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FREQUENCY DEPENDENCES OF COMPLEX MODULI AND COMPLEX POISSON'S RATIO OF REAL SOLID MATERIALS

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The concept of a complex modulus of elasticity is a powerful and widely used tool for characterizing the linear dynamic elastic and damping properties of solid materials in the frequency domain. It is shown in this paper that typical characters of frequency dependences of all complex moduli (shear, Young's etc.), and complex Poisson's ratios of real solid materials can be determined by transforming the causal and real relaxation and creep responses, respectively, from the time-domain into the frequency domain, even without having to specify the processes of relaxation and creep. It is proved that all dynamic moduli monotonically increase, and the dynamic Poisson's ratio monotonically decreases with increasing frequency, and all respective loss factors pass through at least one maximum. These frequency dependences are generally valid for any real solid material regardless of the actual damping mechanism. Some experimental results are presented and interepreted in the light of the theory. The usefulness of theoretical predictions in materials engineering, measurements of dynamic properties and in modelling dynamic behaviour is discussed.

1. INTRODUCTION

All real solid materials possess both elastic and damping properties; damping is the ability to dissipate some mechanical energy during vibration or dynamic deformation of any kind. Both the dynamic elastic and damping properties, briefly referred to as dynamic properties, of real solid materials are frequency dependent to a greater or lesser extent. The complex modulus concept is a powerful, widely used tool for characterizing the dynamic elastic and damping properties in the frequency domain. According to this concept, all elastic moduli (shear, Young's etc.), as well as the Poisson's ratio of any real solid material, whether isotropic or anisotropic, can be regarded as a complex quantity [1]. A knowledge of the complex moduli, and more particularly their components; the dynamic modulus, the loss modulus and the loss factor, and their frequency dependence have been of interest to scientists and engineers for a long time. Acousticians need such data to predict wave propagation, applying a material properly for vibration control, and taking the dynamic properties into account when calculating vibrational response. Furthermore, such information is absolutely necessary to materials engineers when developing and qualifying new materials for vibration and noise control.

The dynamic properties are determined by measurement. Most available experimental data, covering wide frequency ranges, concerns organic polymeric materials usually referred to as viscoelastic ones [1–3]. The experimental data invariably shows that dynamic moduli of polymers increase with increasing frequency, and loss moduli and loss factors pass through at least one maximum [1–4]. The frequency dependence is most significant

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when the loss factor is high, as is characteristic for rubbers and rubber-like materials in shear and uniaxial tension–compression deformation. In contrast to organic polymers the dynamic properties of stiff structural materials, such as metals and wood, are usually considered to be independent of frequency on the basis of measurements performed in the audio frequency range [5]. In comparison with the data available concerning complex moduli, much less data are available on the frequency dependence of the complex Poisson's ratio. All available data appears to be for polymeric materials and, apart from some early contradictory findings, shows that the dynamic Poisson's ratio decreases monotonically with increasing frequency and the relevant loss factor passes through one maximum [1, 2, 4].

The measured frequency dependences of the complex moduli and Poisson's ratios of polymeric materials are usually interpreted by means of viscoelastic models, particularly the standard linear solid [6]. Today, it is quite well known that the classical viscoelastic models can be generalized by means of the fractional calculus to describe weak frequency dependence, and even "frequency independence" over limited frequency ranges for stiff structural materials [7–9]. Nevertheless, it is evident that any frequency dependences calculated by such models reflect only the behaviour of the model and do not represent valid *a priori* predictions.

The aim of this paper is to demonstrate that the typical characters of frequency dependences of all complex moduli and Poisson's ratios of real solid materials can be determined without recourse to any material models. It will be shown that the characters of frequency dependences are the same for any real solid material regardless of the actual damping mechanism. The results of the theoretical investigation presented here may help one to understand, predict and model the dynamic behaviour of solid materials, and help in interpreting experimental data.

2. THEORY

2.1. COMPLEX MODULUS OF ELASTICITY

2.1.1. *Complex modulus as frequency response function*

The linear dynamic elastic and damping properties of any solid material can be characterized in the frequency domain by a complex number, referred to as the complex modulus of elasticity. The complex modulus is usually defined for harmonic vibrations, but a more general definition can be given provided that the Fourier transforms of the stress– and strain–time histories exist: namely,

$$\bar{M}(\mathrm{j}\omega) = \frac{\bar{\sigma}(\mathrm{j}\omega)}{\bar{\varepsilon}(\mathrm{j}\omega)} = M_d(\omega) + \mathrm{j}M_l(\omega) = M_d(\omega)[1 + \mathrm{j}\eta(\omega)], \tag{1}$$

where M is the modulus of elasticity in the general sense, a bar over a symbol represents a complex valued function, $\mathrm{j} = \sqrt{-1}$ is the imaginary unit, $\omega = 2\pi f$; f is the frequency in Hz, $\bar{\sigma}$ and $\bar{\varepsilon}$ are the Fourier transforms of the stress– and strain–times histories, respectively (these are the complex amplitudes if the vibration is harmonic). Furthermore, M_d is the dynamic modulus of elasticity, M_l is the loss modulus, and η is the loss factor,

$$\eta(\omega) = \frac{M_l(\omega)}{M_d(\omega)}. \tag{2}$$

The definition of complex modulus given by equation (1), can also be regarded as general from another point of view, that no restrictions have been placed on the type of deformation such as bulk, shear , tensile, etc. Therefore, $\bar{M}(\mathrm{j}\omega)$ may represent the complex form of any modulus of elasticity of a solid material, whether isotropic or anisotropic Specifically, the usual complex moduli for isotropic material are

$$\overline{B}(\mathrm{j}\omega) = B_d(\omega) + \mathrm{j}B_l(\omega) = B_d(\omega)[1 + \mathrm{j}\eta_B(\omega)], \tag{3}$$

$$\overline{G}(\mathrm{j}\omega) = G_d(\omega) + \mathrm{j}G_l(\omega) = G_d(\omega)[1 + \mathrm{j}\eta_G(\omega)], \tag{4}$$

$$\overline{E}(\mathrm{j}\omega) = E_d(\omega) + \mathrm{j}E_l(\omega) = E_d(\omega)[1 + \mathrm{j}\eta_E(\omega)], \tag{5}$$

$$\overline{L}(\mathrm{j}\omega) = L_d(\omega) + \mathrm{j}L_l(\omega) = L_d(\omega)[1 + \mathrm{j}\eta_L(\omega)], \tag{6}$$

where, B, G, E and L are the bulk, shear, Young's and longitudinal moduli, respectively. The subscripts d and l refer to the dynamic and loss moduli, respectively, and the subscript of η refers to the relevant modulus of elasticity.

Equation (1) implies that the complex modulus can be interpreted as the frequency response function of a linear system, the system being the material itself. Therefore, if $\bar{\sigma}(\mathrm{j}\omega)$ is the excitation, i.e. the input function, then $\bar{\varepsilon}(\mathrm{j}\omega)$ is the output function, the response as illustrated in Figure 1. Recognition that the material can be regarded as a linear physical system, and the complex modulus is its frequency response function, provides one with a very effective tool for understanding the dynamic behaviour of the material. It follows that the well developed methods of linear system theory [10] can be brought to bear. These methods will be used throughout the paper.

2.1.2. *Causality and dispersion relations*

The complex modulus describes the dynamic deformation behaviour of materials in the frequency domain. Both the real and imaginary parts of the complex modulus, being the dynamic and loss moduli, respectively, are expected to be frequency dependent. However, nothing important can be stated about the frequency dependences from simple theoretical considerations, apart from the plausible statement that the loss modulus is zero at zero frequency (no motion, no energy loss), so that $M_l(\omega)$ starts to increase with increasing frequency. On the contrary, a fundamental feature of real material behaviour in the time-domain is known, namely it is causal, i.e. no response can occur before initial application of excitation. It will be shown that, by transforming the causal behaviour of real solid materials into the frequency domain, the typical characters of the frequency dependences of the dynamic properties can be determined.

