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VIBRATIONS OF A CANTILEVER TAPERED BEAM WITH VARYING SECTION PROPERTIES AND CARRYING A MASS AT THE FREE END

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The free vibration frequencies of a beam composed of two tapered beam sections with different physical characteristics with a mass at its end can be determined by using either the exact procedure, for which purpose the solution to the problem can be expressed using Bessel functions, or the approximate Rayleigh–Ritz treatment, with the assumption of orthogonal polynomials as test functions. The results and the numerical comparison between the two methods are provided in diagrams and tables. The effect of different materials on vibrations modes is also provided.

1. INTRODUCTION

Many researchers have provided calculation methods for the determination of the free vibration frequencies of truss beams with masses at the ends in order to offer a simplified model that is capable of simulating the dynamic behaviour of numerous bearing structures used in engineering. Tall buildings, elevated tanks, and movable arms all have behaviours similar to the cantilever beam, and this simplification is useful in the modelling of real structures, especially where tapered beams with variable characteristics of geometry and elasticity are involved. In this case, the dynamic problem can be expressed with the use of variable coefficient differential equations, where a closed-form solution is possible only under particular conditions of taper, while approximative methods are normally preferred under more general tapering conditions.

An exact-type analysis regarding linearly-variable section beams for which the solution can be expressed in terms of Bessel functions was initially proposed by Ward [1], Conway and Dubil [2], and Sanger [3] by solving the differential equation of motion. Mabie and Rogers [4, 5] subsequently extended the problem to the case of cantilever beams with masses at their ends, while Goel [6], although reaching incorrect equations [7, 8], introduced the problem of non-classical constraints. A study on the solution in terms of hypergeometric functions should also be quoted [9] and quite recently, an exact analysis of tapered beams in the presence of attachments has been proposed by Lee and Lin [10], Lee [11]. Interesting solutions regarding intermediate constraints have been provided by Yang [12], Craver and Jampala [13], and Sankaran *et al*. [14].

Numerous approximate methods have been proposed; starting with the classical papers by Morrow [15], in which an integral equation is solved in an iterative way. Most of the approximate studies are based on the Rayleigh–Ritz method in which the test functions are the trigonometric type, such as those by Klein [16], Sato [17] or of the polynomial type, such as those by Grossi *et al*. [18]. Other approximative methods resort to the introduction

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of optimization parameters, such as those by Schmidt [19], Bert [20], Laura and Gutierrez [21], and Alvarez *et al*. [22]. The special features of the latter are results obtained on complex structural patterns in the presence of concentrated masses and discontinuous sections. Finally a comprehensive static, stability and dynamic analysis of tapered beams has been presented by Karabalis and Beskos [23].

This work determines the free vibration of a cantilever beam with a mass at its free end obtained by assembling two linearly-variable section beams with different physical characteristics (Figure 1). Two distinct calculation methods have been adopted: the first is an exact method in which the solution to the problem can be expressed using Bessel functions; the second is the approximate Rayleigh–Ritz treatment with the use of orthogonal polynomials as test functions. The results obtained are provided in figures and tables, and confirm the validity of both methods. The effect of different materials used on the first three vibration modes is also provided.

2. EXACT APPROACH

Consider the tapered beam shown in Figure 1. If the properties of the beam are compatible with Euler–Bernoulli beam theory, and the beam is undergoing a small amplitude bending free vibration, the differential equations may be written in the form

$$\frac{d^2}{dx^2}\left[E_1 I_x \frac{d^2 R(x)}{dx^2}\right] - \rho_1 A_x \omega^2 R(x) = 0 \qquad 0 \leqslant x \leqslant \beta L, \tag{1}$$

$$\frac{d^2}{dy^2}\left[E_2 I_y \frac{d^2 S(x)}{dy^2}\right] - \rho_2 A_y \omega^2 S(y) = 0 \qquad 0 \leqslant y \leqslant (1-\beta) L, \tag{2}$$

where $R(x)$, $S(y)$ are the transverse displacements of the beam axis; ρ_1, ρ_2 are the mass per unit volumes of the beam; A_x, A_y are the cross-section areas; $E_1 \hat{I}_1$ and $E_2 \hat{I}_2$ are the bending rigidities of the beam at $x = 0$ and $x = L$, respectively.

After introducing the dimensionless variables

$$\xi = \left[1 + \frac{\alpha_1 - 1}{\beta L}\right]; \qquad \eta = \left[\alpha_1 + \frac{\alpha_1 \alpha_2 - \alpha_1}{(1-\beta) L} y\right], \tag{3}$$

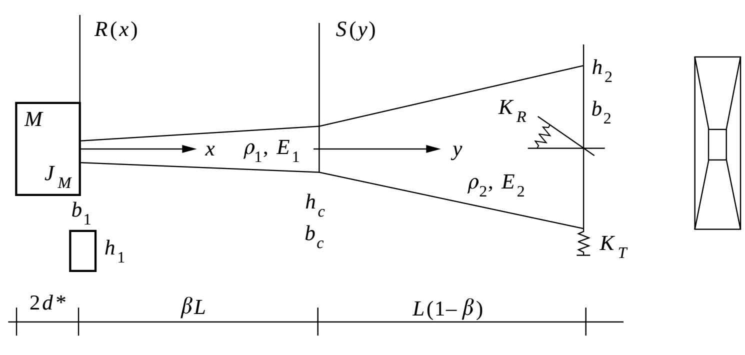

Figure 1. Cantilever beam carrying a heavy tip body.

TABLE 1

Comparison of p_i for $\beta = \alpha_1 = d = k = 0$, $v = 1$ and "cone beam" [5]

		Reference [5]			Exact			Rayleigh–Ritz		
α	μ	p_1	p_2	p_3	p_1	p_2	p_3	p_1	p_2	p_3
1·2	0·2	1·805112	4·531402	7·682832	1·805113	4·531399	7·682832	1·805113	4·531399	7·682992
	0·6	1·519987	4·343869	7·542354	1·519988	4·343868	7·542352	1·519988	4·343868	7·542459
	1	1·373710	4·289382	7·507077	1·373712	4·289384	7·507077	1·373712	4·289384	7·507173
	2	1·180606	4·242900	7·478643	1·180607	4·242896	7·478642	1·180607	4·242897	7·478730
2	0·2	2·392505	5·375537	8·914101	2·392505	5·375542	8·914098	2·392505	5·375542	8·914725
	0·6	1·949585	5·190742	8·804079	1·949585	5·190740	8·804081	1·949585	5·190740	8·804573
	1	1·744162	5·145036	8·779419	1·744161	5·145038	8·779421	1·744162	5·145038	8·779885
	2	1·485715	5·108415	8·760285	1·485718	5·108412	8·760283	1·485718	5·108412	8·760726

the cross-sectional area and moment of inertia are given by

$$A_x = A_1\,\xi^n, \qquad I_x = \hat{I}_1\,\xi^{n+2}, \qquad A_y = A_1\,\eta^n, \qquad I_y = \hat{I}_1\,\eta^{n+2}, \tag{4}$$

where the superscript n describes the taper of the cross-section.

