) Mode localization in the higher modes is more sensitive to the system parameters than in the lower ones.
(5) Weak coupling makes the modes sensitive to mode localization. This effect is especially more pronounced in the lowest modes than in the higher modes.

The second result is well known in the present time. However, the other results are new or are extensions of already established work. From the results of this study, one can say that mode localization may occur in non-periodic structures as well as periodic ones. Extended studies on mode localization phenomena of general non-periodic structures are required.

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