The main consequence of causality is well-known from linear system theory, namely that the real and imaginary parts of the frequency response function are interrelated [10]. The

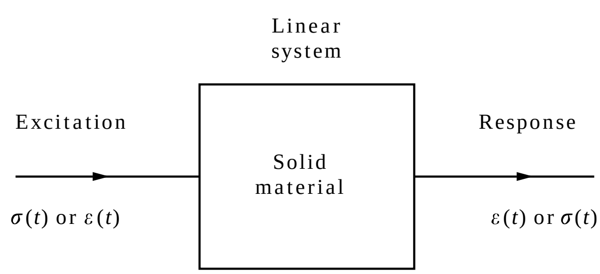

Figure 1. Any real solid material for low amplitude excitation can be regarded as a linear physical system relating the stress to the strain.

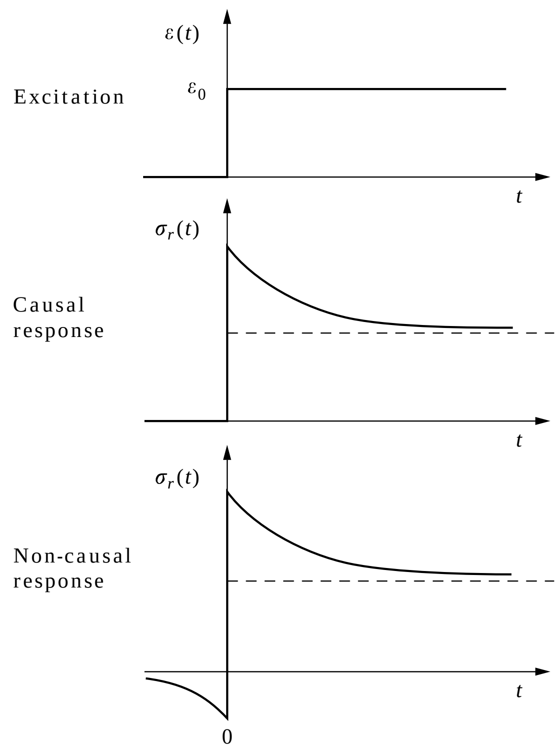

Figure 2. The stress relaxation function is a material response to excitation of strain step function. Causal and non-causal responses.

relations linking the real and imaginary parts of the frequency response function of a real physical system are generally referred to as Kramers–Kronig dispersion relations, or shortly as dispersion relations, after the names of the authors who developed them first in connection with electromagnetic radiation [11, 12]. However, the relations, which in the mathematical context are known as Hilbert transforms, are of general nature and, therefore, have found application in many fields of science such as nuclear physics, electrical engineering, structural dynamics and acoustics. Since the complex modulus can be interpreted as the frequency response function of the material, it follows that the dynamic modulus– and the loss modulus–frequency functions are interrelated too. The derivation of the dispersion relations for the complex modulus is outlined here including only the background and some important details.

For deriving the dispersion relations, the material behaviour in the time-domain is investigated usually as a response to Dirac-impulse or step function excitation. In principle any of these can be used, but from the physical point of view the step function is a reasonable choice. Moreover, the strain step function is chosen here because in that case the stress relaxation is the response which is a common feature of all real solids and quite well-known from many experiments. The strain step function having a magnitude of ε_o, and the causal real response, the $\sigma_r(t)$ relaxation function, are illustrated in Figure 2 along with a non-causal response. Note that in contrast to Figure 2, the stress response for some materials may disappear at long times, but such materials are not of interest for our present purpose. Nevertheless, the theoretical approach presented here may easily be extended to apply for such materials.

With knowledge of the relaxation function the complex modulus can be calculated, as can the reverse. The relationships between $\sigma_r(t)$ and $\bar{M}(\mathrm{j}\omega)$ are easily determined by taking the Fourier transforms of the stress response $\sigma_r(t)$ and the strain excitation $\varepsilon_e(t) = \varepsilon_o 1(t)$:

$$\bar{\sigma}_r(\mathrm{j}\omega) = F\sigma_r(t) = \int_{-\infty}^{\infty} \sigma_r(t)\, \mathrm{e}^{-\mathrm{j}\omega t}\, \mathrm{d}t, \tag{7}$$

$$\bar{\varepsilon}_e(\mathrm{j}\omega) = F\varepsilon_0 1(t) = \varepsilon_o\left[\pi\delta(\omega) + \frac{1}{\mathrm{j}\omega}\right], \tag{8}$$

where F denotes the Fourier transform and δ is the Dirac delta function. By substituting these equations into equation (1) one gets:

$$\bar{M}(\mathrm{j}\omega) = \mathrm{j}\omega \int_{-\infty}^{\infty} m_r(t)\, \mathrm{e}^{-\mathrm{j}\omega t}\, \mathrm{d}t, \tag{9}$$

from which

$$M_d(\omega) = \omega \int_{-\infty}^{\infty} m_r(t) \sin \omega t\, \mathrm{d}t + M_0, \qquad M_l(\omega) = \omega \int_{-\infty}^{\infty} m_r(t) \cos \omega t\, \mathrm{d}t, \tag{10, 11}$$

where $m_r(t)$ is the relaxation modulus,

$$m_r(t) = \frac{\sigma_r(t)}{\varepsilon_0}, \tag{12}$$

and $M_0 = \bar{M}(0) = M_d(0)$ is the static modulus of elasticity. Furthermore, it can be shown that

$$m_r(t) = \frac{M_0}{2} + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\bar{M}(\mathrm{j}\omega)}{\mathrm{j}\omega}\, \mathrm{e}^{\mathrm{j}\omega t}\, \mathrm{d}\omega. \tag{13}$$

One of the basic requirement for deriving the dispersion relations is that the relaxation function be measurable: i.e. a real function which is finite for all times. The mathematical consequence of this can be seen from equations (10) and (11):

$$M_d(\omega) = M_d(-\omega) \qquad \text{and} \qquad M_l(\omega) = -M_l(-\omega). \tag{14, 15}$$

The other requirement is the causality. In order to see the consequence of the causality requirement, the relaxation function is expressed from equation (13) by simple transformations:

$$\sigma_r(t) = \frac{\varepsilon_0 M_0}{2} + \frac{\varepsilon_0}{2\pi}\int_{-\infty}^{\infty}\left[M_d(\omega)\frac{\sin \omega t}{\omega} + M_l(\omega)\frac{\cos \omega t}{\omega}\right]\mathrm{d}\omega$$

$$- \mathrm{j}\,\frac{\varepsilon_0}{2\pi}\int_{-\infty}^{\infty}\left[M_d(\omega)\frac{\cos \omega t}{\omega} + M_l(\omega)\frac{\sin \omega t}{\omega}\right]\mathrm{d}\omega. \tag{16}$$

Bearing in mind equations (14) and (15), one can see that the first and second intergrands are even and odd functions of ω, respectively; therefore

$$\sigma_r(t) = \frac{\varepsilon_0 M_0}{2} + \frac{\varepsilon_0}{\pi}\int_0^\infty M_d(\omega)\frac{\sin\omega t}{\omega}\,\mathrm{d}\omega + \frac{\varepsilon_0}{\pi}\int_0^\infty M_l(\omega)\frac{\cos\omega t}{\omega}\,\mathrm{d}\omega. \tag{17}$$

According to the causality principle: $\sigma_r(t) = 0$ for $t < 0$, and so

$$-\frac{M_0}{2} + \frac{1}{\pi}\int_0^\infty M_d(\omega)\frac{\sin\omega t}{\omega}\,\mathrm{d}\omega = \frac{1}{\pi}\int_0^\infty M_l(\omega)\frac{\cos\omega t}{\omega}\,\mathrm{d}\omega. \tag{18}$$

This equation clearly shows that the dynamic modulus– and the loss modulus–frequency functions are interrelated.