For $n = 1 \rightarrow (b_2/b_1 = b_2/b_c = 1,\ \alpha_1 = h_c/h_1,\ \alpha_2 = b_2/b_c)$ whereas, for $n = 2 \rightarrow (\alpha_1 = b_c/b_1 = h_c/h_1,\ \alpha_2 = b_2/b_c = h_2/h_c)$. Introducing the following non-dimensional quantities

$$v = \frac{\rho_2}{\rho_1}, \qquad \varepsilon = \frac{E_2}{E_1}, \qquad p^4 = \rho_1\,\omega^2\,\frac{A_1 L^4}{E_1 \hat{I}_1}, \tag{5}$$

and substituting equations (3) and (4) into equations (1) and (2), the governing differential equations become

$$\xi^2 R_{,\xi\xi\xi\xi} + 2(n+2)\xi R_{,\xi\xi\xi} + 6nR_{,\xi\xi} - p_a^4\,R = 0 \qquad 1 \leqslant \xi \leqslant \alpha_1, \tag{6}$$

$$\eta^2 S_{,\eta\eta\eta\eta} + 2(n+2)\eta S_{,\eta\eta\eta} + 6nS_{,\eta\eta} - p_b^4\,S = 0 \qquad \alpha_1 \leqslant \eta \leqslant \alpha_1\,\alpha_2, \tag{7}$$

where

$$p_a = p\,\frac{\beta}{\alpha_1 - 1}, \qquad p_b = p\left(\frac{v}{\varepsilon}\right)^{0\cdot25}\frac{1-\beta}{\alpha_1\,\alpha_2 - \alpha_1},$$

and $(,\xi) = \mathrm{d}/\mathrm{d}\xi$, $(,\eta) = \mathrm{d}/\mathrm{d}\eta$.

To simplify the analysis, it is assumed that the tip mass is considered by means of the following parameters

$$d = \frac{d^*}{L}, \qquad \mu = \frac{M}{m_T}, \qquad k^2 = \frac{J_M}{L^2 M}, \tag{8}$$

where

$$m_T = \rho_1\,A_1\,LZ \tag{9}$$

is the whole mass of the beam, J_M is the rotary inertia of the concentrated mass and

$$Z = (n+1)^{-1}\{[(n-1)\alpha_1^2 + \alpha_1 + 1]\beta + (1-\beta)v\alpha_1^n\,[(n-1)\alpha_2^2 + \alpha_2 + 1]\}. \tag{10}$$

The boundary and continuity conditions, in term of dimensionless variables are at $x = 0$, $(\xi = 1)$;

$$R_{,\xi\xi\xi} + (n+2)R_{,\xi\xi} + \mu Z d p^4 \frac{\beta^2}{[\alpha_1 - 1]^2} R_{,\xi} - \mu Z p 4 \frac{\beta^3}{[\alpha_1 - 1]^3} R = 0, \tag{11}$$

$$R_{,\xi\xi} + \mu Z(d^2 + k^2)p^4 \frac{\beta}{[\alpha_1 - 1]} R_{,\xi} - \mu Z d p^4 \frac{\beta^2}{[\alpha_1 - 1]^2} R = 0, \tag{12}$$

at point $x = \beta L$, $y = 0$, $(\xi = \eta = \alpha_1)$;

$$R = S, \tag{13}$$

$$\frac{\alpha_1 - 1}{\beta} R_{,\xi} - \frac{\alpha_1 \alpha_2 - \alpha_1}{1 - \beta} S_{,\eta} = 0, \tag{14}$$

$$\frac{(\alpha_1 - 1)^2}{\beta^2} R_{,\xi\xi} - \frac{(\alpha_1 \alpha_2 - \alpha_1)^2}{(1 - \beta)^2} S_{,\eta\eta} = 0, \tag{15}$$

$$\frac{(\alpha_1 - 1)^3}{\beta^3}[(n+2)R_{,\xi\xi} + \alpha_1 R_{,\xi\xi\xi}] - \frac{(\alpha_1 \alpha_2 - \alpha_1)^3}{(1-\beta)^3}\varepsilon[(n+2)S_{,\eta\eta} + \alpha_1 S_{,\eta\eta\eta}] + \alpha_1 S_{,\eta\eta\eta} = 0, \tag{16}$$

for $y = L(1 - \beta)$, $\eta = \alpha_1 \alpha_2$;

$$\frac{(\alpha_1 \alpha_2 - \alpha_1)^3}{(1-\beta)^3} C_T \left[S_{,\eta\eta\eta} + \frac{n+2}{\alpha_1 \alpha_2} S_{,\eta\eta} \right] - S = 0, \tag{17}$$

$$\frac{\alpha_1 \alpha_2 - \alpha_2}{1 - \beta} C_R S_{,\eta\eta} + S_{,\eta} = 0, \tag{18}$$

where

$$C_R = \frac{E_2 \hat{I}_2}{k_R L}, \qquad C_T = \frac{E_2 \hat{I}_2}{k_T L^3}.$$

TABLE 2

First three dimensionless frequencies for "wedge beam", and $\alpha_1 = \alpha_2 = 1{\cdot}5$, $d = k = 0{\cdot}3$, $\beta = 0{\cdot}5$

		p_1		p_2		p_3	
	μ	R–R	Ex	R–R	Ex	R–R	Ex
	0	2·908698	2·908678	6·197270	6·197194	9·909998	9·909596
	0·2	1·950037	1·950022	3·796142	3·796139	6·834658	6·834583
$\varepsilon = 1$	0·4	1·673839	1·673827	3·555689	3·555322	6·586466	6·586402
$v = 1$	0·6	1·523113	1·523102	3·393224	3·393222	6·438497	6·438438
	0·8	1·422423	1·422412	3·265294	3·265292	6·340378	6·340324
	1	1·348103	1·348093	3·159160	3·159157	6·270748	6·270696
	0	3·598918	3·571805	6·285250	6·238373	10·201526	10·137590
	0·2	1·842931	1·830463	3·960271	3·955555	6·560658	6·512302
$\varepsilon = 3$	0·4	1·558307	1·547941	3·631474	3·623548	6·335685	6·296653
$v = 2{\cdot}88889$	0·6	1·410683	1·401354	3·403380	3·395021	6·231654	6·196897
	0·8	1·313996	1·305332	3·231779	3·223316	6·172279	6·139946
	1	1·243386	1·235203	3·095810	3·087374	6·134031	6·103251

R–R: Rayleigh–Ritz; Ex: Exact.

The general solution to the differential equations (6) and (7) can be written, respectively, as

$$R(\xi) = \xi^{-n0\cdot 5}\{C_1\,\mathrm{J}_n\,(a) + C_2\,\mathrm{Y}_n\,(a) + C_3\,\mathrm{I}_n\,(a) + C_4\,\mathrm{K}_n\,(a)\}, \tag{19}$$

$$S(\eta) = \eta^{-n0\cdot 5}\{C_5\,\mathrm{J}_n\,(b) + C_6\,\mathrm{Y}_n\,(b) + C_7\,\mathrm{I}_n\,(b) + C_8\,\mathrm{K}_n\,(b)\}, \tag{20}$$

where $a = 2p_a\,\xi^{0\cdot 5}$, $b = 2p_b\,\eta^{0\cdot 5}$ and J, Y, I, K are Bessel functions of, respectively, the first second, modified first and modified second kinds.

The C_i are arbitrary constants to be evaluated from the eight boundary conditions. Substituting equations (19) and (20) into equations (11)–(18) one obtains an 8×8 matrix characteristic equation for the eight unknown constants C_i. For a non-trivial solution the determinant of the coefficient matrix is set equal to zero, yielding the frequency equation:

$$\det \mathbf{A} = \mathbf{0} \tag{21}$$

where the elements of matrix **A** are given in the Appendix and were obtained by manipulating some well-known reduction formulae of the Bessel functions. These non-trivial solutions of the characteristic equation may be numerically obtained by utilizing the false position method [24].

3. THE RAYLEIGH–RITZ METHOD

An approximate solution to the problem is obtained with the use of the Rayleigh–Ritz procedure. In this case, the transverse displacements of the beam are approximated with the use of N functions, which should satisfy only the essential conditions at $x = 0$ and $x = L$,

$$w(x) \cong \sum_{i=1}^{N} q_i\,\phi_i\,(x), \tag{22}$$

where q_i are constants.

