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STABILITY ANALYSIS OF DAMPED SDOF SYSTEMS WITH TWO TIME DELAYS IN STATE FEEDBACK

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This paper presents the stability analysis of linear, damped SDOF vibration control systems with two time delays, one in the displacement feedback and the other in the velocity feedback. First, a sufficient and necessary algebraic criterion is proved to check the system stability independent of time delays. According to this criterion, all possible combinations of the feedback gains that guarantee the delay-independent stability are given. Then, the effect of the feedback gains on the system stability is discussed when the time delays are finite. The most dangerous case is found when the time delay in the displacement feedback is much longer than that in the velocity feedback.

1. INTRODUCTION

The active control for attenuating the excessive vibration of mechanical systems has become more and more popular in engineering. One of the limits to the performance and application of this technique is the unavoidable time delays in controllers and actuators, particularly in hydraulic actuators for active suspensions of vehicles, active tendons of tall buildings, etc. These time delays, albeit very short, deteriorate the control performance or even cause the instability of the system, because the actuators may input energy at the exact moment when the controlled system does not need it.

Consider a SDOF system with delayed state feedback, the motion of which yields the following dimensionless difference-differential equation

$$\ddot{x}(t) + 2\zeta\dot{x}(t) + x(t) = f(t) + s_1 x(t-\tau_1) + s_2\dot{x}(t-\tau_2), \tag{1}$$

where $\zeta \geqslant 0$ is the damping ratio as usual, $f(t)$ is the excitation, s_1 and s_2 are the feedback gains, $\tau_1 \geqslant 0$ and $\tau_2 \geqslant 0$ are the time delays in the paths of displacement feedback and velocity feedback, respectively. This system serves as a simple, but widely used model for active suspensions and active tendons in vibration control [1, 2].

To check the asymptotic stability of the steady state motion $x(t)$, it is usual to study the variational equation of equation (1) which governs the small variation $\Delta x(t)$ near $x(t)$

$$\Delta\ddot{x}(t) + 2\zeta\Delta\dot{x}(t) + \Delta x(t) = s_1\Delta x(t-\tau_1) + s_2\Delta\dot{x}(t-\tau_2). \tag{2}$$

Substituting the candidate solution $\Delta x(t) = a\,e^{\lambda t}$ into equation (2) yields a characteristic equation

$$D(\lambda, \tau_1, \tau_2) \equiv \lambda^2 + 2\zeta\lambda + 1 - s_1 \exp(-\lambda\tau_1) - s_2\lambda \exp(-\lambda\tau_2) = 0. \tag{3}$$

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Given the time delays τ_1 and τ_2, the steady state motion of system (1) is asymptotically stable if and only if all the roots of equation (3) have negative real parts. When system (1) is asymptotically stable for arbitrary time delays, it is said to be delay-independent stable.

If there is no time delay in the state feedback, i.e., $\tau_1 = \tau_2 = 0$, equation (3) becomes a quadratic equation in λ

$$D(\lambda, 0, 0) = \lambda^2 + (2\zeta - s_2)\lambda + (1 - s_1) = 0. \tag{4}$$

One can readily write out the stability condition

$$s_1 < 1, \qquad s_2 < 2\zeta. \tag{5}$$

Except for this trivial case, equation (3) is transcendental. It is impossible to check the system stability by solving equation (3) for the infinite number of roots. Thus, it is not an easy task to give simple stability criteria for the retarded differential equation (1).

In general, the criteria of delay-independent stability are simpler than those of delay-dependent stability. Hence, they have received much attention over the past decades [3, 4]. As the simplest case, the systems with single time delay have been intensively studied and the delay-independent stability criteria in terms of pure mathematical parameters have been given in [3, 5]. Yet, fewer successful studies have been made for the systems with multiple time delays and no practical stability criterion has been available. For instance, the sufficient condition given in [6] requires very tedious exponential matrix computation. The concise criteria proposed in [7] are not applicable to vibrating systems.

In practice, the stability independent of time delays may be excessively restrictive. As a matter of fact, most controlled systems are stable only for bounded time delays. Hsu [8] first suggested several numerical methods to study the relationship of a single time delay and a system parameter when the system undergoes instability. Stepan [9] proposed a set of stability criteria for undamped systems. Chen [10] gave a number of algorithms to compute the maximal delay interval, wherein a controlled system is stable. However, if the system is damped or both delayed displacement and delayed velocity are taken as feedback, no simple criterion has been available for stability test.