The dispersion relations are derivable from equation (18). Several forms of the relations have been developed. Here the forms of dispersion relations involving the static modulus M_0, are given [6, p. 429]:

$$M_d(\omega) = M_0 + \frac{2\omega^2}{\pi}\,\mathrm{P}\int_0^\infty \frac{M_l(y)/y}{\omega^2 - y^2}\,\mathrm{d}y, \qquad M_l(\omega) = -\frac{2\omega}{\pi}\,\mathrm{P}\int_0^\infty \frac{M_d(y)}{\omega^2 - y^2}\,\mathrm{d}y. \quad (19, 20)$$

Here y is an integration variable and P means the principal value of the integrals. According to the author's understanding Gross [13] was the first to apply the dispersion relations for the complex modulus. Since that time the relations have been referred to in several books about viscoelasticity, but only rarely in experimental practice [14].

The dispersion relations in the general integral form are not usable to investigate the frequency dependences of dynamic properties. However, beginning with equation (20) and by following a procedure described by O'Donnell *et al*. [15], simple approximate equations may be derived which are usable for our purposes. These approximate equations are local ones in the sense that they relate the value of loss modulus or loss factor at one frequency to the slope of the dynamic modulus–frequency curve at that frequency: namely,

$$M_l(\omega) \approx \frac{\pi}{2}\,\omega\,\frac{\mathrm{d}M_d(\omega)}{\mathrm{d}\omega}, \qquad \text{or} \qquad M_l(\omega) \approx \frac{\pi}{4{\cdot}6}\frac{\mathrm{d}M_d(\omega)}{\mathrm{d}[\log\omega]}, \quad (21\text{a, b})$$

and

$$\eta(\omega) \approx \frac{\pi}{2}\frac{\mathrm{d}[\log M_d(\omega)]}{\mathrm{d}[\log\omega]}. \tag{22}$$

Note that these equations have been known in the theory of viscoelasticity for quite a long time from the work of Staverman and Schwarzl [16], but no dispersion relations were used in the original derivation. Note further that the only assumption made in deriving equations (21) and (22) is that the dynamic properties do not exhibit rapid, resonance-like frequency dependences [15]. All experiments made on solid materials support this assumption.

2.1.3. *The frequency functions*

(*a*) *Qualitative conclusions*

Although equations (21) and (22) are approximate ones, the measurements, in particular those performed on polymers, always support their qualitative validity; moreover, it has been verified by some investigations too [15, 17]. These facts entitle one to use these equations in determining the characters of frequency dependences of dynamic properties.

The first plausible conclusion which can be drawn is that $M_d(\omega)$ is a monotonically increasing frequency function, because its slope is proportional to $M_l(\omega)$ which is positive for all frequencies. The latter follows from the fact that the loss modulus is related to the dissipated energy which is positive in case of real solids, namely [1, p. 10]:

$$M_l(\omega) = \frac{D(\omega)}{\pi \hat{\varepsilon}^2}, \tag{23}$$

where D denotes the energy dissipated during one cycle or harmonic vibration, and $\hat{\varepsilon}$ is the strain amplitude. It is clear from equations (21a, b) that the larger the slope of increase of the dynamic modulus–frequency curve, the larger is the dissipated energy, i.e. the damping. Furthermore, these equations highlight that the frequency dependence of the dynamic modulus, referred to as dispersion, is the direct consequence of damping; the dynamic modulus can be independent of frequency in principle only in the case of ideal elasticity: i.e. no damping.

By bearing in mind the monotonic increase of the dynamic modulus, and equations (21) and (22), moreover, the causal relaxation response, the typical characters of the frequency dependences of dynamic properties can be determined. Figure 3 shows the results in a log–log coordinate system used frequently to present experimental data. The important parts of the frequency curves are discussed as follows.

At zero frequency the loss modulus and loss factor are zero; the dynamic modulus is equal to the static modulus M_0, and the initial slopes of the log M_d–log ω functions are zero. The loss functions start to increase with increasing frequency and, according to equation (22), the slope of the dynamic modulus–frequency function, and therefore the value of M_d, start to increase too. Note that the initial slope of the dynamic modulus-frequency function in a lin–lin coordinate system may be different from zero, as can be seen from equation (21a). In contrast, the high frequency slope of $M_d(\omega)$ must be zero in any coordinate system. The essential reason for this is that the high frequency limit of the dynamic modulus, denoted by M_∞, must be finite. The latter statement follows from the facts that the high frequency limit of the complex modulus is equal to the initial ($t = 0$)

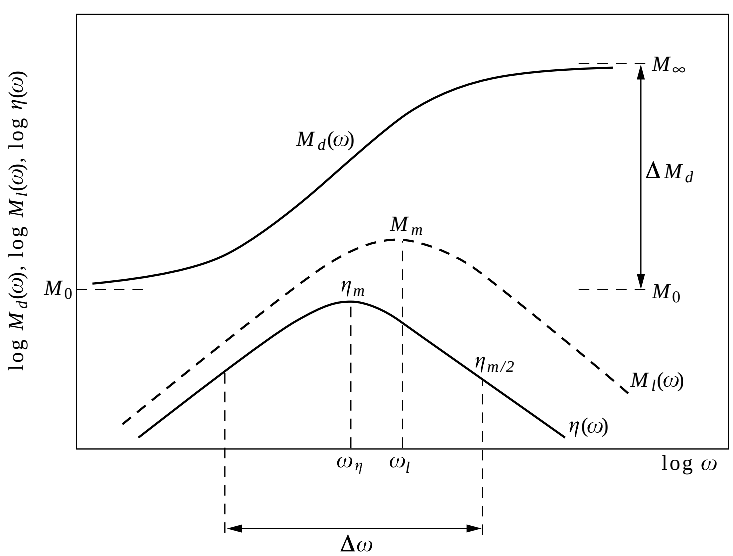

Figure 3. The typical characters of frequency dependences of any dynamic modulus, loss modulus and loss factor of real solid materials plotted in log–log system for the case of one loss maximum.

value of the $m_r(t)$ relaxation modulus [6, p. 109], and $m_r(0)$ must be finite in the case of real solids: i.e. a real relaxation response is required for deriving the dispersion relations (see section 2.1.2). From the finite high frequency limit and the monotonic increase of $M_d(\omega)$, it is clear that the high frequency slope of $M_d(\omega)$ cannot be different from zero. From the latter one can conclude that the loss functions approach zero for infinite frequencies. Furthermore, it follows from the low and high frequency behaviours of loss functions that both must pass through at least one maximum as shown in Figure 3, though in principle many maxima may exist. According to equation (22), the maxima of $\eta(\omega)$ occur at the inflexion points of the log M_d–log ω function. Furthermore, it follows from the definition of the loss factor that the maxima of $\eta(\omega)$ precede those of $M_l(\omega)$.

Frequency curves supporting qualitatively the conclusions drawn above are easy to generate for the case of one loss maximum, by calculating the integrals (10) and (11) with the assumption of exponential relaxation [18]. Moreover, the frequency dependences of dynamic properties shown in Figure 3 qualitatively are in good accord with those predicted by the model theory of viscoelasticity [6, Chap. 3]. Nevertheless, it is important to emphasize that no models have been used here for determining the frequency dependences, only the causal and real relaxation responses have been assumed without having to specify the actual process of relaxation, i.e. the damping mechanism, and no restrictions have been applied with respect to the type of deformation. Causality and relaxation are the features of real solid materials in all deformation modes, and consequently it can be stated that the frequency dependences determined here are valid for all moduli of elasticity of any solid material regardless of the damping mechanism. This last statement is in accord with that of O'Donnell *et al*. [15] that the typical characters of frequency dependences of velocity and attenuation of longitudinal waves propagating in unbounded solids are independent of the damping mechanism.

(*b*) *Quantitative conclusions*

The discussion presented above concerns only the qualitative characteristics of frequency dependences of dynamic properties, and no quantitative conclusions have been drawn. The quantitative characteristics of the frequency curves are: the static modulus M_0, the high frequency dynamic modulus M_∞, the slopes of the log M_d–log ω function at the inflexion points, and moreover the maxima (M_m, η_m) and the relevant frequencies (ω_l, ω_η) of the loss functions, the slopes of the increase and decrease of $M_l(\omega)$ and $\eta(\omega)$ below and above their maximum, respectively, the half-value bandwidth ($\Delta\omega$) of the $M_l(\omega)$ or $\eta(\omega)$ curves (see Figure 3) and their symmetry or asymmetry. These characteristics are dependent on the type of material and the type of deformation, and are determined by measurement. Nevertheless, with knowledge of equations (21) and (22), and from simple theoretical considerations, some quantitative conclusions can be drawn too for the frequency dependences of dynamic properties.

