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TRANSVERSE VIBRATION OF TRIANGULAR PLATE WITH ARBITRARY THICKNESS VARIATION AND VARIOUS BOUNDARY CONDITIONS

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The Rayleigh–Ritz method has been employed to obtain the numerical solution of the vibration problem of a triangular plate with arbitrary thickness variation and various boundary conditions at the three edges. The thickness has been approximated by a polynomial in natural co-ordinates which have been used everywhere as they greatly simplify the calculations. Successive approximations have been worked out until the first three frequencies and mode shapes converge to at least three significant figures. The results are tabulated for selected cases and are compared with known results for uniform and linear thickness variation. Three-dimensional mode shapes have been drawn using the tools for computer graphics.

1. INTRODUCTION

The triangular plates constitute an important part of engineering design. It is, therefore, necessary to know beforehand the frequencies and mode shapes of such plates under different conditions. This is the reason why a lot of information already exists in the literature about triangular plates of different shapes, sizes, thickness variations and the conditions at the edges. But our survey reveals that most of the work has been confined to plates of special shapes such as right-angled, isosceles or equilateral triangles. As far as boundary conditions are concerned, a very large number of papers discuss the cantilever plates only. Again most of the papers deal with plates of uniform thickness. The methods range from experimental to approximate and to purely numerical.

Only recently did a few papers appear on triangular plates of linearly varying thickness. Mirza and Bijlani [1] have given some results for cantilever plates with variable thickness. Singh and Saxena [2] have taken a general triangular plate with linearly varying thickness and with various combinations of clamped, simply-supported and free edges. It contains a wealth of information about triangular plates up to 1995. Extensive tables are given for comparison with the earlier results. Prior to this Singh and Chakraverty [3] studied the most general triangular plates of uniform thickness using boundary characteristic orthogonal polynomials in two variables. Such polynomials have

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interesting properties which remove numerical instability. In both references [2] and [3] the general triangle was first mapped into a standard right-angled triangle on which the solution is generated. The basis functions were so chosen that the essential boundary conditions are satisfied. Other papers which deal with triangular plates of uniform thickness are Gorman [4–8] who discusses right-angled triangular plates, Bhat [9] who studied polygonal plates in general but gives results for isosceles or right-angled plates, Kim and Dickinson [10] and Lam *et al*. [11] who gives some results for right-angled isotropic and orthotropic plates. A few more references related with triangular plates are Strand [12], Christensen [13], Gustafson *et al*. [14], Kuttler and Sigillito [15] and Cowper *et al*. [16]. References [2] and [3] give detailed comparisons of the results given by these authors. Two more papers have been brought to the notice of the authors. One is Liew [17] which deals with the response of plates of arbitrary shape subject to static loading. The other one is Liew *et al*. [18] which investigates the flexural vibration of triangular composite plates influenced by fibre orientation. Although these papers have no direct relevance to the present work, these may be of interest to those readers who wish to extend the present work to composite plates of variable thickness.

The basic aim of the present investigation is to study the problem in its most general form i.e., by (1) taking a general triangle, (2) taking an arbitrary thickness variation, and (3) using different combinations of boundary conditions at the three edges of the plate. It is an extension of reference [2] in two ways. First, the linear thickness variation considered in reference [2] has been generalized to a polynomial variation of arbitrary degree. Second, in place of the Cartesian co-ordinates the natural co-ordinates [19, 20] have been used. This not only simplifies the theoretical discussion but gives simple closed form expressions for the integrals involved. A single computer program, by choosing the parameters properly, gives results for virtually any plate.

A sufficiently large number of approximations have been worked out to ensure convergence. The basis functions are taken so that the essential boundary conditions are satisfied. Comparisons have been made with known results in special cases. Tables are given for frequencies and three-dimensional plots are given for mode shapes in some selected cases.

2. METHOD OF SOLUTION

Let the plate occupy the domain R of an xy-plane with vertices numbered 1, 2 and 3 and having co-ordinates as $(x_1, y_1) = (0, 0)$, $(x_2, y_2) = (a, 0)$ and $(x_3, y_3) = (b, c)$ as shown

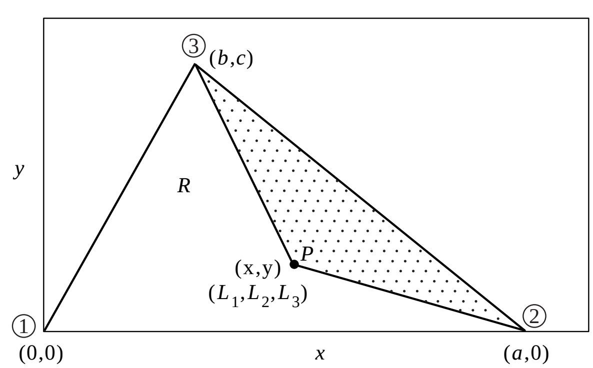

Figure 1. The triangular plate. Shaded area $= A_1$, area of $\triangle 123 = A$, $L_1 = A_1/A$.

in Figure 1. Thus, the numbers a, b and c determine the shape of the triangle completely. The Cartesian co-ordinates (x, y) and the natural co-ordinates (L_1, L_2, L_3) of a point P inside the triangle are related by:

$$x = \sum_{i=1}^{3} L_i x_i, \qquad y = \sum_{i=1}^{3} L_i y_i, \tag{1}$$

and

$$L_i = (\alpha_i + \beta_i x + \gamma_i y)/(2A) \tag{2}$$

where α_i, β_i and γ_i are constants and A is the area of the plate. All these can be expressed in terms of the co-ordinates of the vertices. Note that out of L_1, L_2 and L_3, only two are linearly independent since $L_1 + L_2 + L_3 = 1$. It is known that

$$\iint_R L_1^i L_2^j L_3^k \,\mathrm{d}x\,\mathrm{d}y = \frac{i!j!k!}{(i+j+k+2)!}(2A). \tag{3}$$

This helps in expressing integrals of polynomials in L_1, L_2 and L_3 in closed form.

Let the variable thickness of the plate be expressed as

$$h = ah_o f(L_1, L_2), \tag{4}$$

where h_o is non-dimensional thickness at some standard point and f is a non-dimensional function of L_1 and L_2. As we shall see later, the analysis becomes simpler if f is a polynomial. In fact f can always be approximated by a polynomial by measuring the thickness at a suitably chosen set of points and get the interpolating polynomial of the form

$$f \simeq f_M(L_1, L_2) = \sum_{i=1}^{M} a_i L_1^{m_i} L_2^{n_i}, \tag{5}$$

where M is the number of distinct sample points. The constants a_i will depend upon the location of the sample points and m_i, n_i are non-negative itegers. The first ten values of m_i and n_i are as follows:

i	1	2	3	4	5	6	7	8	9	10,
m_i	0	1	0	2	1	0	3	2	1	0,
n_i	0	0	1	0	1	2	0	1	2	3.

The cases of uniform, linear, quadratic and cubic thickness variations correspond to $M = 1$, 3, 6 and 10, respectively. The constants a_i can be found by measuring thickness at a selected set of points of the plate as already explained. Thus, the function $f_M(L_1, L_2)$ is completely known.

Now, the Rayleigh–Ritz method minimizes

$$\omega^2 = \frac{\iint_R D[[\nabla^2 W]^2 + 2(1-\nu)\,[W_{xy}^2 - W_{xx}\,W_{yy}]]\,\mathrm{d}x\,\mathrm{d}y}{\iint_R \rho h W^2\,\mathrm{d}x\,\mathrm{d}y}, \tag{6}$$

where

$$D = Eh^3/[12(1-\nu^2)] = \text{flexural rigidity},$$

with E, ρ, ν, ω and W as Young's modulus, density, Poisson's ratio, frequency and maximum displacement at (x, y), respectively. In the N-term approximation, we take the displacement of the form

$$W = \sum_{i=1}^{N} c_i\,\phi_i = L_1^{p_1}\,L_2^{p_2}\,L_3^{p_3} \sum_{i=1}^{N} c_i\,L_1^{m_i}\,L_2^{n_i}, \tag{7}$$

where c_i are constants and ϕ_i are the basis functions which are so chosen that the essential boundary conditions are satisfied. For this we take $p_i = 0$, 1 or 2 accordingly as the edge facing vertex i is free, simply-supported or clamped.

Substituting the expressions for h from equation (4) and W from equation (7) in equation (6), one gets after lengthy but straightforward calculations,

$$\sum_{j=1}^{N} (a_{ij} - \lambda^2 b_{ij})c_j = 0, \qquad i = 1, \ldots, N, \tag{8}$$

where

$$\lambda^2 = 12(1-\nu^2)\rho a^2 \omega^2/(Eh_o^2), \tag{9}$$

and a_{ij}, b_{ij} are given by

$$\begin{aligned} a_{ij} = \iint_R f^3[&A_1\,\phi_i^{11}\,\phi_j^{11} + A_2\,(\phi_i^{11}\,\phi_j^{22} + \phi_i^{22}\,\phi_j^{11}) + A_3\,(\phi_i^{11}\,\phi_j^{12} \\ &+ \phi_i^{12}\,\phi_j^{11}) + A_4\,\phi_i^{12}\,\phi_j^{12} + A_5\,(\phi_i^{22}\,\phi_j^{12} + \phi_i^{12}\,\phi_j^{22}) \\ &+ A_6\,\phi_i^{22}\,\phi_j^{22}]\,\mathrm{d}x\,\mathrm{d}y, \end{aligned} \tag{10}$$

$$b_{ij} = \iint_R f\phi_i\,\phi_j\,\mathrm{d}x\,\mathrm{d}y. \tag{11}$$

Here the superscripts 1 and 2 are used for differentiation with respect to L_1 and L_2, respectively. The coefficients A_1 through A_6 are given by

$$\begin{gathered} A_1 = K_1^2, \quad A_2 = K_1\,K_2 - K_3, \quad A_3 = -2K_1\,K_4, \quad A_4 = 4K_4^2 + 2K_3, \\ A_5 = -2K_2\,K_4, \quad A_6 = K_2^2, \end{gathered} \tag{12}$$

where

$$K_1 = 1 + [(\xi - 1)/\eta]^2, \quad K_2 = 1 + (\xi/\eta)^2, \quad K_3 = (1 - \nu)/\eta^2,$$
$$K_4 = K_2 - \xi/\eta^2,$$

with non-dimensional parameters ξ and η defined by

$$\xi = b/a, \qquad \eta = c/a. \tag{13}$$

Substituting expressions for f, ϕ_i and ϕ_j in equations (10) and (11), one gets a fairly lengthy expression which involves the variables L_1 and L_2. The choice of thickness and the shape functions as polynomials leads to expressions involving polynomials only. These could be integrated in closed form using equation (3). Thus, expressions for a_{ij} and b_{ij} are available in closed form and can be computed numerically.