A practical problem in controller design is how to select appropriate feedback gains s_1 and s_2 such that the controlled system is stable if there exist time delays in the controller and actuators. Sometimes, the feedback gains s_1 and s_2 may have already been designed according to a control strategy, say, LQG, but the time delays in controller and the actuators were not taken into account in the previous design. One may want to know whether the controlled system is stable and robust with respect to the variation of the feedback gains. However, the publications mentioned above dealt with the stability criteria in terms of pure mathematical parameters, rather than the feedback gains.

The aim of this paper is to find practical criteria of delay-independent stability for a damped vibrating system (1) with two time delays in state feedback. In section 2, a sufficient and necesssary algebraic condition of delay-independent stability is derived. Then, an equivalent condition in terms of feedback gains s_1 and s_2 is discussed in section 3 and the region of delay-independent stability in the plane of (s_1, s_2) is given. In section 4, the stability of the systems with two finite time delays is analyzed. Finally, some concluding remarks are made in section 5.

2. STABILITY CRITERION INDEPENDENT OF DELAYS

The stability of system (1) is independent of time delays if equation (3) has no root with non-negative real part and $D(i\omega, \tau_1, \tau_2) = 0$ has no real root ω for any given τ_1 and τ_2. It is obvious that the critical condition $D(i\omega, \tau_1, \tau_2) = 0$ results in

$$|1 - \omega^2 + 2i\zeta\omega| = |s_1 \exp(i\omega\tau_2 - i\omega\tau_1) + is_2\omega|, \tag{6}$$

namely

$$(1 - \omega^2)^2 + (2\zeta\omega)^2 = s_1^2 + (s_2\omega)^2 + 2s_1s_2 \sin(\omega\tau_2 - \omega\tau_1). \tag{7}$$

Conversely, if ω, τ_1 and τ_2 satisfy equation (7), thre exists a non-negative number $\theta \in R$ such that $\theta + \tau_1 - \tau_2 \geqslant 0$ and

$$1 - \omega^2 + 2i\zeta\omega = \exp(-i\omega\theta)[s_1 \exp(i\omega(\tau_2 - \tau_1)) + is_2\omega], \tag{8a}$$

or $\theta + \tau_2 - \tau_1 \geqslant 0$ and

$$1 - \omega^2 + 2i\zeta\omega = \exp(-i\omega\theta)\{s_1 + is_2\omega \exp[i\omega(\tau_1 - \tau_2)]\}. \tag{8b}$$

This leads to $D(i\omega, \theta - \tau_2 + \tau_1, \theta) = 0$ or $D(i\omega, \theta, \theta - \tau_1 + \tau_2) = 0$. Thus, for given time delays $\tau_1 \geqslant 0$ and $\tau_2 \geqslant 0$, $D(i\omega, \tau_1, \tau_2) = 0$ has no real root ω if and only if equation (7) has no real root ω. Moreover, equation (7) can be rewritten as

$$\omega^4 + (4\zeta^2 - 2 - s_2^2)\omega^2 + 1 - s_1^2 = 2s_1s_2\omega \sin(\omega\tau_2 - \omega\tau_1). \tag{9}$$

As both sides of equation (9) are even functions in variable ω, it is sufficient hereafter to study the case of $\omega \geqslant 0$ only.

2.1. EQUAL TIME DELAYS

When $\tau_1 = \tau_2 = \tau$, equation (9) becomes

$$\omega^4 + p\omega^2 + q = 0, \tag{10}$$

where

$$p \equiv 4\zeta^2 - 2 - s_2^2, \qquad q \equiv 1 - s_1^2. \tag{11}$$

Equation (10) has four roots in the form

$$\omega_{1,2} = \sqrt{\tfrac{1}{2}(-p \mp \sqrt{p^2 - 4q})}, \qquad \omega_{3,4} = -\sqrt{\tfrac{1}{2}(-p \mp \sqrt{p^2 - 4q})}. \tag{12}$$

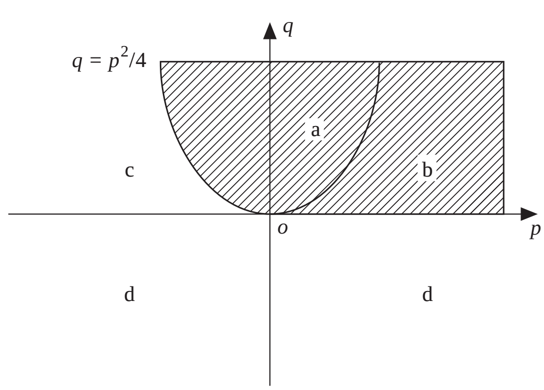

Figure 1. Region of delay-independent stability in the (p, q) plane when $\tau_1 = \tau_2$.