The quantitative conclusions, like the qualitative ones, start with the initial increase of the loss modulus–frequency function. The loss modulus is related to the damping, and the character of its frequency dependence is determined by the physical micromechanism of energy loss operative in the material. If one assumes that the damping in a solid can be attributed to the viscosity, as for fluids, then the loss stress σ_l related to the dissipated energy can simply be calculated by Newton's viscosity law: i.e.

$$\sigma_l(t) = \mu \frac{\mathrm{d}\varepsilon(t)}{\mathrm{d}t}, \tag{24}$$

where μ is the viscosity. The transformation of this equation into the frequency domain results in

$$\bar{\sigma}_l(\mathrm{j}\omega) = \mu \mathrm{j}\omega\bar{\varepsilon}(\mathrm{j}\omega), \tag{25}$$

from which the loss modulus is

$$M_l(\omega) = \mu\omega. \tag{26}$$

Therefore, in the case of pure viscous damping the loss modulus would start to increase linearly with increasing frequency. Viscous damping is the basic assumption of classical viscoelasticity; however, it always leads to discrepancy between the theoretical and experimental frequency curves. The main reason for the discrepancy is that "pure" viscous stresses evidently do not exist in real solid materials. The reasonable assumption is that the loss stresses in solids depend to a "lesser extent" on the rate of variation of strain than in fluids. The "lesser extent" can be expressed mathematically by reducing the order of the time derivative in equation (24). This leads to the fractional calculus intensively used recently in modelling the dynamic behaviour of real solid materials [7–9]. Consequently, instead of equation (24), one can assume that

$$\sigma_l(t) \sim \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\,\varepsilon(t), \tag{27}$$

where $0 < \alpha < 1$, and the fractional derivation $\mathrm{d}^\alpha/\mathrm{d}t^\alpha$ is defined as [8]

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\,\varepsilon(t) = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{\varepsilon(\tau)}{(t-\tau)^\alpha}\,\mathrm{d}\tau, \tag{28}$$

where is the gamma function and τ is an integration variable. The Fourier transform of equation (28) results in a simpler form [8],

$$F\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\,\varepsilon(t) = (\mathrm{j}\omega)^\alpha\bar{\varepsilon}(\mathrm{j}\omega), \tag{29}$$

where

$$(\mathrm{j}\omega)^\alpha = \cos(\alpha\pi/2)\omega^\alpha + \mathrm{j}\sin(\alpha\pi/2)\omega^\alpha, \tag{30}$$

from which one can calculate for the loss modulus that

$$M_l(\omega) \sim \sin(\alpha\pi/2)\omega^\alpha. \tag{31}$$

Consequently, this approach predicts a power function of ω^α for the initial increase of the loss modulus. Clearly the increase of the loss factor can be described by the same power function up to those frequencies where $M_d \approx M_0$. For the sake of completeness it is mentioned that the application of equation (21a) to the loss modulus given by equation (31) results in a power function too for the initial increase of dynamic modulus–frequency function, namely: $M_d(\omega) \approx M_0 + a\omega^\alpha$, where $a > 0$. Furthermore, if one assumes that the dynamic modulus approaches its high frequency limit M_∞ by a function $M_\infty(1 - b/\omega^\beta)$, where $b > 0$ and $\beta > 0$, then it follows from equation (21a) that $M_l(\omega)$ and so $\eta(\omega)$ decrease by a power function of b/ω^β above their maxima.

It is clear from the foregoing that the variation of loss properties as a function of frequency can be very weak if the exponents α and β are small. For example if $\alpha = 0{\cdot}1$, then according to equation (31), the slope of increase of $M_l(\omega)$ in a log–log system would be: $\alpha\sin(\alpha\pi/2) = 0{\cdot}0156$, which predicts an extremely weak increase with increasing frequency.

The other conclusions concern the rate of dispersion: i.e. the frequency dependence of the dynamic modulus. The approximate equation (22) shows that the slope of the increase of the dynamic modulus–frequency curve plotted in a log–log system is the largest at about that frequency where the loss factor has a maximum: namely,

$$\left.\frac{\mathrm{d}\,[\log M_d(\omega)]}{\mathrm{d}\,[\log \omega]}\right|_{max} \approx \frac{2}{\pi}\,\eta_m. \tag{32}$$

The larger the maximum of the loss factor, the larger is the slope of the log M_d–log ω. From this and the monotonic increase of $M_d(\omega)$, it can be concluded that the difference between M_0 and M_∞, characterizing the total dispersion of the dynamic modulus, is proportional to the loss maximum: i.e.

$$M_\infty - M_0 = \Delta M_d \sim M_m \sim \eta_m. \tag{33}$$

The increase of the dynamic modulus in a frequency range, say of ω_1 to ω_2, can be estimated from knowledge of the loss factor by means of equation (22). For this, one takes the integral of equation (22) between ω_1 and ω_2, resulting in

$$\frac{2}{\pi}\int_{\omega_1}^{\omega_2} \eta(\omega)\,\mathrm{d}\log\omega \approx \log M_d(\omega_2) - \log M_d(\omega_1). \tag{34}$$

Furthermore, if the frequency dependence of $\eta(\omega)$ is assumed to be negligible between ω_1 and ω_2, then

$$\log M_d(\omega_2) - \log M_d(\omega_1) \approx \frac{2}{\pi}\,\eta(\log\omega_2 - \log\omega_1), \tag{35a}$$

or

$$\frac{M_d(\omega_2)}{M_d(\omega_1)} \approx \left(\frac{\omega_2}{\omega_1}\right)^{2\eta/\pi}. \tag{35b}$$

In this way one can predict that if the loss is low, say $\eta = 0{\cdot}01$ and $0{\cdot}1$, then the increase of the dynamic modulus is no more than 5 and 55%, respectively, in a frequency range covering three decades. However, if the loss is high, say $\eta = 1{\cdot}0$, then $M_d(\omega_2)/M_d(\omega_1) \approx 80$; i.e. the dynamic modulus increases by about two orders of magnitude in this frequency range.

By bearing in mind the results of quantitative conclusions, the theoretically possible frequency dependences of dynamic modulus and loss factor can be drawn as shown in Figure 4 for the cases of high loss (e.g. $\eta_m \approx 1{\cdot}0$) and low loss (e.g. $\eta_m \approx 0{\cdot}01$) with the assumption of one loss maximum. Moreover, it has been assumed that the low loss is accompanied by small values of the exponents, say $\alpha \approx \beta \approx 0{\cdot}1 \ldots 0{\cdot}2$. For the sake of clarity and interest the frequency curves are drawn in a lin–lin system instead of a log–log one.

Further examples for theoretically possible frequency dependences of dynamic properties predictable by the dispersion theory are shown in Figure 5. The frequency curves of Figure 5(a) are based on the assumption that there are two loss peaks in close proximity to each other, and both maxima of the loss factor are relatively high, (e.g. $\eta_m \approx 1{\cdot}0$) resulting in strong frequency dependence of the dynamic modulus. In contrast to Figure 5(a), all loss peaks seen in Figure 5(b) are assumed to be low (e.g. $\eta_m < 0{\cdot}01$), and therefore

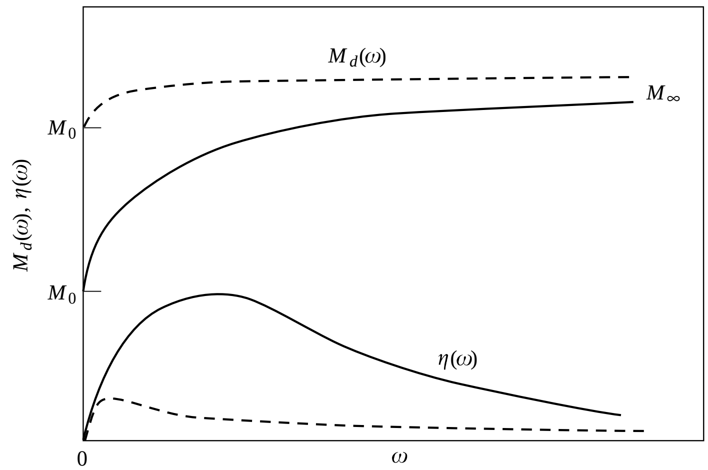

Figure 4. Theoretically possible frequency dependences of dynamic modulus and loss factor plotted in lin–lin coordinate system for the case of one loss maximum. ——, High loss; – – –, low loss.

the dynamic modulus can be regarded as practically frequency independent over a wide range. Note that the loss peaks are the manifestation of different mechanisms of damping operative in the solid material.