System (8) is the standard generalized eigenvalue problem. It has been solved by the Generalized Jacobi method discussed in Wilkinson [21] and Bathe and Wilson [22]. This gives the frequency parameter λ. The associated mode shapes are known from the eigenvector

$$c = [c_1, c_2, \ldots, c_N]^T.$$

3. NUMERICAL WORK AND DISCUSSION

Due to the involvement of a very large number of parameters, it would be a gigantic task to make a detailed study of their effects on the frequencies and mode shapes. The authors, therefore, studied the problem only for a few selected cases. The parameters which have been varied are those which take care of boundary conditions, shape of the plate and thickness variation. The Poisson's ratio has been chosen to be 0·3 because it is this value for which most of the results are available in the literature. In some cases, however, results are given for other values also.

The order of approximation N has been varied from 1 to 36. This is sufficient for convergence of the first three frequencies up to at least three significant figures. All the computations are carried out in double precision arithmetic to avoid numerical instability. The results reported in the tables correspond to $N = 36$.

3.1. GEOMETRY OF THE PLATE

As already explained, this is controlled by parameters ξ and η. We have examined the plates of the following three shapes: (1) equilateral triangle, $\xi = 0{\cdot}5$, $\eta = \sqrt{3}/2$ (Figure 2(a)); (2) isosceles right-angled triangle, $\xi = 0{\cdot}0$, $\eta = 1{\cdot}0$ (Figure 2(b)); and (3) obtuse isosceles triangle with angles 30, 30 and 120°, $\xi = -0{\cdot}5$, $\eta = \sqrt{3}/2$ (Figure 2(c)).

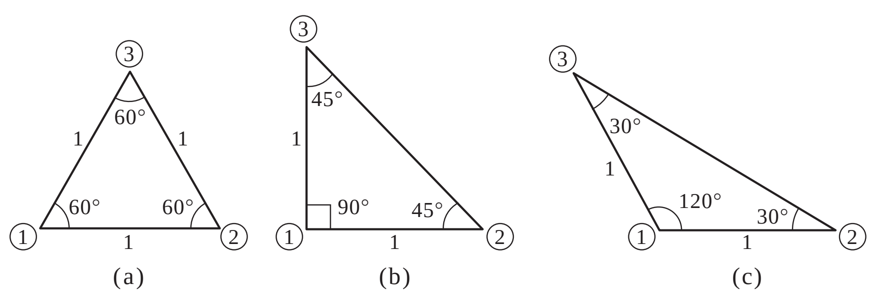

Figure 2. (a) Equilateral triangle. (b) Isosceles right-angled triangle. (c) Obtuse isosceles triangle with angles 30, 30 and 120°.

3.2. BOUNDARY CONDITIONS

All combinations of clamped, simply-supported and free boundary have been considered. Thus, p_1, p_2 and p_3 are given the values 0, 1 or 2. This leads to 27 different combinations in general for a given thickness variation. However, symmetry in shape may reduce this number.

3.3. THICKNESS VARIATION

As already mentioned, M parameters a_1 through a_M control the thickness variation. Numerical work has been carried out for $M = 1, 3, 6$ and 10 which corresponds to uniform, linear, quadratic and cubic variations, respectively. Thus, up to ten parameters can be suitably varied and a large variety of thickness variations can be examined. The following special cases have been investigated in detail.

3.3.1. *Uniform thickness variation*

In this case $M = 1$ and $f = 1$ (Figure 3(a)). This case has been discussed extensively in the literature with a variety of shapes and boundary conditions at the edges. Complete up-to-date information about this case is available in reference [2]. The results for uniform thickness follow, as a special case, from those of quadratic variation which are reported in Table 2. In this case the results agree completely with reference [2] at $N = 28$ and the comparison will not be duplicated here.

3.3.2. *Linear thickness variation*

Here, $M = 3$. In particular, the authors have considered

$$f = \beta' + (1 - \beta')L_1 + (\alpha' - \beta')L_2. \qquad \text{(Figure 3}(b)). \tag{14}$$

This amounts to assigning values 1, α' and β' to f at vertices 1, 2 and 3, respectively. Similar studies have been made in reference [2]. Reference [1] also gives some results in this case.

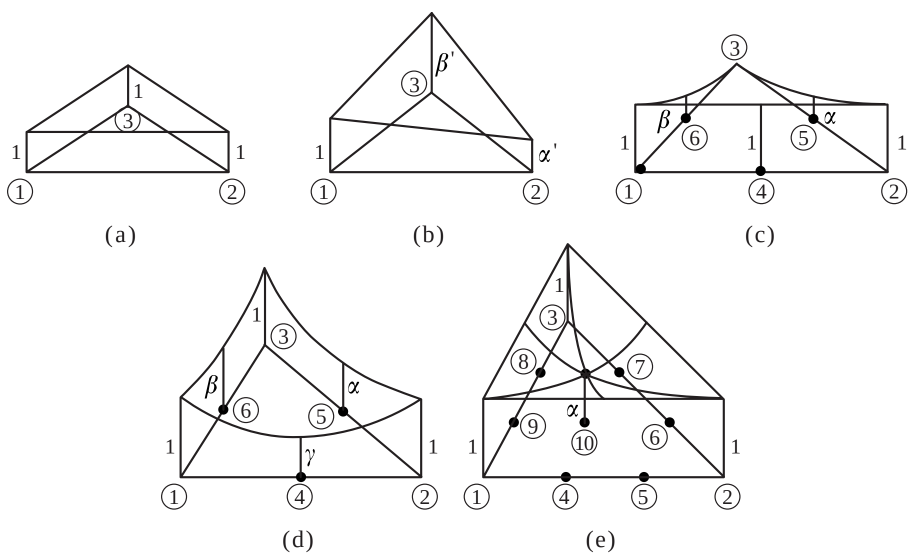

Figure 3. (a) Triangular plate with uniform thickness variation. (b) Triangular plate with linear thickness variation. (c, d) Triangular plate with quadratic thickness variaton. (e) Triangular plate with cubic thickness variation.

TABLE 1

First three frequency parameters for quadratic thickness variation (Figure 3(c))

1 2 3	α	β	$\xi = 0{\cdot}5,\ \eta = \sqrt{3}/2$ λ_1	λ_2	λ_3	$\xi = 0{\cdot}0,\ \eta = 1{\cdot}0$ λ_1	λ_2	λ_3	$\xi = -0{\cdot}5,\ \eta = \sqrt{3}/2$ λ_1	λ_2	λ_3
C	0·3	0·3	38·892	53·674	78·807	40·928	51·869	66·455	66·768	79·049	93·454
C	0·3	0·5	50·032	75·462	108·321	50·470	71·419	95·913	78·793	103·133	129·598
C	0·5	0·3	50·032	75·462	108·321	52·070	71·512	93·342	83·818	105·549	128·161
	0·5	0·5	60·435	96·573	130·658	60·810	89·935	120·402	94·703	127·745	161·188
C	0·3	0·3	35·306	49·317	74·480	38·055	48·556	63·214	62·402	74·569	88·537
C	0·3	0·5	43·528	68·029	100·387	44·834	65·286	89·507	70·201	94·646	120·332
S	0·5	0·3	43·528	68·030	100·382	46·747	65·684	86·923	75·889	97·702	120·019
	0·5	0·5	51·515	86·302	119·026	53·136	81·466	111·039	83·241	116·165	148·177
C	0·3	0·3	25·706	39·029	56·711	28·199	40·620	51·939	39·870	63·805	75·475
C	0·3	0·5	28·137	49·723	74·764	29·750	49·529	69·750	41·252	72·120	92·994
F	0·5	0·3	28·137	49·723	74·763	31·204	50·997	69·233	44·826	77·334	95·819
	0·5	0·5	30·540	60·139	79·884	32·746	59·162	82·471	46·312	84·292	109·162
C	0·3	0·3	32·078	46·211	68·976	31·558	42·485	57·933	47·305	59·739	76·312
S	0·3	0·5	39·737	63·918	95·457	38·486	58·251	82·673	57·234	79·941	106·793
C	0·5	0·3	42·577	66·161	94·690	42·243	60·930	81·925	64·094	85·666	107·528
	0·5	0·5	49·615	82·891	113·378	48·556	75·532	104·121	73·155	103·995	135·754
C	0·3	0·3	28·741	42·034	64·779	29·124	39·435	54·764	44·566	56·151	72·226
S	0·3	0·5	34·075	56·968	88·415	33·990	52·720	76·733	51·730	72·856	98·509
S	0·5	0·3	36·360	59·071	86·907	37·456	55·423	75·824	58·252	78·642	99·786
	0·5	0·5	41·581	73·360	102·650	42·055	67·783	95·331	64·852	93·754	124·248
C	0·3	0·3	16·898	31·566	47·796	19·819	31·824	43·618	31·863	46·018	57·647
S	0·3	0·5	17·781	39·027	58·052	20·961	38·602	57·564	33·606	53·555	73·349
F	0·5	0·3	18·795	41·224	59·295	22·677	41·747	58·788	37·095	59·477	77·663
	0·5	0·5	19·782	48·516	62·843	23·852	47·807	68·707	38·773	65·910	90·272
C	0·3	0·3	16·146	26·105	42·823	13·636	21·602	34·283	16·233	24·585	36·689
F	0·3	0·5	17·959	35·525	53·422	15·939	30·925	51·108	19·423	35·658	57·461
C	0·5	0·3	22·356	38·209	56·915	19·931	33·031	48·475	25·429	40·072	57·465
	0·5	0·5	22·850	44·363	65·766	20·990	39·076	61·379	27·126	47·058	70·990