The ϕ_i functions are a set of orthogonal polynomials in the $\{0, 1\}$ interval and determined through the utilization of the Gram-Schmidt procedure [25]. It is useful to express the geometrical characteristics with respect to a single reference system with origin at $x = 0$; for this reason, the area and inertia of the generic section can be expressed using the following relations:

$$A_x = A_1\left\{1 + \frac{\alpha_1 - 1}{\beta L}\,x\right\}^n, \qquad I_x = \left\{1 + \frac{\alpha_1 - 1}{\beta L}\,x\right\}^{n+2} \qquad 0 \leqslant x \leqslant \beta L, \tag{23, 24}$$

$$\bar{A}_x = A_1\left\{\frac{\alpha_1\,\alpha_2 - \alpha_1}{L(1-\beta)}\,(x - L) + \alpha_2\,\alpha_1\right\}^n, \tag{25}$$

$$\beta L \leqslant x \leqslant L.$$

$$\bar{I}_x = \hat{I}_1\left\{\frac{\alpha_1\,\alpha_2 - \alpha_1}{L(1-\beta)}\,(x - L) + \alpha_2\,\alpha_1\right\}^{n+2} \tag{26}$$

TABLE 3

Frequency coefficients for "wedge beam" for $\alpha_1 = \alpha_2 = 1{\cdot}5$, $d = k = 0{\cdot}3$

	β	$\mu = 0$ p_1	$\mu = 0$ p_2	$\mu = 0$ p_3	$\mu = 0{\cdot}2$ p_1	$\mu = 0{\cdot}2$ p_2	$\mu = 0{\cdot}2$ p_3	$\mu = 0{\cdot}4$ p_1	$\mu = 0{\cdot}4$ p_2	$\mu = 0{\cdot}4$ p_3
Single	0	2·352241	5·426063	8·860973	1·707332	3·457144	6·237358	1·482579	3·239609	6·023052
material	0·2	2·960984	6·623893	10·653342	2·059033	3·925757	7·116501	1·774149	3·666364	6·841862
$\varepsilon = 1$	0·4	2·955257	6·339284	10·116494	1·992281	3·821323	6·926297	1·710449	3·577698	6·665992
$v = 1$	0·6	2·831723	6·092949	9·714487	1·904147	3·765926	6·767201	1·635664	3·527207	6·530251
	0·8	2·610069	5·874837	9·440699	1·806731	3·649276	6·629541	1·581490	3·417562	6·400466
Aluminum	0·2	3·341329	7·266658	10·982127	2·026751	3·661033	6·937752	1·720309	3·405197	6·564168
–steel	0·4	3·638804	6·379341	10·365182	1·889703	3·873329	6·519061	1·597092	3·571705	6·220561
$\varepsilon = 3$	0·6	3·352323	6·436236	9·652822	1·781451	3·937727	6·776444	1·809009	3·602681	6·592916
$v = 2{\cdot}88889$	0·8	2·809352	6·321979	9·990769	1·717485	3·704704	6·940148	1·465313	3·430107	6·694357
Steel–	0·2	2·448228	6·152010	10·232177	1·893668	3·998295	7·349074	1·665636	3·729066	7·163349
aluminum	0·4	2·310298	5·987174	10·093364	1·797208	3·787541	6·992424	1·582539	2·530131	6·769581
$\varepsilon = 0{\cdot}33333$	0·6	2·255394	5·732798	9·537642	1·737999	3·596244	6·596423	1·526666	3·354303	6·375459
$v = 0{\cdot}34615$	0·8	2·258520	5·453641	9·048186	1·707071	3·449192	6·301326	1·492827	3·225488	6·080065
Tungsten–	0·2	2·223087	6·381117	10·567467	1·820271	4·083185	8·013721	1·625852	3·780457	7·869469
aluminum	0·4	2·050444	5·865405	10·710794	1·672813	3·783351	7·233154	1·492634	3·510547	6·992964
$\varepsilon = 0{\cdot}2$	0·6	2·006496	5·539217	9·568246	1·608994	3·559425	6·483186	1·428457	3·305899	6·272004
$v = 0{\cdot}15$	0·8	2·060400	5·308516	8·883911	1·608838	3·367419	6·189325	1·417722	3·138986	5·971895
Aluminum	0·2	3·221447	7·139275	10·91989	1·785804	3·280014	6·294119	1·508671	3·039095	5·892991
–tungsten	0·4	3·797734	6·077439	9·997253	1·627801	3·659843	5·748811	1·371297	3·303439	5·492732
$\varepsilon = 5$	0·6	3·527982	6·354572	9·292952	1·541426	3·746303	6·467649	1·300391	3·326321	6·365088
$v = 6{\cdot}66666$	0·8	2·858571	6·472948	10·206816	1·537534	3·551453	6·907545	1·303834	3·225973	6·677271

	β	$\mu = 0{\cdot}6$ p_1	$\mu = 0{\cdot}6$ p_2	$\mu = 0{\cdot}6$ p_3	$\mu = 0{\cdot}8$ p_1	$\mu = 0{\cdot}8$ p_2	$\mu = 0{\cdot}8$ p_3	$\mu = 1$ p_1	$\mu = 1$ p_2	$\mu = 1$ p_3
Single	0	1·354955	3·103821	5·886694	1·268257	2·998621	5·791332	1·203668	2·911164	5·720927
material	0·2	1·616571	3·496715	6·677581	1·510739	3·364492	6·568507	1·432401	3·255224	6·491042
$\varepsilon = 1$	0·4	1·556513	3·413678	6·511235	1·453651	3·284678	6·408882	1·377719	3·177718	6·336385
$v = 1$	0·6	1·488799	3·366624	6·387887	1·390585	3·240307	6·292928	1·318051	3·135516	6·225254
	0·8	1·420332	3·265895	6·259635	1·297379	3·147232	6·164004	1·258987	3·048733	6·094951
Aluminum	0·2	1·559231	3·223872	6·374655	1·453227	3·082465	6·261217	1·375621	2·967251	6·186011
–steel	0·4	1·445549	3·357531	6·083735	1·346356	3·193892	6·006136	1·273939	3·063028	5·956385
$\varepsilon = 3$	0·6	1·366886	3·375742	6·503363	1·273591	3·205851	6·503363	1·205375	3·071446	6·416133
$v = 2{\cdot}88889$	0·8	1·330597	3·242858	6·561844	1·241334	3·098968	6·561884	1·175735	2·982529	6·423114
Steel–	0·2	1·530193	3·574803	7·040742	1·436299	3·459727	6·950275	1·365538	3·365485	6·880313
aluminum	0·4	1·454497	3·384821	6·621067	1·365572	3·276971	6·512328	1·298486	3·188757	6·429007
$\varepsilon = 0{\cdot}3333$	0·6	1·315215	3·215709	6·228947	1·315215	3·112032	6·122742	1·250144	3·026933	6·042093
$v = 0{\cdot}34615$	0·8	1·368139	3·093281	5·935811	1·282472	2·992849	5·832762	1·218260	2·909834	5·755423
Tungsten–	0·2	1·503841	3·620991	7·773811	1·416958	3·508074	7·699923	1·350434	3·418127	7·640018
aluminum	0·4	1·379993	3·367276	6·831764	1·299934	3·265625	6·710699	1·238701	3·184367	6·615587
$\varepsilon = 0{\cdot}2$	0·6	1·317621	3·168974	6·129826	1·239557	3·069969	6·024162	1·180174	2·990029	5·942125
$v = 0{\cdot}15$	0·8	1·303405	3·008781	5·827753	1·223914	2·912964	5·722981	1·163902	2·834611	5·643202
Aluminum	0·2	1·365292	2·866089	5·707957	1·271493	2·732352	5·602542	1·203038	2·624448	5·534724
–tungsten	0·4	1·239852	3·066207	5·395571	1·154158	2·893622	5·345227	1·091731	2·760135	5·314581
$\varepsilon = 5$	0·6	1·176308	3·069078	6·322726	1·095271	2·887817	6·299683	1·036181	2·749855	6·285215
$v = 6{\cdot}6666$	0·8	1·181491	3·012488	6·569891	1·101066	2·855222	6·507961	1·042215	2·731991	6·467773