2.2. COMPLEX POISSON'S RATIO

The Poisson's ratio is defined by the ratio of lateral strain to axial strain. It is plausible that the lateral strain lags behind the axial one due to material damping. Therefore, the strain to strain ratio, like the stress to strain ratio, can be characterized by a complex number referred to as the complex Poisson's ratio. The general definition of the complex Poisson's ratio, like the complex modulus, is

$$\bar{v}(j\omega) = \frac{\overline{\varepsilon_y}(j\omega)}{\overline{\varepsilon_x}(j\omega)} = v_d(\omega) - jv_l(\omega) = v_d(\omega)[1 - j\eta_v(\omega)], \tag{36}$$

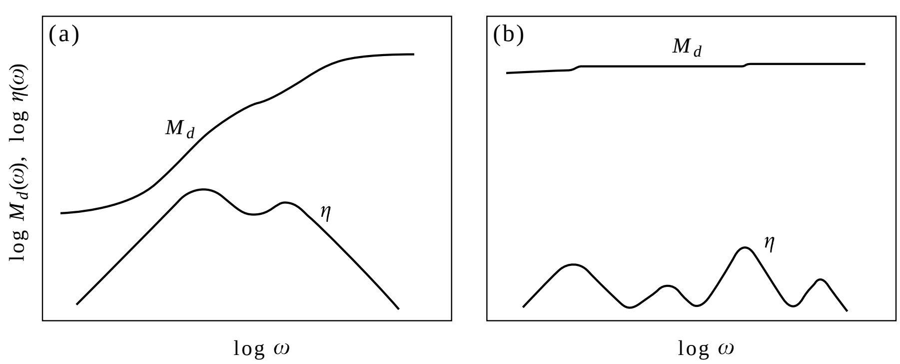

Figure 5. Theoretically possible frequency dependences of dynamic modulus and loss factor plotted in log–log coordinate system. (a) High loss material with two loss peaks. (b) Low loss material with several loss peaks.

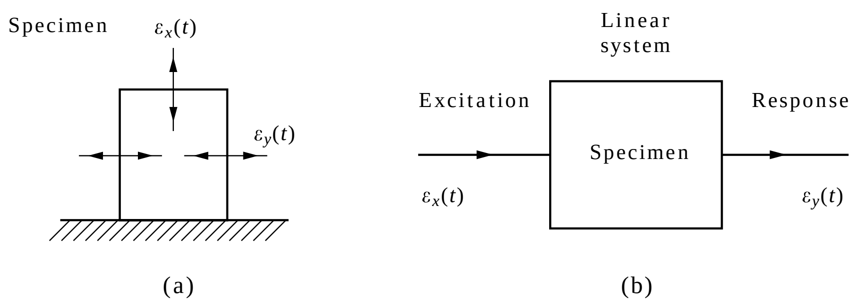

Figure 6. A material specimen (a) for low amplitude excitation can be regarded as a linear physical system (b) relating the lateral strain to the axial strain.

where $\overline{\varepsilon_y}(j\omega)$ and $\overline{\varepsilon_x}(j\omega)$ are the Fourier transforms of the lateral strain– and axial strain–time functions, respectively, ν_d is the dynamic Poisson's ratio, ν_l is the loss part and η_ν is the relevant loss factor:

$$\eta_\nu(\omega) = \frac{\nu_l(\omega)}{\nu_d(\omega)}. \tag{37}$$

Note that the minus sign of the imaginary part is the consequence of the lag of lateral strain, being in the numerator of equation (36), behind the axial strain. In the case of the complex modulus, the strain being in the denominator of equation (1), lags behind the stress.

The complex Poisson's ratio describes the strain to strain ratio in the frequency domain. The frequency dependences of the dynamic Poisson's ratio and the relevant loss part cannot be predicted from simple theoretical considerations. Nevertheless, if one realizes that the complex Poisson's ratio can be interpreted as the frequency response function of a linear system, the system may be a material specimen as shown in Figure 6, then the whole procedure outlined in the previous section can be applied to determine the characters of the frequency dependences in question.

Therefore, one can start the procedure again in the time-domain by investigating the lateral strain response to an axial strain step function excitation having magnitude of ε_{x0} (see Figure 7). It seems to be logical and has been proved by experiments [19] that the time history of lateral strain obeys a creep function $\varepsilon_{yc}(t)$ as illustrated in Figure 7. It can be shown by transforming the causal and real creep response into the frequency domain that the real and imaginary parts of the complex Poisson's ratio are linked through the dispersion relations, namely

$$\nu_d(\omega) = \nu_0 - \frac{2\omega^2}{\pi}\,\mathrm{P}\int_0^\infty \frac{\nu_1(y)/y}{\omega^2 - y^2}\,\mathrm{d}y, \qquad \nu_l(\omega) = \frac{2\omega}{\pi}\,\mathrm{P}\int_0^\infty \frac{\nu_d(y)}{\omega^2 - y^2}\,\mathrm{d}y. \tag{38, 39}$$

Furthermore, it is evident that the relevant forms of the approximate equations (21) and (22) hold true:

$$\nu_l(\omega) \approx -\frac{\pi}{2}\,\omega\,\frac{\mathrm{d}\nu_d(\omega)}{\mathrm{d}\omega}, \qquad \text{or} \qquad \nu_l(\omega) \approx -\frac{\pi}{4{\cdot}6}\,\frac{\mathrm{d}\nu_d(\omega)}{\mathrm{d}\,[\log\omega]} \tag{40a, b}$$

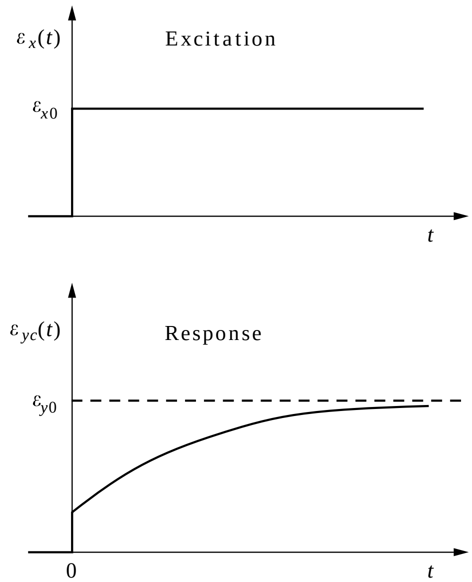

Figure 7. The lateral strain response obeys creep function to excitation of axial strain step function.

and

$$\eta_\nu(\omega) \approx -\frac{\pi}{2}\frac{\mathrm{d}[\log \nu_d(\omega)]}{\mathrm{d}[\log \omega]}. \tag{41}$$

It follows from all approximate equations that the slope of $\nu_d(\omega)$ is negative ($\nu_l > 0, \eta_\nu > 0$), and therefore the dynamic Poisson's ratio of any real solid material, in contrast to the dynamic modulus, decreases monotonically with increasing frequency. The larger the loss, the larger is the rate of decrease of $\nu_d(\omega)$. Equations (40) and (41) enable one to determine the typical characters of frequency dependence for $\nu_d(\omega)$, $\nu_l(\omega)$ and $\eta_\nu(\omega)$, by following the train of thought of section 2.1.3. The frequency functions thus determined are shown in Figure 8 for the case of one loss maximum, though in principle many maxima