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TABLE 1—*continued*

1 2 3	α	β	$\xi = 0{\cdot}5,\ \eta = \sqrt{3}/2$			$\xi = 0{\cdot}0,\ \eta = 1{\cdot}0$			$\xi = -0{\cdot}5,\ \eta = \sqrt{3}/2$		
			λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
C	0·3	0·3	10·842	20·887	36·429	11·150	18·553	30·890	15·315	23·031	34·625
F	0·3	0·5	11·010	27·825	45·991	12·225	25·974	45·555	17·797	32·318	51·893
S	0·5	0·3	14·066	30·747	48·080	15·507	28·259	43·111	23·201	36·593	52·569
	0·5	0·5	14·191	35·356	55·151	16·186	33·117	54·418	24·611	42·467	63·952
C	0·3	0·3	3·859	13·437	23·723	4·823	12·895	21·052	7·789	16·423	24·423
F	0·3	0·5	3·826	14·292	24·922	4·834	14·723	28·141	7·941	19·529	33·416
F	0·5	0·3	4·838	17·859	26·386	6·439	18·040	29·353	11·134	24·435	36·634
	0·5	0·5	4·892	18·360	27·847	6·565	19·038	33·322	11·508	26·158	41·888
S	0·3	0·3	32·078	46·211	68·976	33·059	43·058	56·861	53·352	64·227	77·024
C	0·3	0·5	42·577	66·161	94·690	42·241	61·497	83·061	65·118	87·415	110·327
C	0·5	0·3	39·737	63·918	95·457	39·910	57·058	78·168	63·009	81·182	100·885
	0·5	0·5	49·615	82·891	113·378	48·565	74·676	102·016	74·136	103·282	132·528
S	0·3	0·3	28·741	42·034	64·781	30·407	39·949	53·733	49·334	60·063	72·456
C	0·3	0·5	36·360	59·071	86·906	36·791	55·622	76·837	56·766	79·203	101·631
S	0·5	0·3	34·075	56·968	88·415	35·471	51·868	72·604	56·593	74·498	93·664
	0·5	0·5	41·581	73·360	102·650	41·698	66·952	93·392	63·937	92·804	121·027
S	0·3	0·3	16·898	31·566	47·797	17·395	31·660	41·656	21·524	47·668	57·606
C	0·3	0·5	18·795	41·224	59·295	18·551	39·055	55·380	22·618	52·752	70·780
F	0·5	0·3	17·781	39·027	58·052	18·235	37·647	52·734	22·660	53·532	68·354
	0·5	0·5	19·782	48·516	62·843	19·470	44·620	62·666	23·826	58·021	79·909
S	0·3	0·3	26·539	39·588	58·278	25·326	35·144	48·104	36·914	47·848	62·160
S	0·3	0·5	33·793	55·610	83·114	31·994	49·716	69·927	46·578	66·838	88·869
C	0·5	0·3	33·793	55·610	83·114	31·993	48·317	67·211	46·736	64·653	84·144
	0·5	0·5	40·577	71·196	97·765	38·275	62·282	87·771	55·935	82·768	110·309
S	0·3	0·3	23·423	35·587	53·968	23·079	32·251	44·837	34·448	44·519	57·840
S	0·3	0·5	28·365	49·008	75·914	27·634	44·392	64·017	41·226	59·958	80·551
S	0·5	0·3	28·365	49·008	75·913	27·990	43·409	61·694	42·067	58·720	77·196
	0·5	0·5	33·324	62·330	87·877	32·472	55·216	79·712	48·627	73·565	99·492

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TABLE 1—*continued*

1 2 3	α	β	$\xi = 0{\cdot}5,\ \eta = \sqrt{3}/2$			$\xi = 0{\cdot}0,\ \eta = 1{\cdot}0$			$\xi = -0{\cdot}5,\ \eta = \sqrt{3}/2$		
			λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
S	0·3	0·3	9·822	25·698	40·026	11·513	24·396	33·895	17·617	33·098	42·805
S	0·3	0·5	10·568	32·401	44·885	12·405	29·735	44·219	18·766	38·438	54·683
F	0·5	0·3	10·568	32·401	44·885	12·275	30·099	43·499	18·615	39·551	54·213
	0·5	0·5	11·419	39·235	47·627	13·245	35·085	51·134	19·824	44·387	64·344
S	0·3	0·3	13·749	22·137	35·076	11·027	17·485	27·157	12·186	18·609	27·848
F	0·3	0·5	15·586	30·635	45·985	13·486	26·121	42·083	15·939	29·528	46·288
C	0·5	0·3	17·945	32·566	49·428	14·474	25·630	39·368	15·921	27·144	40·577
	0·5	0·5	18·654	38·412	55·532	15·943	31·861	50·879	18·494	35·252	55·107
S	0·3	0·3	8·862	17·506	29·426	8·700	14·716	23·946	11·339	17·074	25·721
F	0·3	0·5	9·010	23·681	38·780	9·835	21·495	36·701	14·197	26·030	40·670
S	0·5	0·3	10·680	25·438	41·137	10·773	21·331	34·545	14·317	24·295	36·428
	0·5	0·5	10·791	29·918	46·045	11·620	26·381	44·466	16·370	31·089	48·477
S	0·3	0·3	10·630	15·440	19·839	8·832	14·748	20·302	10·339	15·820	24·169
F	0·3	0·5	10·914	16·792	27·066	9·860	19·605	26·117	12·323	22·945	34·599
F	0·5	0·3	13·538	17·039	28·642	11·322	19·456	25·953	13·026	21·692	31·843
	0·5	0·5	13·647	18·568	34·371	12·023	22·627	31·687	14·488	26·714	40·217
F	0·3	0·3	16·146	26·105	42·823	14·574	20·461	29·699	21·736	27·373	34·252
C	0·3	0·5	22·356	38·209	56·915	20·047	32·155	45·682	28·489	41·861	55·477
C	0·5	0·3	17·959	35·525	53·422	14·853	25·063	39·233	20·678	29·165	40·792
	0·5	0·5	22·850	44·363	65·766	19·591	34·355	50·924	27·086	42·172	58·004
F	0·3	0·3	10·842	20·887	36·428	9·875	16·774	24·866	13·331	23·001	28·965
C	0·3	0·5	14·066	30·747	48·080	12·044	25·435	37·979	15·092	31·869	44·411
S	0·5	0·3	11·010	27·825	45·991	9·432	19·482	32·173	12·677	23·078	32·726
	0·5	0·5	14·191	35·356	55·151	11·762	26·879	41·851	14·673	31·915	46·025
F	0·3	0·3	3·859	13·437	23·723	3·582	11·787	18·033	3·810	14·857	22·722
C	0·3	0·5	4·838	17·859	26·386	4·146	14·812	25·134	4·211	16·655	28·467
F	0·5	0·3	3·826	14·292	24·922	3·591	11·608	21·489	3·924	14·468	22·945
	0·5	0·5	4·892	18·360	27·847	4·213	14·781	26·140	4·353	16·677	28·624

continued overleaf

TABLE 1—*continued*

1 2 3	α	β	$\xi = 0{\cdot}5,\ \eta = \sqrt{3}/2$ λ_1	λ_2	λ_3	$\xi = 0{\cdot}0,\ \eta = 1{\cdot}0$ λ_1	λ_2	λ_3	$\xi = -0{\cdot}5,\ \eta = \sqrt{3}/2$ λ_1	λ_2	λ_3
F	0·3	0·3	13·749	22·137	35·076	11·127	16·483	24·119	13·322	18·602	25·425
S	0·3	0·5	17·945	32·566	49·428	14·589	25·532	37·682	17·385	29·365	41·519
C	0·5	0·3	15·586	30·635	45·985	11·833	21·010	32·554	13·430	21·777	31·716
	0·5	0·5	18·654	38·412	55·532	14·383	27·618	42·618	16·575	29·791	43·741
F	0·3	0·3	8·862	17·506	29·426	7·300	13·062	19·806	8·828	14·569	20·193
S	0·3	0·5	10·680	25·438	41·137	8·313	19·224	30·449	9·407	20·522	31·530
S	0·5	0·3	9·010	23·681	38·780	6·928	15·894	26·230	8·036	15·929	24·461
	0·5	0·5	10·791	29·918	46·045	7·956	20·720	34·179	8·701	20·861	33·172
F	0·3	0·3	10·630	15·440	19·839	8·919	13·940	17·790	8·674	14·612	20·466
S	0·3	0·5	13·538	17·039	28·642	10·493	18·631	22·034	9·385	20·137	28·477
F	0·5	0·3	10·914	16·792	27·066	8·944	16·738	19·180	8·568	16·062	23·585
	0·5	0·5	13·647	18·568	34·371	10·565	19·320	23·954	9·432	20·540	29·284
F	0·3	0·3	9·057	15·198	21·722	6·253	10·349	16·450	5·443	9·099	15·112
F	0·3	0·5	10·126	21·554	26·098	7·215	15·523	23·053	6·475	14·388	23·582
C	0·5	0·3	10·126	21·554	26·098	6·770	13·678	20·663	5·816	11·504	18·972
	0·5	0·5	10·529	25·427	28·341	7·286	16·971	24·313	6·500	15·190	24·876
	Reference [1]					7·2571	16·902	24·580	6·5534	15·337	25·158
F	0·3	0·3	11·031	17·266	19·899	7·390	12·553	16·883	6·203	10·631	17·703
F	0·3	0·5	15·188	18·645	29·777	10·328	17·450	22·850	8·708	16·115	25·119
S	0·5	0·3	15·188	18·645	29·777	9·298	15·887	20·235	7·263	13·239	21·006
	0·5	0·5	18·093	19·406	37·107	11·290	18·597	25·402	9·105	17·304	27·649
F	0·3	0·3	11·748	21·072	21·815	7·844	14·346	17·545	6·523	11·478	17·537
F	0·3	0·5	16·765	23·375	25·321	11·554	18·235	23·581	9·343	15·147	23·022
F	0·5	0·3	16·765	23·375	25·321	10·218	17·871	21·532	7·780	13·588	20·118
	0·5	0·5	20·465	26·100	26·802	12·962	20·111	27·108	9·949	16·548	25·313

A comparison of the results of references [1] and [2] can be found in reference [2]. Calculations for various values of α' and β' have been carried out. The results agree completely with reference [2] in all the cases at $N = 28$. So to avoid duplication of results, this comparison has been omitted here. In Table 1, results are given for quadratic variation. Some of these reduce to linear variation in special cases.