With the approximation (22) the functional of the problem takes the form

$$\Pi = \tfrac{1}{2}\left\{\int_0^{\beta L} E_1\, I_x \left[\left(\sum_{i=1}^{N} q_i\,\phi_i\right)''\right]^2 \mathrm{d}x + \int_{\beta L}^{L} E_2\, \bar{I}_x \left[\left(\sum_{i=1}^{N} q_i\,\phi_i\right)''\right]^2 \mathrm{d}x\right\}$$

$$+ \frac{k_R}{2}\left[\left(\sum_{i=1}^{N} q_i\,\phi_i\,(L)\right)'\right]^2 + \omega^2 M d^*\left(\sum_{i=1}^{N} q_i\,\phi_i\,(0)\right)$$

$$- \frac{\omega^2}{2}\left\{\int_0^{\beta L} \rho_1\, A_x \left(\sum_{i=1}^{N} q_i\,\phi_i\right)^2 \mathrm{d}x + \int_{\beta L}^{L} \rho_2\, \bar{A}_x \left(\sum_{i=1}^{N} q_i\,\phi_i\right)^2 \mathrm{d}x\right\}$$

$$- \omega^2\,\frac{M d^{*2} + J_M}{2}\left(\sum_{i=1}^{N} q_i\,\phi_i\,(0)\right)'^2 - \omega^2\,\frac{M}{2}\left(\sum_{i=1}^{N} q_i\,\phi_i\,(0)\right)^2, \tag{27}$$

where (') = d/dx.

This is the function, quadratic in q_i, which has to be minimized, the necessary conditions for which are that

$$\frac{\partial \Pi}{\partial q_i} = 0. \tag{28}$$

Equation (28) yields a set of N homogeneous equations linear with regard to the unknown constants q_j. This set of equations has the form

$$(\mathbf{K} - \omega^2 \mathbf{m})\mathbf{q} = \mathbf{0} \tag{29}$$

TABLE 4

First three dimensionless for "cone beam", and $\alpha_1 = \alpha_2 = 1{\cdot}5$, $d = k = 0{\cdot}3$

		p_1		p_2		p_3	
		R–R	Ex	R–R	Ex	R–R	Ex
	0	3·242137	3·242074	6·443742	6·443560	10·078169	10·072212
	0·2	1·932664	1·932621	3·840273	3·840260	6·799172	6·798796
$\varepsilon = 1$	0·4	1·643317	1·643282	3·569174	3·569157	6·533887	6·533553
$v = 1$	0·6	1·490420	1·490388	3·378417	3·378398	6·393477	6·393163
	0·8	1·389566	1·389536	3·230088	3·230068	6·307267	6·306966
	1	1·315632	1·315604	3·109389	3·109368	6·249181	6·248887
	0	3·970436	3·936340	6·528690	6·486210	10·422317	10·342960
	0·2	1·726143	1·715648	3·969292	3·954941	6·518772	6·485023
$\varepsilon = 3$	0·4	1·455223	1·446492	3·555054	3·540319	6·342611	6·316978
$v = 2{\cdot}88889$	0·6	1·316606	1·308200	3·291399	3·277077	6·271695	6·249200
	0·8	1·225252	1·217950	3·1026115	3·088788	6·233707	6·212851
	1	1·159067	1·152169	2·957667	2·944287	6·210089	6·190237

TABLE 5
Frequency coefficients for "cone beam" for $\alpha_1 = \alpha_2 = 1{\cdot}5$, $d = k = 0{\cdot}3$

		$\mu = 0$			$\mu = 0{\cdot}2$			$\mu = 0{\cdot}4$		
	β	p_1	p_2	p_3	p_1	p_2	p_3	p_1	p_2	p_3
Single	0	2·493745	5·529994	8·927155	1·735675	3·461755	6·216654	1·497497	3·244322	5·987237
material	0·2	3·241992	6·890704	10·872190	2·068727	3·875903	7·018439	1·765293	3·605407	6·712642
$\varepsilon = 1$	0·4	3·295517	6·585544	10·284771	1·979450	3·843161	6·855248	1·682952	3·573860	6·570203
$v = 1$	0·6	3·135124	6·355970	9·883609	1·887290	3·814184	6·781485	1·606400	3·544996	6·529558
	0·8	2·826031	6·098033	9·655944	1·804884	3·672274	6·668325	1·543512	3·424297	6·418686
Aluminum	0·2	3·601942	7·570925	11·289006	1·944836	3·641679	6·799336	1·642494	3·354608	6·420367
–steel	0·4	4·013659	6·625189	10·579703	1·773741	3·908384	6·364059	1·494782	3·528107	6·113427
$\varepsilon = 3$	0·6	3·653699	6·759004	9·818821	1·674785	3·895979	6·867924	1·413566	3·490150	6·713386
$v = 2{\cdot}88889$	0·8	3·005513	6·525742	10·241486	1·648233	3·654176	6·903785	1·398120	3·338763	6·659011
Steel–	0·2	2·719027	6·366531	10·431558	1·980940	3·998783	7·247505	1·721109	3·737392	7·030027
aluminum	0·4	2·591143	6·257891	10·208067	1·896179	3·787051	6·964831	1·648045	3·539363	6·716333
$\varepsilon = 0{\cdot}33333$	0·6	2·520643	5·982285	9·700203	1·829771	3·608648	6·646516	1·587743	3·375870	6·395523
$v = 0{\cdot}34615$	0·8	2·484248	5·647080	9·224965	1·776400	3·481009	6·337592	1·537680	3·262114	6·093694
Tungsten–	0·2	2·485453	6·604193	10·794795	1·952698	4·135567	7·910581	1·723607	3·847410	7·742398
aluminium	0·4	2·303951	6·171426	10·786371	1·810238	3·816238	7·207479	1·597569	3·553881	6·950312
$\varepsilon = 0{\cdot}2$	0·6	2·247777	5·799427	9·727984	1·734207	3·570670	6·561262	1·523479	3·328375	6·325515
$v = 0{\cdot}15$	0·8	2·280355	5·486273	9·059468	1·706268	3·387884	6·223840	1·488628	3·168331	5·983732
Aluminum	0·2	3·444017	7·392360	10·993261	1·676904	3·285560	6·110438	1·412793	3·007518	5·732135
–tungsten	0·4	4·142424	6·344458	10·200137	1·499219	3·631035	5·607285	1·261640	3·196992	5·439036
$\varepsilon = 5$	0·6	3·816945	6·688564	9·456933	1·420405	3·577169	6·653282	1·196319	3·115381	6·583743
$v = 6{\cdot}66666$	0·8	3·048076	6·658406	10·461855	1·439460	3·423103	6·821924	1·216112	3·053286	6·625030