The number of real roots depends on the combination of p and q as follows.

(a) $p^2 - 4q < 0$: none of the roots is real;

(b) $p^2 - 4q \geqslant 0$, $p > 0$, $q > 0$: none of the roots is real;

(c) $p^2 - 4q \geqslant 0$, $p < 0$, $q \geqslant 0$: all of the roots are real;

(d) $p^2 - 4q \geqslant 0$, $q < 0$: two roots ω_2 and ω_4 are real.

These cases are shown in the (p, q) plane in Figure 1, where the shaded region represents the parameter combinations that guarantee the system stability independent of time delays. From Figure 1, the conditions (a) and (b) can be expressed in a simpler form

$$p > 0, \quad q > 0, \qquad \text{or} \qquad p < 0, \quad p^2 - 4q < 0. \tag{13}$$

2.2. UNEQUAL TIME DELAYS

In the case of $\tau_1 \neq \tau_2$, one defines three functions

$$g(\omega) \equiv \omega^4 + p\omega^2 + q - r_0\omega \sin(\omega\tau_2 - \omega\tau_1),$$
$$\underline{g}(\omega) \equiv \omega^4 + p\omega^2 + q + r\omega, \qquad \bar{g}(\omega) \equiv \omega^4 + p\omega^2 + q - r\omega, \tag{14}$$

where

$$r_0 = 2s_1s_2, \qquad r = -|r_0|. \tag{15}$$

It is easy to prove that for all $\omega \geqslant 0$

$$\underline{g}(\omega) \leqslant g(\omega) \leqslant \bar{g}(\omega), \tag{16}$$

and

$$\underline{g}(0) = g(0) = \bar{g}(0) = q \geqslant 0. \tag{17}$$

As τ_1 and τ_2 take arbitrary non-negative values, one can choose suitable τ_1 and τ_2 such that $g(\omega) = \bar{g}(\omega)$ or $g(\omega) = \underline{g}(\omega)$ for any given $\omega \geqslant 0$. Thus, $g(\omega)$ has no root in $(0, +\infty)$ if $\underline{g}(\omega) > 0$ for all $\omega \geqslant 0$ or $\bar{g}(\omega) < 0$ for all $\omega \geqslant 0$. However, the second case is obviously impossible.

Now, one focuses on the condition of $\underline{g}(\omega) > 0$ for all $\omega \geqslant 0$. Consider the derivative of $\underline{g}(\omega)$ with respect to ω

$$\underline{g}'(\omega) = 4\omega^3 + 2p\omega + r. \tag{18}$$

By letting

$$\Delta = (p/6)^3 + (r/8)^2, \tag{19}$$

one has three cases according to the solution of a cubic algebraic equation [11].

(a) If $\Delta > 0$, $\underline{g}'(\omega) = 0$ has one real root and a pair of conjugate complex roots

$$\omega_1 = u + v, \qquad \omega_2 = u\theta_1 + v\theta_2, \qquad \omega_3 = u\theta_2 + v\theta_1. \tag{20}$$

where

$$u = \sqrt[3]{-r/8 + \sqrt{\Delta}}, \qquad v = \sqrt[3]{-r/8 - \sqrt{\Delta}}, \qquad \theta_1 = \frac{-1 + \mathrm{i}\sqrt{3}}{2}, \qquad \theta_2 = \frac{-1 - \mathrm{i}\sqrt{3}}{2}. \tag{21}$$

(b) If $\Delta = 0$, $\underline{g}'(\omega) = 0$ has three real roots

$$\omega_1 = 2\sqrt[3]{-r/8}, \qquad \omega_2 = \omega_3 = -\sqrt[3]{-r/8}. \tag{22}$$