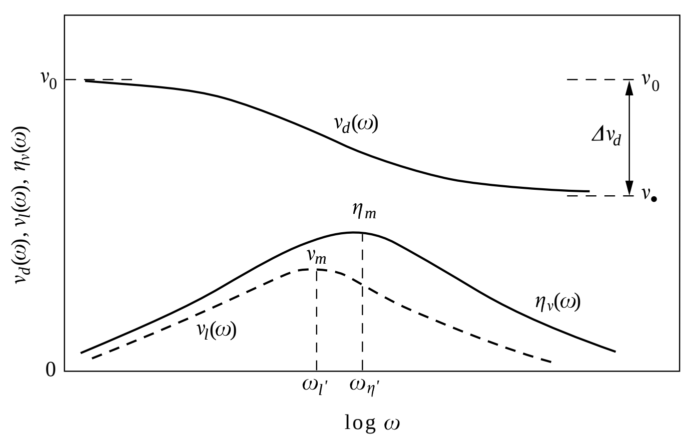

Figure 8. The typical characters of frequency dependences of dynamic Poisson's ratio and the relevant loss functions plotted in lin–log system for case of one loss maximum.

may exist. Note that the frequency curves are drawn in a lin–log coordinate system instead of a log–log one because of the small rate of decrease of $\nu_d(\omega)$ discussed later on. It can be seen that the characters of the frequency dependences of the loss functions $\nu_l(\omega)$ and $\eta_\nu(\omega)$, are the same as those of the loss functions of the complex moduli, with the exception that the maximum of $\nu_l(\omega)$ precedes that of $\eta_\nu(\omega)$.

The quantitative characteristics of the frequency functions $\nu_d(\omega)$, $\nu_l(\omega)$ and $\eta_\nu(\omega)$ cannot be predicted, but are known from measurements. The only plausible prediction is that the rate of decrease of dynamic Poisson's ratio, likewise the rate of increase of dynamic modulus, is proportional to the loss maximum: i.e.

$$\nu_0 - \nu_\infty = \Delta\nu_d \sim \nu_m \sim \eta_m. \tag{42}$$

The decreasing character of the dynamic Poisson's ratio–frequency function and the existence of one loss maximum, are in accord with a previous theoretical investigation based on the assumption of exponential creep [18], moreover, these are predictable by means of the model theory of linear viscoelasticity [6, pp. 528–532]. More than one maximum for $\nu_l(\omega)$ and $\eta_\nu(\omega)$, however, has never been predicted theoretically.

3. EXPERIMENTAL EVIDENCE

3.1. COMPLEX MODULI OF ELASTICITY

In this section experimental findings will be compared with the theoretical predictions, and interpreted in the light of dispersion relations. Although the relations concern all solid materials and any frequency range, here the engineering materials are considered with special respect to those which are used for or may play a role in sound and vibration control. Therefore, the material behaviour in the audio frequency range is in focus, but the variations of dynamic properties in a frequency range as wide as possible are considered to illustrate all important details.

(*a*) *Rubbers and rubberlike materials*

Of the engineering materials known up to now, rubbers and rubberlike materials, the elastomers exhibit the largest damping in shear and tension-compression deformation. The maximum of loss factors in question, η_G and η_E, is usually around 1·0. This high loss explains the strong frequency dependence experienced for the dynamic shear and Young's moduli; their increase extends to two to four orders of magnitude in a frequency range covering 6 to 10 decades. Dozens of experiments made on rubbers and rubberlike materials to determine the variations of their dynamic properties for a wide frequency range, are in good accord with the predictions of the dispersion relations. By way of example the frequency dependences of shear dynamic properties are given in Figure 9 for a filled natural rubber having a loss maximum well above the audio frequency range [20]. The loss peaks of other rubberlike materials developed for vibration damping occur in the audio range [2, 3]. Figure 9 shows one loss peak; more than one peak is characteristic of polymeric blends [2]. The slope of the initial increase of the loss modulus–frequency curve is about 0·2. This slope implies that the damping mechanism even in this lossy rubber does not obey the viscosity law.

The dynamic Young's modulus and its loss factor of rubbers do not differ appreciably from the relevant dynamic properties in shear deformation. On the contrary, the dynamic bulk and longitudinal moduli and the relevant loss factors of the nearly incompressible rubbers are much larger and smaller than the dynamic shear modulus and shear loss factor, respectively. The bulk and longitudinal loss factor peaks are of the order of 0·01 [21, 22].

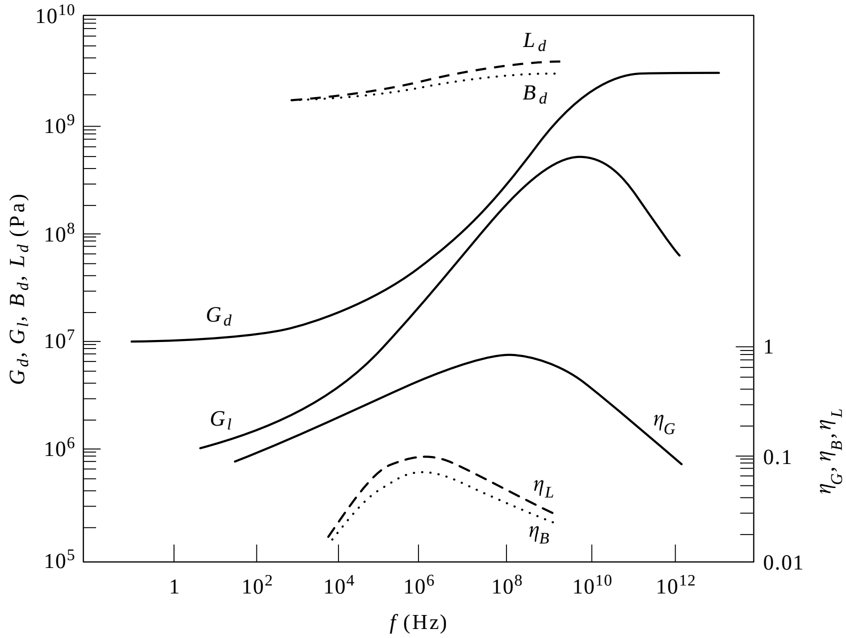

Figure 9. Frequency dependences of dynamic shear, bulk and longitudinal moduli, and the relevant loss functions of rubbery materials. ——, Shear dynamic properties of a natural rubber filled with carbon black at 0°C. (Data from Payne and Scott [20].) Bulk (. . .) and longitudinal (- - -) dynamic properties of a styrene-butadiene rubber at 20°C. (Data from Wada *et al*. [21]. The loss factors η_B and η_L have been calculated from the published values of the relevant loss and dynamic moduli.)

The frequency dependences of dynamic bulk and longitudinal moduli and the relevant loss factors found for a synthetic rubber are given in Figure 9 too by way of illustration [21]. The weak increase of these dynamic moduli with increasing frequency is the consequence of the low losses in bulk and longitudinal deformation modes. Figure 9 supports the predictions of the dispersion relations: namely that the higher the loss factor, the larger is the slope of the increase of dynamic modulus-frequency curve. Moreover, it is clear that the characters of frequency dependences of dynamic properties are the same in different deformation modes.

(*b*) *Rigid plastics*

It is known that the damping of rigid plastics in shear and tension-compression deformation is much smaller than that of the rubberlike materials. The maximum of loss factors η_G and η_E is, in general, about or smaller than 0·1. The maximum of bulk loss factor is of the order of 0·01. These low losses predict weak frequency dependence for all dynamic moduli. Frequency dependences of dynamic shear, Young's and bulk moduli, and the relevant loss factors measured for polymethylmethacrylate (Perspex), are shown in Figure 10 by way of example [1, pp. 181, 183, 187]. Similar frequency curves have been found for other rigid plastics [1]. All of them are in accord with the predictions of the dispersion relations.

The slopes of the loss factor-frequency curves are even smaller than those of the natural rubber shown in Figure 9. These small slopes indicate that the damping mechanisms in the rigid plastics is rather far from the viscous one; however, it is usual to assume the latter for all kinds of polymeric materials.