		$\mu = 0{\cdot}6$			$\mu = 0{\cdot}8$			$\mu = 1$		
	β	p_1	p_2	p_3	p_1	p_2	p_3	p_1	p_2	p_3
Single	0	1·365236	3·102096	5·846668	1·276237	2·990602	5·751502	1·210283	2·897871	5·682955
material	0·2	1·602944	3·418782	6·547340	1·495357	3·273474	6·444656	1·184205	2·860899	5·632830
$\varepsilon = 1$	0·4	1·526303	3·384034	6·420021	1·422986	3·236181	6·328091	1·149950	2·875243	5·618532
$v = 1$	0·6	1·457470	3·356701	6·394129	1·359099	3·210373	6·310177	1·111408	2·867848	5·646070
	0·8	1·402724	3·254002	6·277478	1·309146	3·120930	6·187055	1·071910	2·798227	5·576787
Aluminum	0·2	1·486251	3·152610	6·248163	1·384077	2·998755	6·150785	1·309525	2·875872	6·088407
–steel	0·4	1·351662	3·276970	6·013777	1·258315	3·094255	5·960893	1·190297	2·952763	5·928233
$\varepsilon = 3$	0·6	1·278885	3·233484	6·647104	1·190875	3·049622	6·610482	1·126679	2·908292	6·587294
$v = 2{\cdot}88889$	0·8	1·267047	3·127637	6·541794	1·180852	2·970097	6·473414	1·117766	2·845609	6·428710
Steel–	0·2	1·573275	3·575332	6·890734	1·472764	3·450579	6·792691	1·397851	3·347402	6·719936
aluminum	0·4	1·506648	3·387086	6·557554	1·410465	3·269991	6·446765	1·338760	3·173087	6·365184
$\varepsilon = 0{\cdot}3333$	0·6	1·450587	3·231030	6·236890	1·357518	3·119087	6·127207	1·288230	3·026248	6·046991
$v = 0{\cdot}34615$	0·8	1·403547	3·122880	5·941740	1·312859	3·014361	5·837633	1·245479	2·924062	5·761988
Tungsten–	0·2	1·585779	3·687121	7·627747	1·489669	3·569359	7·540752	1·416991	3·473634	7·471957
aluminum	0·4	1·469673	3·408812	6·778338	1·380516	3·302139	6·652509	1·313111	3·215198	6·556171
$\varepsilon = 0{\cdot}2$	0·6	1·398778	3·190139	6·169312	1·312509	3·086868	6·056676	1·247576	3·002154	5·971565
$v = 0{\cdot}15$	0·8	1·362908	3·036739	5·829343	1·276890	2·936183	5·720806	1·212554	2·852922	5·640386
Aluminum	0·2	1·277395	2·814320	5·574330	1·189116	2·669183	5·488645	1·124803	2·554634	5·435027
–tungsten	0·4	1·140306	2·935211	5·381843	1·061307	2·753546	5·353338	1·003797	2·616752	5·336309
$\varepsilon = 5$	0·6	1·081571	2·851033	6·557370	1·006783	2·670630	6·543519	0·952308	2·535849	6·534987
$v = 6{\cdot}6666$	0·8	1·100603	2·823890	6·543074	1·025029	2·660884	6·498304	0·969869	2·536133	6·470121

where **K** is the global stiffness matrix, **m** is the global mass matrix, and **q** is the vector of the unknown q_j.

It is noted at this point that the accuracy of the Rayleigh–Ritz method (henceforth R–R) results depends on the number N of the assumed mode shape functions ϕ_i.

4. THE EIGENFUNCTIONS

The mass at the free end leads to a non-homogeneous boundary condition at $x = 0$, and the appearance of the eigenvalue ω^2 is responsible for the non-self-adjointness of the problem. Consequently, the vibration modes are no longer mutually orthogonal in the usual sense. It is therefore convenient to adopt a transformation as suggested by Morgan [26], according to which the orthogonality condition reads:

$$\int_0^{\beta L} m(x)\psi_i\,\psi_j\,\mathrm{d}x + M\psi_i\,(0)\psi_j\,(0) + J_M\,\psi_i'\,(0)\psi_j'\,(0) = 0. \tag{30}$$

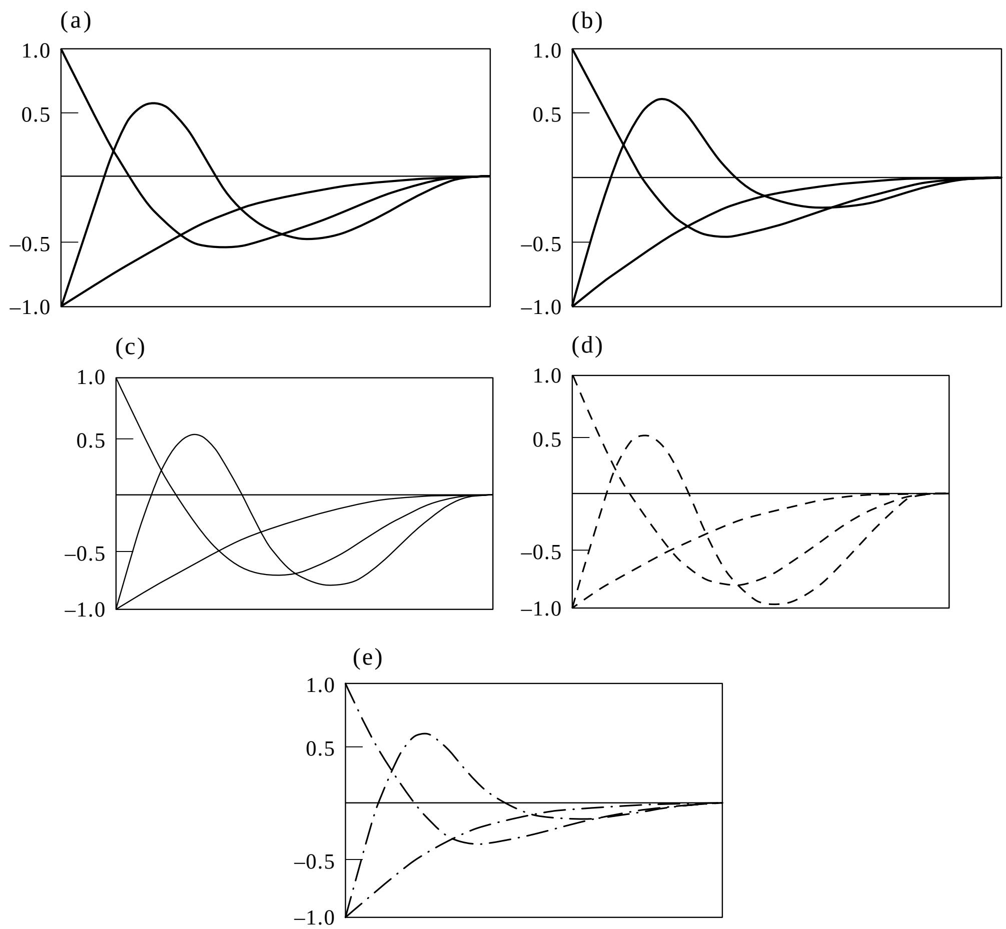

Figure 2. First three mode shapes for wedge-beam; $\alpha 1 = 1{\cdot}5$, $\beta = 0{\cdot}5$, $\mu = 0$. (a) Single material, $\varepsilon = 1$, $v = 1$; (b) aluminum–steel, $\varepsilon = 3$, $v = 2{\cdot}88889$; (c) steel–aluminum, $\varepsilon = 0{\cdot}33333$, $v = 0{\cdot}34615$; (d) tungsten alloy–aluminum, $\varepsilon = 0{\cdot}2$, $v = 0{\cdot}15$; (e) aluminum–tungsten alloy, $\varepsilon = 5$, $v = 6{\cdot}66667$.