(c) If $\Delta < 0$, $\underline{g}'(\omega) = 0$ has three real roots

$$\omega_1 = 2\sqrt{-\frac{p}{6}}\cos\left(\frac{\alpha}{3}\right), \qquad \omega_2 = 2\sqrt{-\frac{p}{6}}\cos\left(\frac{\alpha}{3}+\frac{2\pi}{3}\right), \qquad \omega_3 = 2\sqrt{-\frac{p}{6}}\cos\left(\frac{\alpha}{3}+\frac{4\pi}{3}\right), \tag{23}$$

where

$$\cos\alpha = 6r/8p\sqrt{-p/6}, \qquad 0 < \alpha < \pi. \tag{24}$$

Noting $r \leqslant 0$, one can find that only the root ω_1 in these cases is positive so that

$$\underline{g}'(\omega) = \begin{Bmatrix} <0, & \text{for } \omega \in (0, \omega_1) \\ >0, & \text{for } \omega \in (\omega_1, +\infty) \end{Bmatrix}. \tag{25}$$

This implies that $\underline{g}(\omega_1)$ is the minimum of $\underline{g}(\omega)$ in $[0, +\infty]$. Hence, $\underline{g}(\omega) > 0$ holds for all $\omega \geqslant 0$ provided that $\underline{g}(\omega_1) > 0$. If this is the case, $g(\omega) > 0$ holds for all $\omega \geqslant 0$, $\tau_1, \tau_2 \geqslant 0$.

In summary, the delay-independent stability criterion can be stated as follows.

Criterion 1

The system (1) is delay-independent stable for any time delays if and only if either

$$p > 0, \qquad q > 0, \qquad \underline{g}(\omega_1) > 0 \qquad \text{or} \qquad p < 0, \qquad p^2 - 4q < 0, \qquad \underline{g}(\omega_1) > 0 \tag{26}$$

holds, where

$$\omega_1 = \begin{Bmatrix} \sqrt[3]{-r/8+\sqrt{\Delta}} + \sqrt[3]{-r/8-\sqrt{\Delta}}, & \text{for } \Delta \geqslant 0 \\ 2\sqrt{-(p/6)}\cos\alpha/3, & \text{for } \Delta < 0 \end{Bmatrix}. \tag{27}$$

For a given system, the stability test based on this criterion is an easy task including very simple algebraic computation. Table 1 shows 4 numerical examples.

TABLE 1

Delay-independent stability of four illustrative examples

Examples	System parameters	Stability tests	Conclusions
2.1	$\zeta = 0{\cdot}1$ $s_1 = 0{\cdot}1$, $s_2 = 0{\cdot}05$	$p = -1{\cdot}963$, $q = 0{\cdot}9900$ $p^2 - 4q = -0{\cdot}1086$, $\underline{g}(\omega_1) = 0{\cdot}0173$	Delay-independent stable
2.2	$\zeta = 0{\cdot}1$ $s_1 = 0{\cdot}3$, $s_2 = 0{\cdot}05$	$p = -1{\cdot}963$, $q = 0{\cdot}9100$ $p^2 - 4q = 0{\cdot}2114$, $\underline{g}(\omega_1) = -0{\cdot}0824$	**Not** delay-independent stable
2.3	$\zeta = 0{\cdot}5$ $s_1 = 0{\cdot}5$, $s_2 = 0{\cdot}5$	$p = -1{\cdot}250$, $q = 0{\cdot}7500$ $p^2 - 4q = -1{\cdot}436$, $\bar{g}(\omega_1) = 0{\cdot}0652$	Delay-independent stable
2.4	$\zeta = 0{\cdot}5$ $s_1 = 0{\cdot}6$, $s_2 = 0{\cdot}3$	$p = -1{\cdot}090$, $q = 0{\cdot}6400$ $p^2 - 4q = -1{\cdot}372$, $\bar{g}(\omega_1) = 0{\cdot}1345$	Delay-independent stable

3. DELAY-INDEPENDENT STABILITY CRITERION IN TERMS OF FEEDBACK GAINS

In this section, all possible combinations of feedback gains that guarantee the delay-independent stability of system (1) are determined. Let D denote the region of those combinations in the (s_1, s_2) plane, and D_0 denote the region where the system with two equal time delays is delay-independent stable. For simplicity, these regions are referred to as the regions of delay-independent stability. It is obvious that $(0, 0) \in D \subset D_0$ if $\zeta > 0$.