3.3.3. *Quadratic thickness variation*

Here, $M = 6$. So six parameters are at our disposal. The following special quadratic variations have been examined in detail:

$$(1)\ f = (4\beta - 1)L_1 + (4\alpha - 1)L_2 + (2 - 4\beta)L_1^2 + 4(1 - \alpha - \beta)L_1 L_2 + (2 - 4\alpha)L_2^2 \quad \text{(Figure 3(c))}. \quad (15)$$

The results for this case are given in Table 1. If $\alpha = \beta = 0{\cdot}5$, this reduces to linear variation given above with $\alpha' = 1$ and $\beta' = 0$.

$$(2)\ f = 1 - 4(1 - \beta)L_1 - 4(1 - \alpha)L_2 + 4(1 - \beta)L_1^2 + 4(1 - \alpha - \beta + \gamma)L_1 L_2 + 4(1 - \alpha)L_2^2 \quad \text{(Figure 3(d))}. \quad (16)$$

The results are given in Table 2. If $\alpha = \beta = \gamma = 1$, this reduces to uniform thickness. These choices have been made by assigning appropriate values to f at vertices and mid-points of sides. Comparison has also been made with references [3–5, 7–16] for the case of uniform thickness.

3.3.4. *Cubic thickness variation*

In this case $M = 10$ and so one has the freedom of choosing ten parameters. Numerical results for the following special case have been obtained

$$f = 1 + 27(\alpha - 1)L_1 L_2 (1 - L_1 - L_2) = 1 + 27(\alpha - 1)L_1 L_2 L_3 \quad \text{(Figure 3(e))}. \quad (17)$$

The results are given in Table 3. This corresponds to $f = 1$ on the boundary and $f = \alpha$ at the centroid of the triangle. Not that $\alpha = 1$ corresponds to uniform thickness. In Table 3 results are given for $\alpha = 0{\cdot}5$, $1{\cdot}5$, for the three types of triangle and all combinations of boundary conditions. The results for uniform thickness are not given because these are the same as in Table 2.

A large number of approximations (up to $N = 36$) have been worked out to ensure convergence of results up to at least three significant figures in all cases. This is clearly illustrated in Table 4 in which results for fundamental frequency parameters are reported for various values of N for the set of parameters specified in the table. An interesting special case arises for the isosceles right-angled triangle of uniform thickness when all sides are simply-supported. As reported by Gorman [8], the exact values of the first three frequency parameters in this case are $5\pi^2$, $10\pi^2$ and $13\pi^2$ giving 49·348, 98·696 and 128·305, respectively. The corresponding values obtained here for $N = 21$, 28, 35 and 36 are as follows:

N	λ_1	λ_2	λ_3
21	49·359	98·948	130·296
28	49·348	98·833	128·624
35	49·348	98·832	128·433
36	49·348	98·700	128·433

It is clear that the first frequency parameter has converged to all the five significant figures—the second differs from the exact value only in the last figure by 4. So it can be

TABLE 2

*First three frequency parameters for quadratic thickness variation (Figure 3(d)) (U for upper bound, L for lower bound, * for $v = 0.333$)*

1 2 3	α	β	γ	$\xi = 0.5, \eta = \sqrt{3}/2$ λ_1	λ_2	λ_3	$\xi = 0.0, \eta = 1.0$ λ_1	λ_2	λ_3	$\xi = -0.5, \eta = \sqrt{3}/2$ λ_1	λ_2	λ_3
C	0·5	0·5	1·0	62·674	117·079	119·717	59·123	97·150	124·938	89·713	129·477	174·550
C	0·5	1·0	0·5	62·674	117·079	119·717	59·123	97·150	124·938	89·713	129·477	174·550
C	1·0	0·5	0·5	62·674	117·079	119·717	62·902	100·326	125·751	100·814	138·526	178·484
	0·5	0·5	0·5	46·972	87·027	87·027	45·878	72·436	93·672	72·814	100·242	134·550
	1·0	1·0	1·0	99·020	189·007	189·007	93·790	157·789	194·822	140·172	207·830	272·217
	Reference [3]			99·022	189·05	189·22	93·800					
	Reference [10]						93·79	157·8	194·8			
	Reference [5]						93·86	157·7	194·8			
	Reference [15]			99·044	189·116	U	94·155	U				
	Reference [15]			98·989	188·892	L	93·404	L				
C	0·5	0·5	1·0	49·831	98·870	106·681	49·182	84·245	113·634	75·947	113·513	155·389
C	0·5	0·1	0·5	26·359	47·216	53·652	29·712	42·350	62·390	50·685	64·029	89·483
S	1·0	0·5	0·5	52·847	103·483	105·225	54·022	89·371	113·092	85·488	123·219	161·518
	0·5	0·5	0·5	38·928	74·955	77·915	38·969	64·121	84·531	60·630	88·514	121·264
	1·0	1·0	1·0	81·601	164·989	165·319	78·893	138·848	173·738	118·670	183·025	243·064
	Reference [3]			81·604	165·12	165·52	78·89					
	Reference [10]						78·89	138·8	173·7			
	Reference [5]						78·91	138·9	176·7			
C	0·5	0·5	1·0	27·576	60·529	72·042	29·479	55·667	79·065	41·333	76·664	103·917
C	0·5	0·1	0·5	19·007	33·224	44·178	22·099	33·655	49·230	31·558	52·407	66·744
F	1·0	0·5	0·5	29·928	65·554	67·048	32·678	60·857	75·294	47·461	83·501	109·105
	0·5	0·5	0·5	23·610	48·403	53·809	25·009	44·838	59·406	34·630	61·799	82·246
	1·0	1·0	1·0	40·016	95·827	101·791	41·111	86·698	109·028	58·484	111·128	155·072
	Reference [3]			40·022	95·891	101·85	41·12					
	Reference [10]						41·11	86·7	109·0			
C	0·5	0·5	1·0	52·847	103·483	105·225	50·169	86·393	111·133	74·107	113·762	159·644
S	0·5	1·0	0·5	49·831	98·870	106·681	49·182	84·245	113·634	75·947	113·513	155·399
C	1·0	0·5	0·5	52·847	103·483	105·225	54·022	89·371	113·092	85·488	123·219	161·518

continued overleaf

TABLE 2—*continued*

1 2 3	α	β	γ	$\xi = 0{\cdot}5,\ \eta = \sqrt{3}/2$ λ_1	λ_2	λ_3	$\xi = 0{\cdot}0,\ \eta = 1{\cdot}0$ λ_1	λ_2	λ_3	$\xi = -0{\cdot}5,\ \eta = \sqrt{3}/2$ λ_1	λ_2	λ_3
	0·5	0·5	0·5	38·928	74·955	77·915	38·969	64·121	84·531	60·630	88·516	121·268
	1·0	1·0	1·0	81·601	164·989	165·319	78·893	138·848	173·738	118·670	183·025	243·064
	Reference [3]			81·604	165·12	165·52	78·89					
	Reference [10]						78·89	138·8	173·7			
	Reference [5]						78·91	138·9	176·7			
C	0·5	0·5	1·0	41·233	86·490	92·974	40·926	74·232	100·368	62·152	97·973	141·738
S	0·5	1·0	0·5	41·233	86·490	92·974	40·926	74·232	100·368	62·152	97·973	141·738
S	1·0	0·5	0·5	43·931	90·950	91·612	45·583	78·727	101·097	72·018	107·039	146·425
	0·5	0·5	0·5	31·735	64·892	67·727	32·290	55·921	75·357	49·739	75·574	109·944
	1·0	1·0	1·0	66·177	142·740	143·475	65·791	121·085	154·608	100·766	159·275	219·544
	Reference [3]			66·180	142·96	143·73	65·81					
	Reference [10]						65·79	121·1	154·5			
	Reference [5]						65·79	121·1	154·5			
C	0·5	0·5	1·0	18·072	49·543	57·317	21·542	47·256	66·490	34·559	61·864	92·312
S	0·5	0·1	0·5	11·394	29·024	33·968	13·696	30·391	41·185	22·326	40·943	59·451
F	1·0	0·5	0·5	18·775	49·725	56·531	23·461	51·060	64·286	39·771	68·697	97·007
	0·5	0·5	0·5	13·707	38·549	41·503	16·566	37·524	49·006	27·312	48·446	73·132
	1·0	1·0	1·0	26·561	75·314	84·350	31·783	72·801	94·375	51·280	95·310	136·363
	Reference [3]			26·565	75·36	84·432	31·78					
	Reference [10]						31·78	72·81	94·36			
C	0·5	0·5	1·0	29·928	65·554	67·048	29·166	57·618	70·924	38·918	72·597	101·479
F	0·5	1·0	0·5	27·576	60·528	72·042	29·479	55·668	79·067	41·333	76·664	103·909
C	1·0	0·5	0·5	29·928	65·554	67·048	32·678	60·857	75·294	47·461	83·501	109·105
	0·5	0·5	0·5	23·610	48·403	53·809	25·009	44·838	59·405	34·630	61·805	82·252
	1·0	1·0	1·0	40·016	95·827	101·791	41·111	86·698	109·028	58·484	111·128	155·072
	Reference [3]			40·022	95·891	101·85	41·12					
	Reference [10]						41·11	86·7	109·0			
C	0·5	0·5	1·0	17·717	48·758	55·183	20·616	46·782	61·220	32·588	59·920	89·623
F	0·5	1·0	0·5	18·072	49·543	57·317	21·542	47·256	66·490	34·559	61·864	92·312