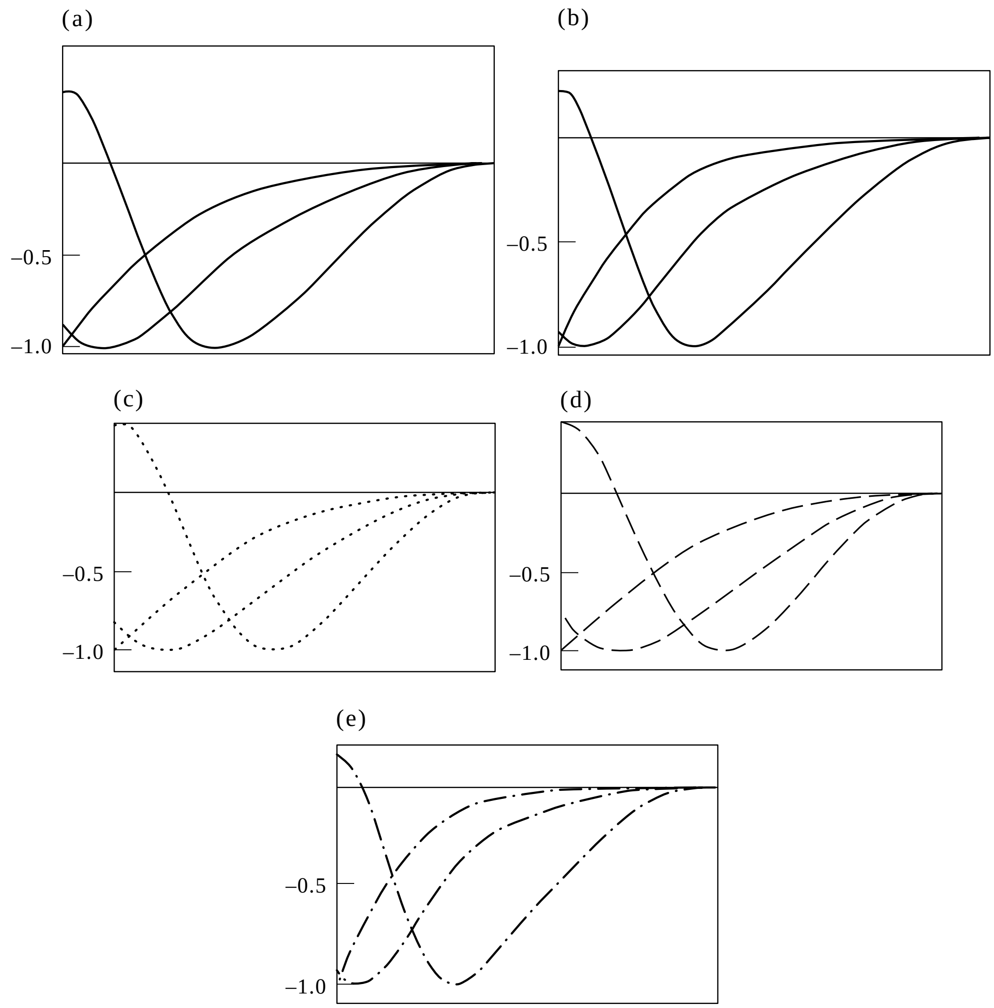

Figure 3. As fig. 2 for $\mu = 1$.

From equation (22) the first three orthogonal vibration modes can be obtained, by using a Schmidt-like procedure. It will be:

$\psi_1 = R_1$,

$$\psi_2 = R_2 - \frac{\int_0^{\beta L} \rho_1 A(x)\psi_1 \psi_2 \, dx + M\psi_1(0)\psi_2(0) + J_M \psi_1'(0)\psi_2'(0)}{\int_0^{\beta L} \rho_1 A(x)\psi_1 \psi_1 \, dx + M\psi_1(0)\psi_1(0) + J_M \psi_1'(0)\psi_1'(0)} \psi_1, \quad (31)$$

$$\psi_3 = R_3 + \tilde{b}\psi_1 + \tilde{c}\psi_2,$$

where R_i, $i = 1, 2, 3$ are the vibration modes (19, 20) corresponding to p_i, and the constants $\tilde{b}$, $\tilde{c}$ can be determined by imposing the orthogonality conditions of ψ_3 with respect to ψ_2 and ψ_1.

If we put

$$\hat{a}_{ij} = \int_0^{\beta L} \rho_1 A(x) \psi_i \psi_j \,\mathrm{d}x + M\psi_i(0)\psi_j(0) + J_M \psi_i'(0)\psi_j'(0), \tag{32}$$

they will be:

$$\tilde{b} = \frac{\hat{a}_{12}\hat{a}_{23} - \hat{a}_{13}\hat{a}_{22}}{\hat{a}_{11}\hat{a}_{22} - \hat{a}_{12}\hat{a}_{21}}, \qquad \tilde{c} = \frac{\hat{a}_{13}\hat{a}_{21} - \hat{a}_{11}\hat{a}_{23}}{\hat{a}_{11}\hat{a}_{22} - \hat{a}_{12}\hat{a}_{21}}.$$

5. NUMERICAL RESULTS AND DISCUSSION

The beam is composed of two assembled tapered beams that may have different physical characteristics that are represented by the two coefficients ε and ν. For the purpose of making the problem real and the subsequent numerical examples, the following characteristics regarding the materials have been adopted. aluminium: $E = 6{\cdot}867 \times 10^{10}$ N/

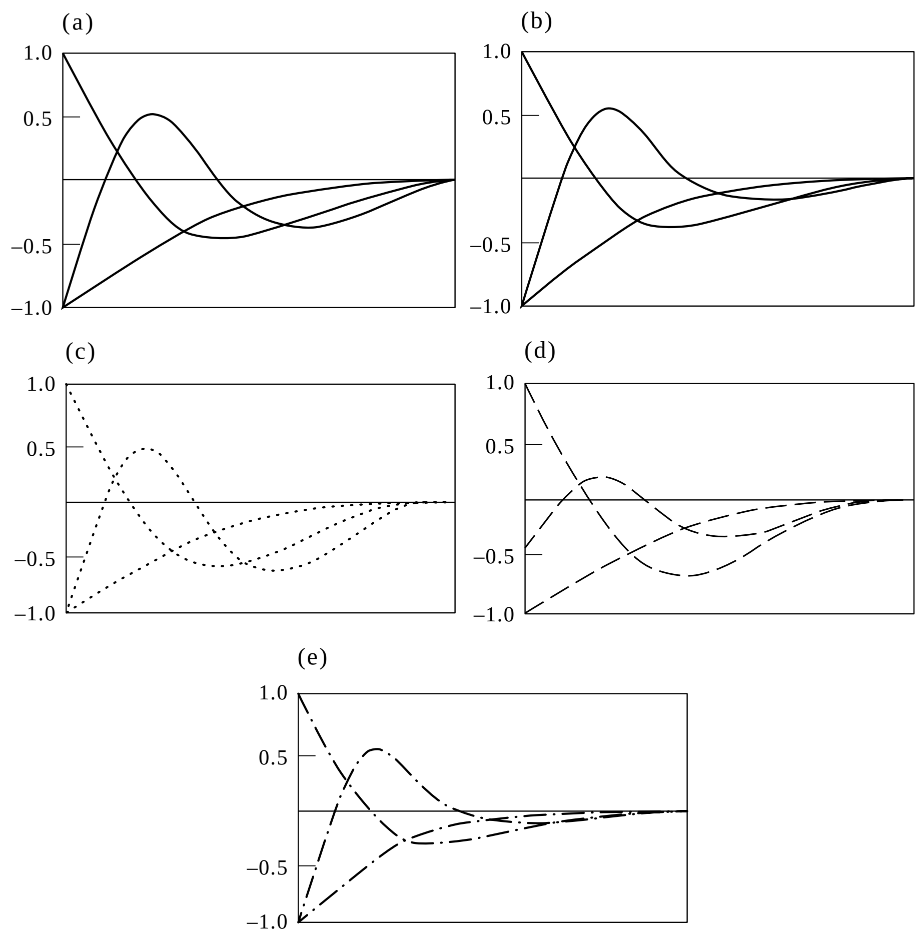

Figure 4. First three mode shapes for cone-beam; $\alpha_1 = \alpha_2 = 1{\cdot}5$, $\beta = 0{\cdot}5$, $\mu = 0$. (a) single material, $\varepsilon = 1$, $\nu = 1$; (b) aluminum–steel, $\varepsilon = 3$, $\nu = 2{\cdot}88889$; (c) steel–aluminum, $\varepsilon = 0{\cdot}33333$, $\nu = 0{\cdot}34615$; (d) tungsten alloy–aluminum, $\varepsilon = 0{\cdot}2$, $\nu = 0{\cdot}15$; (e) aluminum–tungsten alloy, $\varepsilon = 5$, $\nu = 6{\cdot}66667$.