(*c*) *Stiff structural materials*

In contrast to the organic polymers much fewer, less detailed experimental data are available on the dynamic properties of stiff structural materials such as metals, concrete and wood. The damping in these materials is small; the values of the loss factors η_G and η_E are of the order of 0·001 to 0·01 [23]. Furthermore, the loss factors and the dynamic moduli of elasticity for these structural materials are usually believed to be frequency independent; however, this belief violates the dispersion relations, i.e. the causality. The reason for belief of "frequency independence" can be found partly in the nature of these materials, the frequency range and the accuracy of measurements of their dynamic properties; and can be well explained by means of the dispersion relations.

With knowledge of the low loss the "frequency independence" measured for dynamic moduli is not a surprise (see Figure 4). Furthermore, the damping mechanism in most of these structural materials is certainly far from the viscous one, and therefore the frequency dependences of their loss factor may be so weak (see Figure 4) that they could be determined only by very precise, detailed measurements made in a wide frequency range. In contrast, the belief in frequency independent damping, called solid damping, structural damping or hysteretic damping, is essentially based on some early measurements made in such a narrow frequency range (e.g. 2–200 Hz [24]) that the variations of loss factor would be difficult to determine for these materials, Actually, the frequency dependences of damping and the existence of damping peaks in metals like those illustrated in Figure 5(b), have been known for quite a long time [7, 25]. Moreover, subsequent measurements performed on non-metallic structural materials such as concrete and wood in a wider frequency range, show the definite, albeit weak variation of their loss factors [26]. Notwithstanding, in order to reveal the frequency dependences of loss factor for these materials, predicted by the dispersion relations, more refined measurements in a much

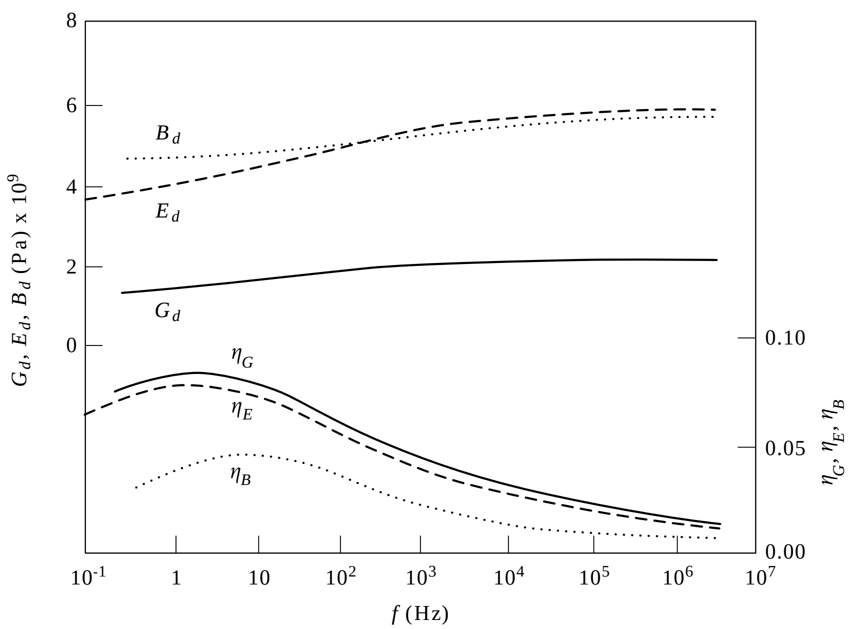

Figure 10. Frequency dependences of dynamic shear (—), Young's (- - -) and bulk (. . .) moduli and the relevant loss factors of a rigid plastic (polymethylmethacrylate) at 21°C. (Data from Read and Dean [1, pp. 181, 183, 187].)

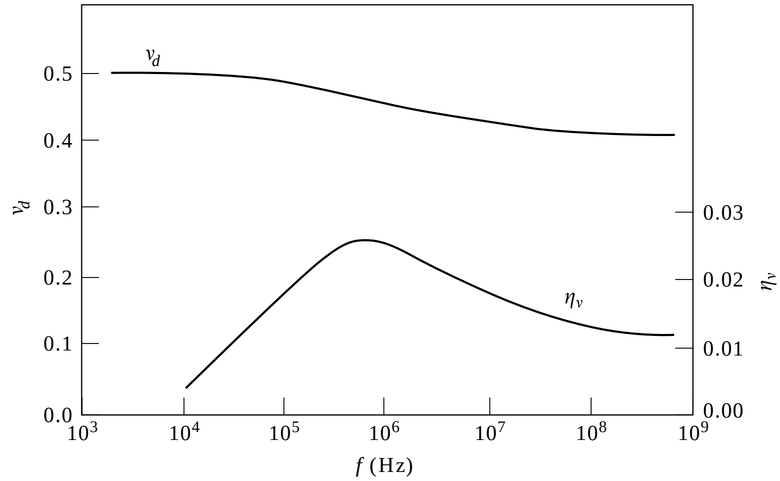

Figure 11. Frequency dependences of dynamic Poisson's ratio and the relevant loss factor of the styrene-butadiene rubber mentioned in Figure 9, at 20°C. (The frequency curves have been calculated from the values of dynamic bulk and longitudinal moduli and their loss factors measured by Wada *et al*. [21].)

wider frequency range should be made. The better understanding of the damping mechanism operating in the non-metallic structural materials would be the other benefit of such measurements.

3.2. COMPLEX POISSON'S RATIO

In contrast to the complex modulus data, only a few published experimental results can be found with respect to the frequency dependence of the complex Poisson's ratio, and the data concerns only polymeric materials. According to the dispersion relations, the greatest frequency dependences are to be expected for lossy polymers, the elastomers. In the absence of direct measurements for these materials, the author has calculated the values of ν_d and η_ν for a styrene-butadiene rubber, based on published experimental values of the dynamic bulk and longitudinal properties, given in Figure 9 [21]. The results supporting the theoretical statements are shown in Figure 11. It can be seen that the maximum value of $\eta_\nu(\omega)$ for this rubber, at room temperature, occurs in the ultrasonic frequency range, as is the case for the maximum of loss factors $\eta_B(\omega)$ and $\eta_L(\omega)$ as seen in Figure 9, so the decrease of $\nu_d(\omega)$ is not great in the audio frequency range. Other measurements [2, p. 77] performed on a vibration damping elastomer show a decrease of the dynamic Poisson's ratio which occurs in the audio frequency range. Unfortunately, no data appears to have been published on the loss factor η_ν for this material. Other experiments on a rigid plastic (polymethylmethacrylate, referred to in section 3.1) support the theoretically predicted frequency dependences for both ν_d and η_ν [1, p. 186]. More than one maximum of $\eta_\nu(\omega)$ has not been observed, but their possible existence can be explained for polymer blends.

It is worth mentioning that the weak dispersion of the dynamic Poisson's ratio shown in Figure 11 is related to the low values of η_ν, the maximum of which ($\sim$0·025) is much smaller than the shear loss factor of rubber (see Figure 9). It is probable that η_ν is even lower for stiff structural materials, so that their dynamic Poisson's ratios, like the dynamic moduli, can be regarded as practically frequency independent. Nevertheless, the latter has never been investigated experimentally.

4. USEFULNESS OF DISPERSION RELATIONS

The great advantage of the dispersion relations is in their generality, and thus they can help one to understand dynamic behaviour of solid material of any kind, in different deformation modes. Moreover, the relations enable one to predict material behaviour and interpret experimental data. Therefore, the consequences of this theory can be useful in both materials engineering, measurements of dynamic properties and in modelling dynamic bevaviour.

(*a*) *Materials engineering*

Modern damping technology requires materials with loss factors as high as possible. With knowledge of the dispersion relations the realizable value of the loss factor can be estimated, the principal way of increasing the loss factor can be found, and the bounds of the increase can be clarified.