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TABLE 2—*continued*

1 2 3	α	β	γ	$\xi = 0{\cdot}5,\ \eta = \sqrt{3}/2$ λ_1	λ_2	λ_3	$\xi = 0{\cdot}0,\ \eta = 1{\cdot}0$ λ_1	λ_2	λ_3	$\xi = -0{\cdot}5,\ \eta = \sqrt{3}/2$ λ_1	λ_2	λ_3
S	0·1	0·5	0·5	10·275	26·800	32·905	10·887	26·152	35·315	15·842	32·052	53·767
	0·5	0·5	0·5	13·707	38·549	41·503	16·566	37·524	49·006	27·312	48·446	73·132
	1·0	1·0	1·0	26·561	75·314	84·350	31·783	72·801	94·375	51·280	95·310	136·363
	Reference [3]			26·565	75·36	84·432	31·78					
	Reference [10]						31·78	72·81	94·36			
C	0·5	0·5	1·0	5·247	23·171	25·440	7·065	23·389	34·025	12·118	31·240	53·009
F	0·5	1·0	0·5	5·247	23·171	25·440	7·065	23·389	34·025	12·118	31·240	53·009
F	1·0	0·5	0·5	6·032	24·429	24·470	8·548	25·495	34·613	15·603	36·636	57·969
	0·5	0·5	0·5	3·872	17·776	17·943	5·212	18·074	24·587	8·885	24·575	42·015
	1·0	1·0	1·0	8·921	35·095	38·484	12·645	35·935	52·933	23·077	50·277	79·644
	Reference [3]			8·922	35·155	38·503	12·64					
	Reference [9]			8·922	35·132	38·505						
	Reference [10]						12·65	35·94	52·93			
S	0·5	0·5	1·0	52·847	103·483	105·225	47·061	81·764	106·523	68·575	104·202	146·765
C	0·5	1·0	0·5	52·847	103·483	105·225	47·061	81·764	106·523	68·575	104·202	146·765
C	1·0	0·5	0·5	49·831	98·870	106·681	44·819	81·857	101·948	66·414	102·871	139·882
	0·5	0·5	0·5	38·928	74·955	77·915	34·662	59·984	77·588	51·723	77·049	109·883
	1·0	1·0	1·0	81·601	164·989	165·319	73·395	131·582	165·049	105·263	165·723	223·536
	Reference [3]			81·604	165·12	165·52	73·4					
	Reference [10]						73·39	131·6	165·1			
	Reference [5]						73·39	131·7	165·0			
	Reference [11]						73·388	132·95	168·89			
S	0·5	0·5	1·0	41·233	86·490	92·974	38·462	69·913	96·232	57·277	89·671	129·711
C	0·5	1·0	0·5	43·931	90·950	91·612	39·342	72·274	93·900	55·384	91·015	132·666
S	1·0	0·5	0·5	41·233	86·490	92·974	37·824	72·163	90·455	55·195	90·280	125·288
	0·5	0·5	0·5	31·735	64·892	67·727	29·060	52·530	69·177	42·481	67·298	98·367
	1·0	1·0	1·0	66·177	142·740	143·475	60·538	114·565	145·833	87·101	143·808	198·311
	Reference [3]			66·18	142·96	143·73	60·54					
	Reference [10]						60·54	114·6	145·9			
	Reference [5]						60·57	114·7	145·8			

continued overleaf

TABLE 2—*continued*

1 2 3	α	β	γ	$\xi = 0·5,\ \eta = \sqrt{3}/2$			$\xi = 0·0,\ \eta = 1·0$			$\xi = -0·5,\ \eta = \sqrt{3}/2$		
				λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
S	0·5	0·5	1·0	18·072	49·543	57·317	17·766	41·264	60·789	21·774	52·161	75·094
C	0·5	1·0	0·5	18·775	49·725	56·531	16·196	40·515	56·026	18·382	45·273	73·572
F	1·0	0·5	0·5	17·717	48·758	55·183	16·577	42·368	54·560	20·252	50·374	72·635
	0·5	0·5	0·5	13·707	38·549	41·503	12·945	31·651	43·325	15·386	38·404	56·115
	1·0	1·0	1·0	26·561	75·314	84·350	23·920	62·721	85·744	28·517	71·483	112·432
	Reference [3]			26·565	75·36	84·432	23·93					
	Reference [10]						23·92	62·73	85·77			
S	0·5	0·5	1·0	43·931	90·950	91·612	39·342	72·274	93·900	55·384	91·015	132·666
S	0·5	1·0	0·5	41·233	86·490	92·974	38·462	69·913	96·232	57·277	89·671	129·711
C	1·0	0·5	0·5	41·233	86·490	92·974	37·824	72·163	90·455	55·195	90·280	125·288
	0·5	0·5	0·5	31·735	64·892	67·727	29·060	52·530	69·177	42·481	67·298	98·367
	1·0	1·0	1·0	66·177	142·740	143·475	60·538	114·565	145·833	87·101	143·808	198·311
	Reference [3]			66·18	142·96	143·73	60·54					
	Reference [10]						60·54	114·6	145·9			
	Reference [5]						60·57	114·7	145·8			
S	0·5	0·5	1·0	33·601	74·820	80·294	31·463	60·929	84·085	45·631	76·620	117·160
S	0·5	1·0	0·5	33·601	74·820	80·294	31·463	60·929	84·085	45·631	76·620	117·160
S	1·0	0·5	0·5	33·601	74·820	80·294	31·247	62·962	79·597	45·112	77·271	112·912
	0·5	0·5	0·5	25·483	56·823	56·823	23·729	45·388	60·834	34·110	56·634	89·063
	1·0	1·0	1·0	52·638	122·823	122·823	49·348	98·700	128·433	72·121	123·225	176·386
	Reference [3]			52·638	122·91	124·11	49·36					
	Reference [10]						49·35	98·76	128·4			
	Reference [4]						49·35	98·70	128·3			
	Reference [11]						49·323	98·395	128·55			
	Reference [8]						49·348	98·696	128·305			
S	0·5	0·5	1·0	10·650	40·720	43·372	12·124	33·832	50·615	18·052	40·459	64·640
S	0·5	1·0	0·5	10·518	34·600	45·603	11·139	30·601	47·957	15·543	35·905	61·223
F	1·0	0·5	0·5	10·518	34·600	45·603	11·025	33·711	45·605	16·045	39·718	62·287
	0·5	0·5	0·5	7·283	27·325	32·530	7·958	24·457	35·508	12·003	28·069	47·686
	1·0	1·0	1·0	16·092	57·630	68·330	17·316	51·036	72·982	24·327	59·999	95·398

continued overleaf

TABLE 2—*continued*

1 2 3	α	β	γ	$\xi = 0{\cdot}5,\ \eta = \sqrt{3}/2$ λ_1	λ_2	λ_3	$\xi = 0{\cdot}0,\ \eta = 1{\cdot}0$ λ_1	λ_2	λ_3	$\xi = -0{\cdot}5,\ \eta = \sqrt{3}/2$ λ_1	λ_2	λ_3
	Reference [3]			16·092	57·709	68·593	17·31					
	Reference [10]						17·32	51·04	73·00			
S	0·5	0·5	1·0	18·775	49·725	56·531	16·196	40·515	56·026	18·382	45·274	73·573
F	0·5	1·0	0·5	18·072	49·543	57·317	17·766	41·264	60·789	21·774	52·161	75·093
C	1·0	0·5	0·5	17·717	48·758	55·183	16·577	42·368	54·560	20·252	50·374	72·635
	0·5	0·5	0·5	13·707	38·549	41·503	12·945	31·651	43·325	15·386	38·404	56·115
	1·0	1·0	1·0	26·561	75·314	84·350	23·920	62·721	85·744	28·517	71·483	112·432
	Reference [3]			26·565	75·36	84·432	23·93					
	Reference [10]						23·92	62·73	85·77			
S	0·5	0·5	1·0	10·518	34·600	45·603	11·139	30·601	47·957	15·543	35·905	61·223
F	0·5	1·0	0·5	10·650	40·720	43·372	12·124	33·832	50·615	18·052	40·459	64·640
S	1·0	0·5	0·5	10·518	34·600	45·603	11·025	33·711	45·605	16·045	39·718	62·287
	0·5	0·5	0·5	7·283	27·325	32·530	7·958	24·457	35·508	12·003	28·069	47·686
	1·0	1·0	1·0	16·092	57·630	68·330	17·316	51·036	72·982	24·327	59·999	95·398
	Reference [3]			16·092	57·709	68·593	17·31					
	Reference [10]						17·32	51·04	73·00			
S	0·5	0·5	1·0	13·682	17·267	41·462	11·715	22·131	36·283	13·538	27·939	48·668
F	0·5	1·0	0·5	13·682	17·267	41·462	11·715	22·131	36·283	13·538	27·939	48·668
F	1·0	0·5	0·5	13·428	16·989	41·784	14·285	18·400	40·490	16·244	27·550	47·943
	0·5	0·5	0·5	10·220	10·411	32·754	8·417	14·053	28·135	9·679	20·228	35·149
	1·0	1·0	1·0	22·646	26·659	69·418	19·596	34·802	61·107	22·646	43·965	73·514
	Reference [3]			22·666	26·717	71·033	19·61					
	Reference [10]						19·60	34·8	61·62			
F	0·5	0·5	1·0	29·928	65·554	67·048	21·207	42·833	60·188	25·202	44·752	68·597
C	0·5	1·0	0·5	29·928	65·554	67·048	21·207	42·833	60·188	25·202	44·752	68·597
C	1·0	0·5	0·5	27·576	60·528	72·042	17·723	43·085	54·301	19·299	40·416	63·206
	0·5	0·5	0·5	23·610	48·403	53·809	15·593	32·743	43·538	17·808	31·945	51·730
	1·0	1·0	1·0	40·016	95·827	101·791	29·093	63·567	89·866	34·641	66·498	99·863
	Reference [3]			40·022	95·891	101·85	29·09					
	Reference [10]						29·09	63·57	89·87			