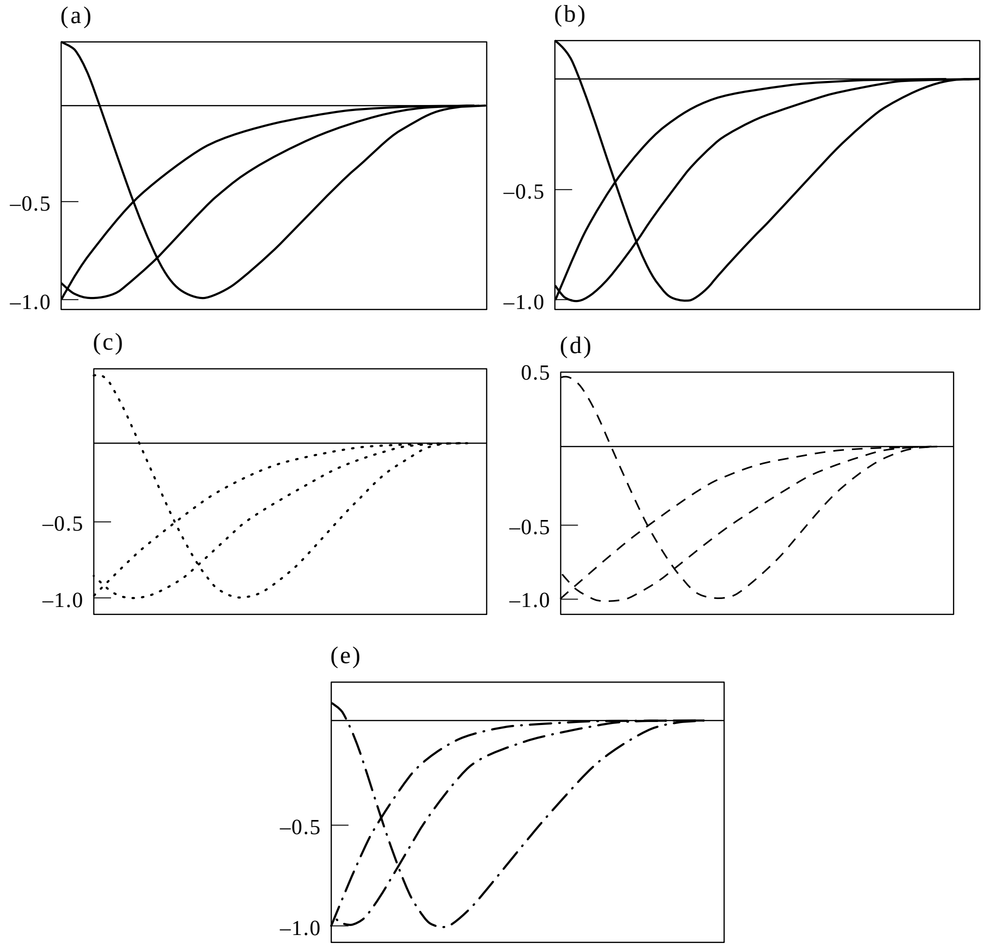

Figure 5. As fig. 4 for $\mu = 1$.

m^2, $\rho = 2{\cdot}7 \times 10^4$ N/m^3; steel: $E = 2{\cdot}06 \times 10^{11}$ N/m^2, $\rho = 7{\cdot}8 \times 10^4$ N/m^3; tungsten alloy: $E = 3{\cdot}4335 \times 10^{11}$ N/m^2, $\rho = 18 \times 10^4$ N/m^3.

Given the particular geometrical pattern studied and the difficulty experienced by the authors in obtaining a current bibliography of similar results, a preliminary numerical analysis was performed with the primary purpose of comparing the obtained free vibration values with the exact values procedure (Ex) and the approximate Rayleigh–Ritz (R–R) results, and for this reason, seven test functions were employed.

For the sake of comparison, the first three non-dimensional frequency coefficients are given in Table 1 for the particular case $\beta = \alpha_1 = k = d = 0$ and $v = 1$, together with the Mabie and Rogers results [5]. As can be seen, the discrepancies with the exact values are negligible, whereas the R–R method gives, as expected, an upper bound.

The results for the first three free frequencies are summarized in the p values provided in Table 2 for $\alpha_1 = \alpha_2 = 1{\cdot}5$, $d = k = 0{\cdot}3$, $\beta = 0{\cdot}5$, which also shows how the approximative values coincide with the exact values. In fact the maximum error is attained for $\mu = 1$, $\varepsilon = 3$ and $v = 2{\cdot}888$, and is less than 0·6% with respect to the exact values.

Tables 3, 4 and 5 provide the first three non-dimensional frequencies for $\alpha_1 = \alpha_2 = 1{\cdot}5$, $d = k = 0{\cdot}3$ and for various values of the parameters ε, v, μ and β.

For the purpose of demonstrating the different dynamic behaviour of the structures as a function of the various combinations of materials used, the beam's first three vibration modes were compared at fixed d, k and β parameter values at increasing μ values. The results are shown in Figures 2 and 3 for wedge-type beams and Figures 4 and 5 for cone-beams. It may be observed how the transverse displacements are highly influenced by the physical characteristics of the material involved, while significant differences in dynamic behaviour also depend on the order in which these materials are combined together, all other conditions being equal. In demonstration of the above, it is sufficient to compare the deformations in the tungsten–aluminium combination with those of the aluminium–tungsten combination in order to realize just how much the transverse displacements of the latter are reduced.

6. CONCLUSIONS

In this paper, the Rayleigh–Ritz approximate method has been employed by using orthogonal polynomials as trial functions. The results obtained are in excellent agreement with the exact results, even for the higher order frequencies of complex structural patterns. Obviously, the Rayleigh–Ritz approach becomes invaluable for all the cases in which analytical solutions cannot be obtained.

The results achieved also permit interventions on geometric parameters so allowing a structural optimization of the beam with respect to its dynamic behaviour.

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APPENDIX

$$a_{11} = -\mu z p^2 J_n(r) - \mu z d p^3 J_{n+1}(r) + (n+1) q_1 J_{n+1}(r) - p J_{n+3}(r),$$

$$a_{12} = -\mu z p^2 Y_n(r) - \mu z d p^3 Y_{n+1}(r) + (n+1) q_1 Y_{n+1}(r) - p Y_{n+3}(r),$$

$$a_{13} = -\mu z p^2 I_n(r) + \mu z d p^3 I_{n+1}(r) + (n+1) q_1 I_{n+1}(r) + p I_{n+3}(r),$$

$$a_{14} = -\mu z p^2 K_n(r) - \mu z d p^3 K_{n+1}(r) + (n+1) q_1 K_{n+1}(r) - p K_{n+3}(r),$$

$$a_{21} = -\mu z d p^2 J_n(r) - \mu z (d^2 + k^2) p^3 J_{n+1}(r) + J_{n+2}(r),$$

$$a_{22} = -\mu z d p^2 Y_n(r) - \mu z (d^2 + k^2) p^3 Y_{n+1}(r) + Y_{n+2}(r),$$

$$a_{23} = -\mu z d p^2 I_n(r) + \mu z (d^2 + k^2) p^3 I_{n+1}(r) + I_{n+2}(r),$$