The dispersion relations state that the loss factor in a frequency range is proportional to the rate of modulus increase in that range. This statement is formulated by equation (35b), which offers a rough estimation of the η_m maximum of loss factor. Namely, if one assumes that the frequency range of ω_1 to ω_2 is needed approximately for the dynamic modulus to increase from about M_0 up to M_∞, then

$$\eta_m \approx \frac{\pi}{2}\frac{\log\,[M_d(\omega_2)/M_d(\omega_1)]}{\log\,(\omega_2/\omega_1)} \approx \frac{\pi}{2}\frac{\log\,(M_\infty/M_0)}{\log\,(\omega_2/\omega_1)}\,. \tag{43}$$

By means of equation (43) one can explain the reason why the loss factors, η_G and η_E, for the effective elastomeric damping materials have a maximum of around 1·0, and why no loss factors larger than 2·0 to 3·0 have been realized [2, 3].

It is known that the high frequency limit of the dynamic shear (and Young's) modulus, known as the glassy modulus, is about the same ($\sim 10^9$ Pa) for all elastomers. The low frequency modulus, the static modulus, is dependent very much on the type of material, the filler content, etc., and it may be as low as 10^5 Pa. The width of the frequency range needed for the total increase of dynamic modulus is at least six decades (see Figure 9). With these data equation (43) gives $\eta_m \approx 1$, which seems to explain the experimental findings. In principle there are no bounds to increasing the loss factor maximum; it could be done by either increasing the glassy modulus, or decreasing the static modulus. However, the glassy modulus of elastomeric materials, known up to now, has an upper theoretical bound determined by the molecular structure. The decrease of the static modulus has not only theoretical, but practical bounds too, because a certain stiffness is required for engineering applications.

Similarly, the dynamic behaviour of composite materials can be explained and the principal way of improving their damping properties can be shown. Plunkett [27] has raised the following question: what is the reason that, in spite of a lot of efforts, the damping properties of polymer matrix composites, in the linear range, have not been significantly improved? Is there any physical limitation of it? The answer is definitely: yes, the limitation follows from the dispersion relations.

It is known that the damping properties of composites are essentially dependent on the matrix material, which is a rigid plastic [28]. The maximum of the loss factor, e.g. η_E, of rigid plastics is usually smaller than 0·1, resulting in weak dispersion of the dynamic Young's modulus (see Figure 10). As a result of the fibres, particles added into the polymer matrix, the "stiffness" of the material, i.e. the static modulus of elasticity, increases. It is evident that the high frequency dynamic modulus increases too, but there is no physical reason that this increase should differ considerably from the increase of the static modulus.

Consequently, the dispersion of the dynamic modulus, and therefore the maximum of the loss factor, do not differ appreciably from those of the polymer matrix. The key to improving the damping properties of composites is, therefore, in the considerable increase of dispersion of the dynamic modulus. Moreover, the increase of the high frequency modulus is the feasible way to increase the dispersion, because the static modulus must not be reduced to keep the stiffness properties required, as illustrated in Figure 12. If one could find how to increase considerably the high frequency modulus—by finding either a new matrix material or creating a new composite structure, then that composite material could be lossy as well as have high stiffness.

(*b*) *Dynamic properties measurements*

The dynamic properties are determined by measurement. The knowledge of consequences of dispersion relations gives a powerful tool in planning the measurement, and moreover in interpreting and evaluating the measured data. With knowledge of the approximate value of the loss factor (it can easily be determined by a simple preliminary measurement of resonance type), one can decide on what detailed measurements are necessary to perform for determining the frequency dependences of dynamic properties. For example if the loss factor is low, say $\eta < 0{\cdot}1$, then it is evidently useless to search for the frequency dependences of the dynamic modulus even in a wider frequency range; however, it is frequently done. Furthermore, the consequences of the dispersion relations may help one to interpret the measurement results in a case of their large scatter when the frequency dependences could hardly be established. Finally, if one seeks the mathematical form of the measured frequency dependences for use in the vibration calculus without causality problem, then the dispersion relation, the general integral form, has to be taken into account in fitting the frequency curves to experimental data. Good examples for this are given in the work of Kennedy and Tomlinson [14].

(*c*) *Modelling dynamic behaviour*

A lot of efforts have been and are being made in viscoelasticity, acoustics, structural dynamics, etc., to find models describing the dynamic behaviour of solid materials. The

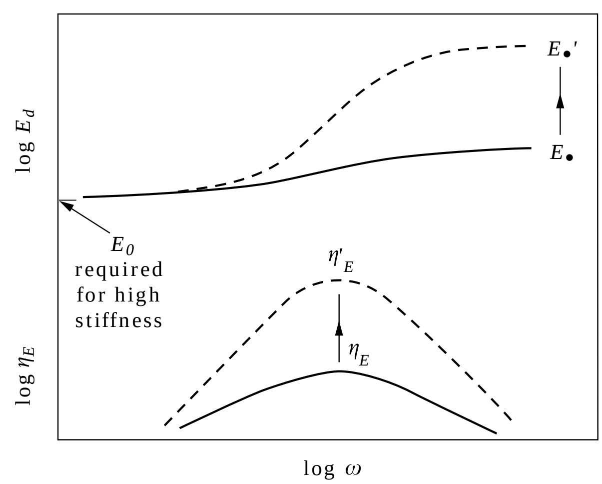

Figure 12. The key of increasing the damping of composites is to have considerable dispersion of the dynamic modulus. In order to increase the dispersion, the high frequency modulus should be increased without decreasing the static modulus to keep the stiffness properties required. ——, Low loss composite; – – –, high loss composite.

models are evaluated by different methods, e.g. by using the principles of thermodynamics or by comparing the model behaviour with experimental data either in the frequency- or the time-domain. The dispersion relations offer an exact method for this evaluation; a model aiming to describe dynamic behaviour of real solids, should satisfy the dispersion relations.

In this way one can easily find the reason for the non-causal behaviour of the hysteretic damping model, which has led to a long-standing discussion from 1970, especially in this journal. In the hysteretic model one assumes that the loss factor, and implicitly the dynamic modulus, are independent of frequency [2]. In contrast, according to the dispersion relations, the loss factor and the dynamic modulus of a real, and therefore causal solid material cannot be constant together in even a narrow frequency range! The same applies to the loss factor and dynamic stiffness of passive mechanical structures too.

5. CONCLUSIONS

This paper has dealt with the frequency dependence of the complex moduli of elasticity and the complex Poisson's ratios of real solid materials. It has been shown that the typical characters of the frequency dependence of both the complex moduli and the complex Poisson's ratios can be determined by transforming the causal and real relaxation and creep responses, respectively, from the time-domain into the frequency-domain, without having to specify the precise processes governing relaxation and creep. The transformation results in the dispersion relations which link the dynamic elastic and damping properties. The simplified, approximate, local forms of the dispersion relations, together with some simple theoretical considerations were used to determine the frequency dependences. The advantage of the method described in this paper is that no restrictions need be applied with respect to either the type of deformation (shear, extension, etc.) or the type of material, so that the conclusions are of general nature. The main conclusions are summarized as follows.

(a) All the dynamic moduli (shear, Young's etc.) of real solid materials increase monotonically with increasing frequency, and the high frequency limit is finite.

(b) The dynamic Poisson's ratios of real solid materials decrease monotonically with increasing frequency.

(c) All loss functions pass through at least one maximum with respect to frequency, though in principle many loss maxima may exist.

(d) The larger the loss maximum, the larger the rate of increase of dynamic modulus and the rate of decrease of dynamic Poisson's ratio, respectively.

(e) These characters of the frequency dependences are physically inevitable regardless of the damping mechanism operating in solid materials.

The frequency dependence of the dynamic properties is strongest for organic polymers with high loss factors (rubbers and other elastomers). In contrast, the rate of variation of the dynamic moduli and loss factors is much lower for most structural materials having low loss, and can be neglected over fairly wide frequency ranges for many practical applications. The consequences of the dispersion relations can effectively be used in materials engineering to predict dynamic behaviour and the realizable loss factor of vibration damping materials and composites. Furthermore, the dispersion relations may be useful in interpreting and evaluating the results of measurements of dynamic properties, and in modelling the dynamic behaviour of materials.

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