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TABLE 2—*continued*

1 2 3	α	β	γ	$\xi = 0{\cdot}5,\ \eta = \sqrt{3}/2$ λ_1	λ_2	λ_3	$\xi = 0{\cdot}0,\ \eta = 1{\cdot}0$ λ_1	λ_2	λ_3	$\xi = -0{\cdot}5,\ \eta = \sqrt{3}/2$ λ_1	λ_2	λ_3
	Reference Ours						28·872	62·969	89·671			
	Reference [3]						28·87	62·97	89·66			
	Reference [7]						28·87	62·97	89·65			
F	0·5	0·5	1·0	17·717	48·758	55·183	12·157	30·471	49·452	13·806	30·193	49·310
C	0·5	1·0	0·5	18·775	49·725	56·531	12·740	31·722	49·321	13·398	31·431	52·439
S	1·0	0·5	0·5	18·072	49·543	57·317	11·059	32·739	44·954	11·226	29·555	48·652
	0·5	0·5	0·5	13·707	38·549	41·503	8·846	23·525	35·903	9·591	22·241	37·986
	1·0	1·0	1·0	26·561	75·314	84·350	17·967	47·949	73·629	18·951	46·791	75·941
	Reference [3]			26·565	75·36	84·432	17·96					
	Reference [10]						17·97	47·95	73·63			
F	0·5	0·5	1·0	5·247	23·171	25·440	3·963	15·194	24·638	3·968	15·355	26·680
C	0·5	1·0	0·5	6·032	24·429	24·470	4·115	15·673	21·668	3·617	14·430	24·645
F	1·0	0·5	0·5	5·247	23·171	25·440	3·670	14·737	21·358	3·579	13·363	23·239
	0·5	0·5	0·5	3·872	17·776	17·943	2·796	11·069	16·890	2·681	10·854	18·907
	1·0	1·0	1·0	8·921	35·095	38·484	6·167	23·458	32·682	5·701	21·501	36·269
	Reference [3]			8·9219	35·155	38·503	6·173					
	Reference [9]			8·9221	35·132	38·505	6·1732	23·477	32·716			
	Reference [10]						6·168	23·46	32·69			
	Reference [14]						5·93	23·4	32·7			
	Reference [13]						6·16	23·7	32·54			
	Reference [12]						6·1215	23·02	31·853			
	Reference [11]						6·1575	23·436	32·735			
	Reference [16]						6·157	23·415	32·621			
F	0·5	0·5	1·0	18·775	49·725	56·531	12·740	31·722	49·321	13·398	31·431	52·439
S	0·5	1·0	0·5	17·717	48·758	55·183	12·157	30·471	49·452	13·806	30·193	49·310
C	1·0	0·5	0·5	18·072	49·543	57·317	11·059	32·739	44·954	11·226	29·555	48·652
	0·5	0·5	0·5	13·707	38·549	41·503	8·846	23·525	35·903	9·591	22·241	37·986
	1·0	1·0	1·0	26·561	75·314	84·350	17·967	47·949	73·629	18·951	46·791	75·941
	Reference [3]			26·565	75·36	84·432	17·96					
	Reference [10]						17·97	47·95	73·63			

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TABLE 2—*continued*

1 2 3	α	β	γ	$\xi = 0{\cdot}5,\ \eta = \sqrt{3}/2$ λ_1	λ_2	λ_3	$\xi = 0{\cdot}0,\ \eta = 1{\cdot}0$ λ_1	λ_2	λ_3	$\xi = -0{\cdot}5,\ \eta = \sqrt{3}/2$ λ_1	λ_2	λ_3
F	0·5	0·5	1·0	10·518	34·600	45·603	7·136	20·750	38·573	7·778	18·579	35·476
S	0·5	1·0	0·5	10·518	34·600	45·603	7·136	20·750	38·573	7·778	18·579	35·476
S	1·0	0·5	0·5	10·650	40·720	43·372	6·051	24·248	35·750	5·912	20·161	36·578
	0·5	0·5	0·5	7·283	27·325	32·530	5·023	15·383	28·076	6·150	13·168	26·593
	1·0	1·0	1·0	16·092	57·630	68·330	9·798	34·633	58·574	8·974	30·652	55·214
	Reference [3]			16·092	57·709	68·593	9·798					
	Reference [10]						9·798	34·63	58·56			
F	0·5	0·5	1·0	13·682	17·267	41·462	9·366	17·641	25·060	8·101	18·021	28·630
S	0·5	1·0	0·5	13·428	16·989	41·784	8·266	16·735	24·869	6·499	17·853	25·874
F	1·0	0·5	0·5	13·682	17·267	41·462	9·139	14·491	29·753	7·076	17·329	27·581
	0·5	0·5	0·5	10·220	10·411	32·754	6·191	11·212	18·738	5·130	12·956	19·388
	1·0	1·0	1·0	22·646	26·659	69·418	14·561	24·738	42·039	11·380	27·175	41·995
	Reference [3]			22·666	26·717	71·033	14·56					
	Reference [10]						14·56	24·74	42·07			
F	0·5	0·5	1·0	6·032	24·429	24·470	4·115	15·673	21·668	3·617	14·430	24·645
F	0·5	1·0	0·5	5·247	23·171	25·440	3·963	15·194	24·638	3·968	15·355	26·680
C	1·0	0·5	0·5	5·247	23·171	25·440	3·670	14·737	21·358	3·579	13·363	23·239
	0·5	0·5	0·5	3·872	17·776	17·943	2·796	11·069	16·890	2·681	10·854	18·907
	1·0	1·0	1·0	8·921	35·095	38·484	6·167	23·458	32·682	5·701	21·501	36·269
	Reference [1]						6·1575	23·059	33·288	5·7617	21·099	35·952
	Reference [3]			8·9219	35·155	38·503	6·173			5·7167	21·524	37·456
	Reference [9]			8·9221	35·132	38·505				5·717	21·525	37·455
	Reference [10]						6·168	23·46	32·69			
F	0·5	0·5	1·0	13·428	16·989	41·784	8·266	16·735	24·869	6·499	17·853	25·874
F	0·5	1·0	0·5	13·682	17·267	41·462	9·366	17·641	25·060	8·101	18·021	28·630
S	1·0	0·5	0·5	13·682	17·267	41·462	9·139	14·491	29·753	7·076	17·329	27·581
	0·5	0·5	0·5	10·220	10·411	32·754	6·191	11·212	18·738	5·130	12·956	19·388
	1·0	1·0	1·0	22·646	26·659	69·418	14·561	24·738	42·039	11·380	27·175	41·995
	Reference [3]			22·666	26·717	71·033	14·56					
	Reference [10]						14·56	24·74	42·07			

continued overleaf

TABLE 2—*continued*

1 2 3				$\xi = 0{\cdot}5,\ \eta = \sqrt{3}/2$			$\xi = 0{\cdot}0,\ \eta = 1{\cdot}0$			$\xi = -0{\cdot}5,\ \eta = \sqrt{3}/2$		
	α	β	γ	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
F	0·5	0·5	1·0	16·842	23·100	23·926	10·211	18·258	28·727	7·592	16·331	27·588
F	0·5	1·0	0·5	16·842	23·100	23·926	10·211	18·258	28·727	7·592	16·331	27·588
F	1·0	0·5	0·5	16·842	23·100	23·926	12·340	18·753	23·101	8·496	16·643	31·437
	0·5	0·5	0·5	12·881	14·611	14·611	7·523	11·933	18·615	5·522	11·018	19·761
	1·0	1·0	1·0	34·283	36·072	36·072	19·072	29·129	45·450	13·434	25·220	46·639
	Reference [3]			34·962	36·331	36·337	19·17					
	Reference [10]						19·08	29·25	46·03			

TABLE 3

First three frequency parameters for cubic thickness variation (Figure 3(e))

		$\xi = 0{\cdot}5,\ \eta = \sqrt{3}/2$			$\xi = 0{\cdot}0,\ \eta = 1{\cdot}0$			$\xi = -0{\cdot}5,\ \eta = \sqrt{3}/2$		
123	α	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
CCC	0·5	82·271	146·797	146·797	78·082	124·327	152·642	117·376	169·781	226·417
	1·5	114·742	226·112	226·112	108·565	188·103	232·175	161·874	245·717	319·652
CCS	0·5	65·572	124·219	128·907	63·757	107·620	136·221	95·414	146·486	200·893
	1·5	97·241	199·186	201·379	93·404	167·854	208·470	140·163	219·253	287·931
CCF	0·5	34·642	78·607	81·352	35·588	70·614	90·054	49·254	92·280	128·273
	1·5	45·362	111·354	125·315	46·622	101·935	130·335	67·215	129·280	183·897
CSC	0·5	65·572	124·219	128·907	63·757	107·620	136·221	95·414	146·485	200·892
	1·5	97·241	199·186	201·379	93·404	167·854	208·470	140·163	219·253	287·932
CSS	0·5	51·259	104·807	110·298	51·309	91·584	120·933	77·635	123·667	177·544
	1·5	81·538	175·312	177·462	79·861	149·175	186·607	121·564	194·652	262·148
CSF	0·5	21·551	62·068	63·324	26·041	57·267	77·028	41·956	74·759	111·256
	1·5	31·754	89·739	106·409	37·208	87·953	113·931	59·569	114·117	163·445
CFC	0·5	34·642	78·607	81·352	35·588	70·614	90·054	49·254	92·279	128·271
	1·5	45·362	111·354	125·315	46·622	101·935	130·335	67·215	129·280	183·896
CFS	0·5	21·551	62·068	63·324	26·041	57·267	77·028	41·956	74·759	111·256
	1·5	31·754	89·739	106·409	37·208	87·953	113·931	59·569	114·117	163·445
CFF	0·5	6·562	26·995	30·469	9·037	27·960	41·818	15·650	38·528	64·948
	1·5	10·946	44·024	47·238	15·669	44·531	64·577	29·131	62·123	95·652
SCC	0·5	65·572	124·219	128·907	57·667	101·790	123·247	81·469	129·404	179·783
	1·5	97·241	199·186	201·379	88·823	160·244	201·883	128·880	202·986	269·128
SCS	0·5	51·259	104·807	110·298	46·050	86·875	108·176	64·366	109·868	157·291
	1·5	81·538	175·312	177·462	75·473	141·876	180·122	110·046	179·306	241·056
SCF	0·5	21·551	62·068	63·324	19·386	50·251	65·960	22·735	57·872	87·556
	1·5	31·754	89·739	106·409	29·024	75·438	107·498	35·293	85·599	138·330
SSC	0·5	51·259	104·807	110·298	46·050	86·875	108·176	64·366	109·868	157·291
	1·5	81·538	175·312	177·462	75·473	141·876	180·122	110·046	179·306	241·056