$$a_{24} = -\mu z d p^2 K_n(r) - \mu z (d^2 + k^2) p^3 K_{n+1}(r) + K_{n+2}(r),$$

$$a_{31} = J_n(s), \qquad a_{32} = Y_n(s), \qquad a_{33} = I_n(s), \qquad a_{34} = K_n(s),$$

$$a_{35} = -J_n(\bar{r}), \qquad a_{36} = -Y_n(\bar{r}), \qquad a_{37} = -I_n(\bar{r}), \qquad a_{38} = -K_n(\bar{r}),$$

$$a_{41} = -J_{n+1}(s), \qquad a_{42} = -Y_{n+1}(s), \qquad a_{43} = I_{n+1}(s), \qquad a_{42} = -K_{n+1}(s),$$

$$a_{45} = \left(\frac{v}{\varepsilon}\right)^{0\cdot25} J_{n+1}(\bar{r}), \qquad a_{46} = \left(\frac{v}{\varepsilon}\right)^{0\cdot25} Y_{n+1}(\bar{r}), \qquad a_{47} = -\left(\frac{v}{\varepsilon}\right)^{0\cdot25} I_{n+1}(\bar{r}),$$

$$a_{48} = \left(\frac{v}{\varepsilon}\right)^{0\cdot25} K_{n+1}(\bar{r}),$$

$$a_{51} = \mathrm{J}_{n+2}(s), \qquad a_{52} = \mathrm{Y}_{n+2}(s), \qquad a_{53} = \mathrm{I}_{n+2}(s), \qquad a_{54} = \mathrm{K}_{n+2}(s),$$

$$a_{55} = -(v\varepsilon)^{0\cdot5}\mathrm{J}_{n+2}(\bar{r}), \qquad a_{56} = -(v\varepsilon)^{0\cdot5}\mathrm{Y}_{n+2}(\bar{r}), \qquad a_{57} = -(v\varepsilon)^{0\cdot5}\mathrm{I}_{n+2}(\bar{r}),$$

$$a_{58} = -(v\varepsilon)^{0\cdot5}\mathrm{K}_{n+2}(\bar{r}),$$

$$a_{61} = (n+2)q_1\,\mathrm{J}_{n+2}(s) - \alpha_1^{0\cdot5}\,p\mathrm{J}_{n+3}(s), \qquad a_{62} = (n+2)q_1\,\mathrm{Y}_{n+2}(s) - \alpha_1^{0\cdot5}\,p\mathrm{Y}_{n+3}(s),$$

$$a_{63} = (n+2)q_1\,\mathrm{I}_{n+2}(s) + \alpha_1^{0\cdot5}\,p\mathrm{I}_{n+3}(s), \qquad a_{64} = (n+2)q_1\,K_{n+2}(s) - \alpha_1^{0\cdot5}\,p\mathrm{K}_{n+3}(s),$$

$$a_{65} = -(n+2)\,(v\varepsilon)^{0\cdot5}q_2\,\mathrm{J}_{n+2}(\bar{r}) + \alpha_1^{0\cdot5}\,\varepsilon^{0\cdot25}v^{0\cdot75}p\mathrm{J}_{n+3}(\bar{r}),$$

$$a_{66} = -(n+2)\,(v\varepsilon)^{0\cdot5}q_2\,\mathrm{Y}_{n+2}(\bar{r}) + \alpha_1^{0\cdot5}\,\varepsilon^{0\cdot25}v^{0\cdot75}p\mathrm{Y}_{n+3}(\bar{r}),$$

$$a_{67} = -(n+2)\,(v\varepsilon)^{0\cdot5}q_2\,\mathrm{I}_{n+2}(\bar{r}) - \alpha_1^{0\cdot5}\,\varepsilon^{0\cdot25}v^{0\cdot75}p\mathrm{I}_{n+3}(\bar{r}),$$

$$a_{68} = -(n+2)\,(v\varepsilon)^{0\cdot5}q_2\,\mathrm{K}_{n+2}(\bar{r}) + \alpha_1^{0\cdot5}\,\varepsilon^{0\cdot25}v^{0\cdot75}p\mathrm{K}_{n+3}(\bar{r}),$$

$$a_{75} = -(\alpha_1\,\alpha_2)^2\mathrm{J}_n(\bar{s}) + (n+2)\left(\frac{v}{\varepsilon}\right)^{0\cdot5} q_2\,C_T\,p^2\mathrm{J}_{n+2}(\bar{s}) - (\alpha_1\,\alpha_2)^{0\cdot5}\left(\frac{v}{\varepsilon}\right)^{0\cdot75} C_T\,p^3\mathrm{J}_{n+3}(\bar{s}),$$

$$a_{76} = -(\alpha_1\,\alpha_2)^2 Y_n(\bar{s}) + (n+2)\left(\frac{v}{\varepsilon}\right)^{0\cdot5} q_2\,C_T\,p^2\mathrm{Y}_{n+2}(\bar{s}) - (\alpha_1\,\alpha_2)^{0\cdot5}\left(\frac{v}{\varepsilon}\right)^{0\cdot75} C_T\,p^3\mathrm{Y}_{n+3}(\bar{s}),$$

$$a_{77} = -(\alpha_1\,\alpha_2)^2\mathrm{I}_n(\bar{s}) + (n+2)\left(\frac{v}{\varepsilon}\right)^{0\cdot5} q_2\,C_T\,p^2\mathrm{I}_{n+2}(\bar{s}) + (\alpha_1\,\alpha_2)^{0\cdot5}\left(\frac{v}{\varepsilon}\right)^{0\cdot75} C_T\,p^3\mathrm{I}_{n+3}(\bar{s}),$$

$$a_{78} = -(\alpha_1\,\alpha_2)^2\mathrm{K}_n(\bar{s}) + (n+2)\left(\frac{v}{\varepsilon}\right)^{0\cdot5} q_2\,C_T\,p^2\mathrm{K}_{n+2}(\bar{s}) - (\alpha_1\,\alpha_2)^{0\cdot5}\left(\frac{v}{\varepsilon}\right)^{0\cdot75} C_T\,p^3\mathrm{K}_{n+3}(\bar{s}),$$

$$a_{85} = -(\alpha_1\,\alpha_2)^{0\cdot5}\mathrm{J}_{n+1}(\bar{s}) + \left(\frac{v}{\varepsilon}\right)^{0\cdot25} C_R\,p\mathrm{J}_{n+2}(\bar{s}),$$

$$a_{86} = -(\alpha_1\,\alpha_2)^{0\cdot5}\mathrm{Y}_{n+1}(\bar{s}) + \left(\frac{v}{\varepsilon}\right)^{0\cdot25} C_R\,p\mathrm{Y}_{n+2}(\bar{s}),$$

$$a_{87} = (\alpha_1\,\alpha_2)^{0\cdot5}\mathrm{I}_{n+1}(\bar{s}) + \left(\frac{v}{\varepsilon}\right)^{0\cdot25} C_R\,p\mathrm{I}_{n+2}(\bar{s}),$$

$$a_{88} = -(\alpha_1\,\alpha_2)^{0\cdot5}\mathrm{K}_{n+1}(\bar{s}) + \left(\frac{v}{\varepsilon}\right)^{0\cdot25} C_R\,p\mathrm{K}_{n+2}(\bar{s}),$$

where

$$r = 2p_a, \qquad s = 2p_a\,\alpha_1^{0\cdot5}, \qquad \bar{r} = 2p_b\,\alpha_1^{0\cdot5}, \qquad \bar{s} = 2p_b\,(\alpha_1\,\alpha_2)^{0\cdot5},$$

$$q_1 = \frac{(\alpha_1 - 1)}{\beta}, \qquad q_2 = \frac{(\alpha_1\,\alpha_2 - \alpha_1)}{(1-\beta)}.$$