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TABLE 3—*continued*

123	α	$\xi = 0{\cdot}5,\ \eta = \sqrt{3}/2$			$\xi = 0{\cdot}0,\ \eta = 1{\cdot}0$			$\xi = -0{\cdot}5,\ \eta = \sqrt{3}/2$		
		λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
SSS	0·5	39·064	89·929	89·929	35·929	72·736	94·237	50·244	90·372	137·980
	1·5	67·620	154·373	154·373	63·717	125·028	160·161	94·128	157·623	216·816
SSF	0·5	12·071	44·745	49·927	13·020	38·794	55·076	18·665	44·641	73·169
	1·5	20·737	71·665	88·555	22·076	63·866	93·015	30·612	74·741	120·230
SFC	0·5	21·551	62·068	63·324	19·386	50·251	65·960	22·735	57·872	87·557
	1·5	31·754	89·739	106·409	29·024	75·438	107·498	35·293	85·599	138·330
SFS	0·5	12·071	44·745	49·927	13·020	38·794	55·076	18·665	44·641	73·169
	1·5	20·737	71·665	88·555	22·076	63·866	93·015	30·612	74·741	120·230
SFF	0·5	16·750	18·627	53·160	14·131	24·650	44·388	16·077	33·371	57·219
	1·5	29·417	35·900	86·044	26·090	46·007	79·198	30·615	56·387	90·744
FCC	0·5	34·642	78·607	81·352	24·472	52·482	68·430	28·479	53·788	81·966
	1·5	45·362	111·354	125·315	33·020	75·322	109·881	38·991	79·207	118·696
FCS	0·5	21·551	62·068	63·324	14·327	38·357	55·463	15·070	36·929	61·151
	1·5	31·754	89·739	106·409	21·528	58·430	91·124	22·601	57·253	92·027
FCF	0·5	6·562	26·995	30·469	4·579	18·372	25·632	4·227	17·310	28·882
	1·5	10·946	44·024	47·238	7·607	28·811	40·417	7·230	25·727	43·832
FSC	0·5	21·551	62·068	63·324	14·327	38·357	55·463	15·070	36·929	61·151
	1·5	31·754	89·739	106·409	21·528	58·430	91·124	22·601	57·253	92·027
FSS	0·5	12·071	44·745	49·927	7·721	25·866	43·833	7·920	22·343	43·314
	1·5	20·737	71·665	88·555	12·429	44·619	73·605	10·661	40·397	69·124
FSF	0·5	16·750	18·627	53·160	10·562	18·448	31·173	8·426	21·369	30·365
	1·5	29·417	35·900	86·044	19·332	31·487	54·572	15·035	33·445	55·805
FFC	0·5	6·562	26·995	30·469	4·579	18·372	25·632	4·227	17·310	28·882
	1·5	10·946	44·024	47·238	7·607	28·811	40·417	7·230	25·727	43·832
FFS	0·5	16·750	18·627	53·160	10·562	18·448	31·173	8·426	21·369	30·365
	1·5	29·417	35·900	86·044	19·332	31·487	54·572	15·035	33·445	55·805
FFF	0·5	23·804	25·050	25·050	13·384	20·718	32·038	9·561	19·234	32·240
	1·5	45·646	49·317	49·317	25·850	38·601	60·052	18·169	31·916	56·700

TABLE 4

Convergence of results for quadratic thickness variation (*Figure* 3(*c*), $\alpha = 0{\cdot}3$, $\beta = 0{\cdot}5$)

123	ξ	η	$N = 10$	15	21	28	34	35	36
CCC	0·5	$\sqrt{3}/2$	50·860	50·354	50·152	50·068	50·038	50·032	50·032
	0·0	1·0	50·102	50·669	50·529	50·486	50·472	50·472	50·470
	−0·5	$\sqrt{3}/2$	80·088	79·449	79·002	78·827	78·804	78·794	78·793
CCS	0·5	$\sqrt{3}/2$	44·273	43·790	43·619	43·554	43·534	43·529	43·528
	0·0	1·0	45·492	45·027	44·886	44·846	44·837	44·835	44·834
	−0·5	$\sqrt{3}/2$	72·189	70·946	70·526	70·276	70·207	70·203	70·201
CCF	0·5	$\sqrt{3}/2$	28·228	28·176	28·147	28·140	28·137	28·137	28·137
	0·0	1·0	29·847	29·788	29·769	29·757	29·750	29·750	29·750
	−0·5	$\sqrt{3}/2$	42·389	41·773	41·432	41·334	41·267	41·260	41·252
CSC	0·5	$\sqrt{3}/2$	41·478	40·486	40·041	39·836	39·738	39·738	39·737
	0·0	1·0	39·787	38·994	38·685	38·547	38·489	38·488	38·486
	−0·5	$\sqrt{3}/2$	59·468	58·391	57·556	57·349	57·264	57·235	57·234
CSS	0·5	$\sqrt{3}/2$	35·666	34·699	34·310	34·147	34·077	34·076	34·075
	0·0	1·0	35·318	34·476	34·169	34·042	33·993	33·993	33·990
	−0·5	$\sqrt{3}/2$	54·090	52·886	52·121	51·849	51·740	51·731	51·730
CSF	0·5	$\sqrt{3}/2$	17·900	17·819	17·794	17·785	17·781	17·781	17·781
	0·0	1·0	21·112	21·004	20·974	20·965	20·962	20·962	20·961
	−0·5	$\sqrt{3}/2$	34·955	33·723	33·717	33·622	33·606	33·606	33·606
CFC	0·5	$\sqrt{3}/2$	19·578	18·848	18·403	18·135	17·963	17·960	17·959
	0·0	1·0	17·402	16·741	16·341	16·100	15·942	15·940	15·939
	−0·5	$\sqrt{3}/2$	21·427	20·526	19·930	19·618	19·425	19·423	19·423
CFS	0·5	$\sqrt{3}/2$	11·856	11·418	11·199	11·079	11·011	11·010	11·010
	0·0	1·0	13·392	12·806	12·505	12·328	12·227	12·227	12·225
	−0·5	$\sqrt{3}/2$	20·019	18·970	18·359	18·003	17·802	17·800	17·797
CFF	0·5	$\sqrt{3}/2$	3·930	3·860	3·838	3·830	3·826	3·826	3·826
	0·0	1·0	5·029	4·917	4·868	4·846	4·834	4·834	4·834
	−0·5	$\sqrt{3}/2$	8·461	8·126	8·017	7·964	7·941	7·941	7·941
SCC	0·5	$\sqrt{3}/2$	43·047	42·752	42·644	42·596	42·581	42·579	42·577
	0·0	1·0	42·685	42·367	42·274	42·250	42·243	42·242	42·241
	−0·5	$\sqrt{3}/2$	67·399	65·497	65·396	65·145	65·128	65·121	65·118
SCS	0·5	$\sqrt{3}/2$	36·818	36·509	36·413	36·374	36·363	36·362	36·360
	0·0	1·0	37·194	36·921	36·820	36·799	36·792	36·792	36·791
	−0·5	$\sqrt{3}/2$	59·434	57·353	57·130	56·851	56·774	56·769	56·766
SCF	0·5	$\sqrt{3}/2$	18·873	18·819	18·801	18·797	18·795	18·795	18·795
	0·0	1·0	18·619	18·578	18·568	18·557	18·551	18·551	18·551
	−0·5	$\sqrt{3}/2$	23·706	23·108	22·821	22·693	22·624	22·621	22·618
SSC	0·5	$\sqrt{3}/2$	34·752	34·115	33·914	33·830	33·799	33·793	33·793
	0·0	1·0	32·450	32·213	32·064	32·015	31·997	31·995	31·994
	−0·5	$\sqrt{3}/2$	47·779	47·256	46·837	46·623	46·578	46·578	46·578
SSS	0·5	$\sqrt{3}/2$	29·310	28·656	28·464	28·393	28·370	28·365	28·365
	0·0	1·0	28·163	27·826	27·700	27·652	27·637	27·634	27·634
	−0·5	$\sqrt{3}/2$	42·632	41·845	41·510	41·310	41·229	41·227	41·226
SSF	0·5	$\sqrt{3}/2$	10·615	10·572	10·569	10·568	10·568	10·568	10·568
	0·0	1·0	12·449	12·410	12·406	12·405	12·405	12·405	12·405
	−0·5	$\sqrt{3}/2$	19·239	18·974	18·840	18·801	18·766	18·766	18·766

continued overleaf

TABLE 4—*continued*

123	ξ	η	$N = 10$	15	21	28	34	35	36
SFC	0·5	$\sqrt{3}/2$	15·961	15·752	15·663	15·613	15·588	15·588	15·586
	0·0	1·0	13·715	13·610	13·545	13·507	13·490	13·490	13·486
	−0·5	$\sqrt{3}/2$	16·262	16·185	16·038	15·963	15·943	15·940	15·939
SFS	0·5	$\sqrt{3}/2$	9·151	9·077	9·037	9·019	9·010	9·010	9·010
	0·0	1·0	10·000	9·908	9·870	9·848	9·837	9·837	9·835
	−0·5	$\sqrt{3}/2$	14·480	14·348	14·292	14·229	14·212	14·199	14·197
SFF	0·5	$\sqrt{3}/2$	11·620	11·136	11·016	10·947	10·918	10·916	10·914
	0·0	1·0	10·144	9·990	9·921	9·882	9·867	9·862	9·860
	−0·5	$\sqrt{3}/2$	12·580	12·526	12·391	12·341	12·330	12·323	12·323
FCC	0·5	$\sqrt{3}/2$	22·509	22·395	22·372	22·361	22·360	22·359	22·356
	0·0	1·0	20·207	20·099	20·055	20·047	20·047	20·047	20·047
	−0·5	$\sqrt{3}/2$	29·321	28·630	28·566	28·500	28·495	28·494	28·489
FCS	0·5	$\sqrt{3}/2$	14·102	14·077	14·071	14·067	14·067	14·067	14·066
	0·0	1·0	12·106	12·056	12·044	12·044	12·044	12·044	12·044
	−0·5	$\sqrt{3}/2$	15·844	15·432	15·234	15·141	15·092	15·092	15·092
FCF	0·5	$\sqrt{3}/2$	4·852	4·843	4·840	4·839	4·839	4·839	4·838
	0·0	1·0	4·189	4·165	4·155	4·149	4·146	4·146	4·146
	−0·5	$\sqrt{3}/2$	4·302	4·249	4·226	4·217	4·212	4·212	4·211
FSC	0·5	$\sqrt{3}/2$	18·211	18·006	17·961	17·949	17·946	17·945	17·945
	0·0	1·0	14·633	14·606	14·595	14·590	14·590	14·589	14·589
	−0·5	$\sqrt{3}/2$	17·813	17·513	17·455	17·403	17·385	17·385	17·385
FSS	0·5	$\sqrt{3}/2$	10·766	10·694	10·684	10·681	10·681	10·680	10·680
	0·0	1·0	8·319	8·315	8·314	8·313	8·313	8·313	8·313
	−0·5	$\sqrt{3}/2$	10·038	9·761	9·597	9·484	9·407	9·407	9·407
FSF	0·5	$\sqrt{3}/2$	14·031	13·730	13·581	13·546	13·540	13·540	13·538
	0·0	1·0	10·585	10·521	10·498	10·495	10·494	10·493	10·493
	−0·5	$\sqrt{3}/2$	9·483	9·440	9·404	9·394	9·386	9·386	9·385
FFC	0·5	$\sqrt{3}/2$	10·197	10·143	10·132	10·127	10·126	10·126	10·126
	0·0	1·0	7·235	7·228	7·218	7·216	7·215	7·215	7·215
	−0·5	$\sqrt{3}/2$	6·538	6·493	6·485	6·478	6·475	6·475	6·475
FFS	0·5	$\sqrt{3}/2$	16·467	15·535	15·298	15·217	15·198	15·188	15·188
	0·0	1·0	10·572	10·401	10·342	10·331	10·329	10·328	10·328
	−0·5	$\sqrt{3}/2$	8·873	8·726	8·713	8·710	8·709	8·708	8·708
FFF	0·5	$\sqrt{3}/2$	22·228	18·432	17·276	16·886	16·784	16·780	16·765
	0·0	1·0	14·292	11·988	11·674	11·572	11·556	11·556	11·554
	−0·5	$\sqrt{3}/2$	10·852	9·536	9·362	9·351	9·345	9·345	9·343

said that four digits are significant and in the last we have three digits, namely, 1, 2 and 8, agreeing with the exact result. This particular case serves as a test of the accuracy of our results.

The effects of the various parameters, α, β, γ, α', β', and the boundary conditions are self-explanatory from Tables 1 through 4. A general observation can be made that as the overall thickness increases, the frequency also increases. Further, as one goes from all free edges to all simply-supported edges and then to all clamped edges, the frequencies increase. It is difficult to comment when the boundary conditions are mixed since some have increasing, while others decreasing, effects on the frequency.

4. MODE SHAPES

Figure 4 gives the first three mode shapes of an equilateral triangle with all sides clamped and uniform thickness (Table 2, CCC, $\alpha = \beta = \gamma = 1$, $\xi = 0{\cdot}5$, $\eta = \sqrt{3}/2$). Figure 5 gives the first three mode shapes of an isosceles triangle with angles of 30, 30 and 120°, again with all sides clamped but thickness varying quadratically (Table 1, CCC, $\alpha = 0{\cdot}3$, $\beta = 0{\cdot}5$, $\xi = -0{\cdot}5$, $\eta = \sqrt{3}/2$). Figure 6 gives mode shapes for the same triangle as in Figure 5 but sides facing vertices 1, 2 and 3 are simply-supported, clamped and free, respectively. The thickness varies quadratically (Table 2, SCF, $\alpha = 0{\cdot}5$, $\beta = 1{\cdot}0$, $\gamma = 0{\cdot}5$, $\xi = -0{\cdot}5$, $\eta = \sqrt{3}/2$). For want of space, only a few selected cases have been given here but the program can generate mode shapes for a triangle of any shape and arbitrary thickness variation which can be approximated up to a cubic variation and for any combination of boundary conditions.

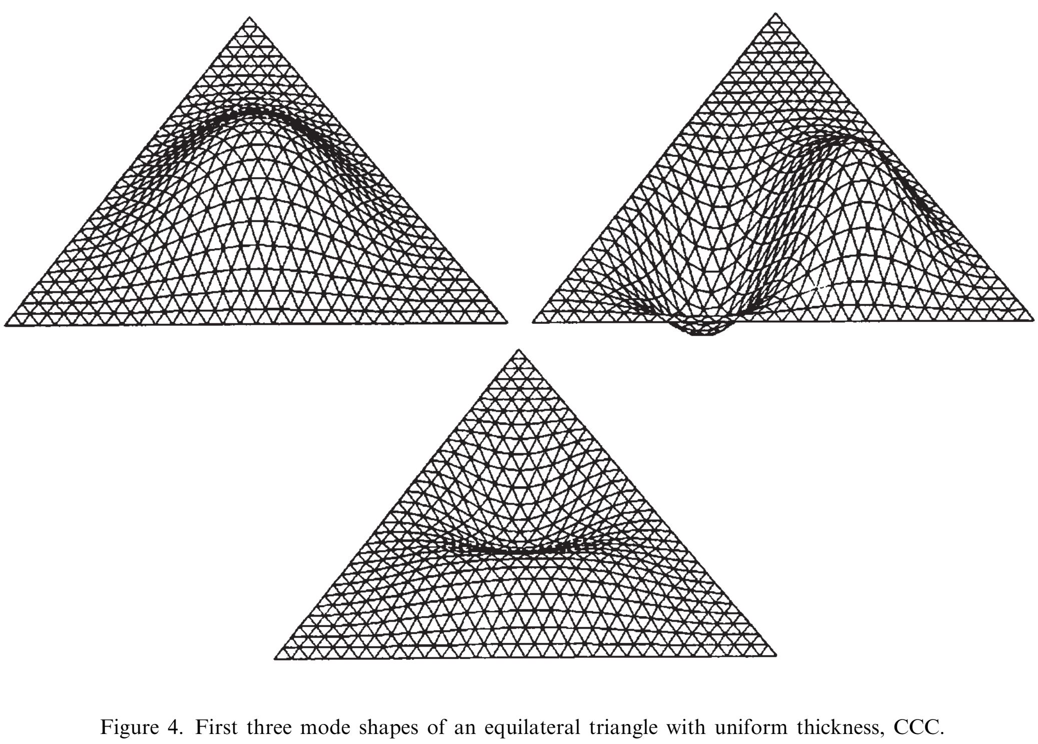

Figure 4. First three mode shapes of an equilateral triangle with uniform thickness, CCC.

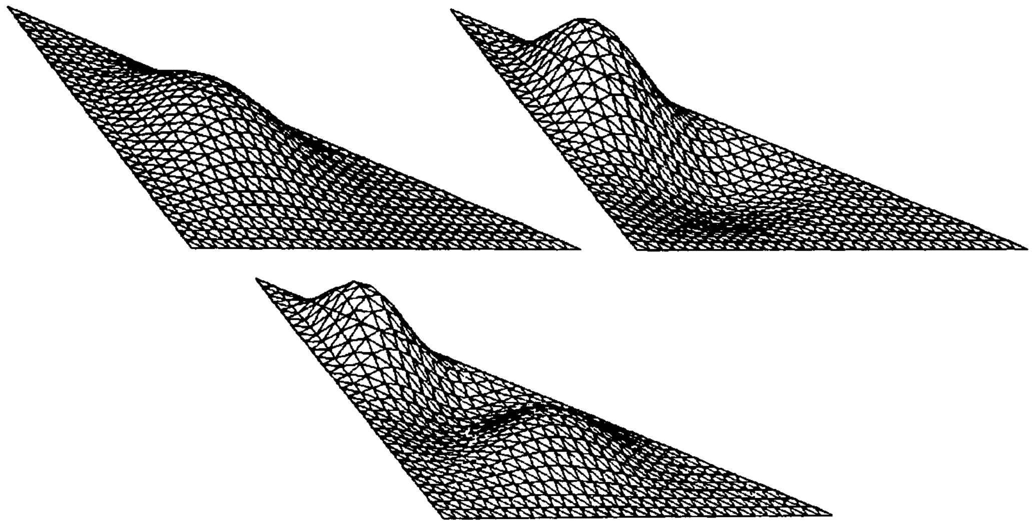

Figure 5. First three mode shapes of an isosceles triangle with angles 30, 30 and 120° and quadratic thickness variation, CCC.

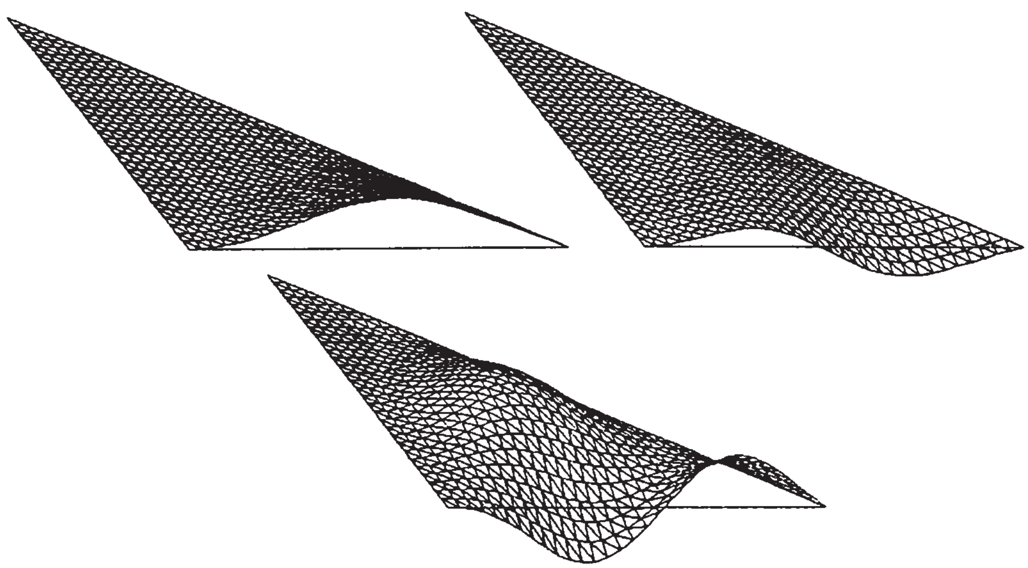

Figure 6. First three mode shapes of isosceles triangle with angles 30, 30 and 120° and quadratic thickness variation, SCF.

5. CONCLUSION

The present method gives numerical results for frequency and mode shapes of practically any triangular plate with arbitrary thickness variation and any combination of boundary conditions. The crux of the matter lies in approximating the thickness variation by a polynomial of suitable degree. This can be done by choosing a set of sample points and carrying out measurements for thickness at these points and fitting a polynomial by interpolation. The convergence is ensured by working out a large number of approximations using the Rayleigh–Ritz method and suitable basis functions satifying the essential boundary conditions. The use of natural co-ordinates has simplified the computations to a great extent.

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