3.1. EQUAL TIME DELAYS

By substituting equation (12) into equation (13), one has

$$p = 4\zeta^2 - 2 - s_2^2 > 0, \qquad q = 1 - s_1^2 > 0, \tag{28a}$$

or

$$p = 4\zeta^2 - 2 - s_2^2 < 0, \qquad p^2 - 4q = (4\zeta^2 - 2 - s_2^2)^2 - 4(1 - s_1^2) < 0; \tag{28b}$$

namely,

$$s_1^2 < 1, \qquad s_2^2 < 4\zeta^2 - 2, \tag{29a}$$

or

$$s_2^2 > 4\zeta^2 - 2, \qquad 4s_1^2 + (s_2^2 + 2 - 4\zeta^2)^2 < 4. \tag{29b}$$

The second inequality in equation (29a) and the first inequality in equation (29b) are a pair of contradictory bounds for s_2^2, holding for different damping. To gain insight in the second inequality in equation (29b), one can find all intersections of the curve

$$P(s_1, s_2) \equiv 4s_1^2 + (s_2^2 + 2 - 4\zeta^2)^2 - 4 = 0 \tag{30}$$

with the axes of s_1 and s_2. They are

$$s_1 = 0, \quad s_2 = \pm 2\zeta, \pm 2\sqrt{\zeta^2 - 1}; \qquad s_2 = 0, \quad s_1 = \pm 2\zeta\sqrt{1 - \zeta^2}. \tag{31a, 31b}$$

From equation (31a), it is easy to see that this curve, like an ellipse, has a pair of intersections on the s_1 and s_2 axes, if and only if the system is underdamped in the usual sense. Once the system is overdamped, the curve has no intersection on the s_1-axis, but four intersections on the s_2-axis. In fact, two separated ellipses appear in this case. By imposing

$$\partial P(s_1, s_2)/\partial s_2 = 4s_2(s_2^2 + 2 - 4\zeta^2) = 0, \tag{32}$$

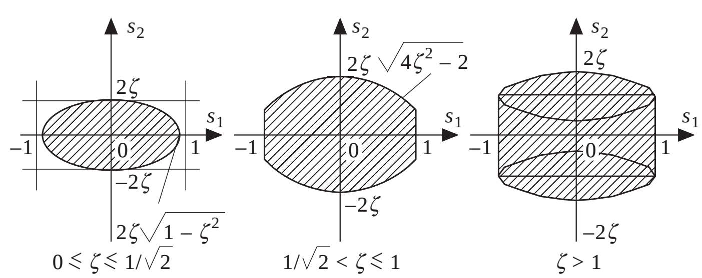

Figure 2. Regions of delay-independent stability in the (s_1, s_2) plane for different damping ratios when $\tau_1 = \tau_2$. (a) $\zeta = 0{\cdot}5$; (b) $\zeta = 1{\cdot}0$; (c) $\zeta = 1{\cdot}5$.

one obtains the extrema of s_1 on these two ellipses

$$s_1 = \pm 1. \tag{33}$$

On the basis of the above analysis, one can determine the criterion for delay-independent stability as the following and then plot the corresponding region of delay-independent stability as shown in Figure 2.

(a) If $0 \leqslant \zeta \leqslant 1/\sqrt{2}$, D_0 is surrounded by an ellipse, i.e.,

$$4s_1^2 + (s_2^2 + 2 - 4\zeta^2)^2 < 4. \tag{34}$$

(b) If $1/\sqrt{2} < \zeta \leqslant 1$, the boundary of D_0 consists of two arcs of an ellipse and two sides of a rectangular, i.e.,

$$s_1^2 \leqslant 1, \qquad s_2^2 \leqslant 4\zeta^2 - 2,$$

$$4s_1^2 + (s_2^2 + 2 - 4\zeta^2)^2 < 4, \qquad s_2^2 > 4\zeta^2 - 2. \tag{35}$$

(c) If $\zeta > 1$, the boundary of D_0 is composed of two sides of a rectangular and two arcs from two ellipses governed by equation (35), too.

No matter what case happens, D_0 is bounded in the rectangle $\{(s_1, s_2)|\, |s_1| \leqslant 1, |s_2| \leqslant 2\zeta\}$. So, it is the damping that makes the delay-independent stability possible.

3.2. UNEQUAL TIME DELAYS

If $\tau_1 \neq \tau_2$, the boundary of D yields $\underline{g}(\omega_1) = 0$. However, it is almost impossible to solve $\underline{g}(\omega_1) = 0$ for the explicit expression of the boundary. Thus, a qualitative analysis of region $\bar{D}$ will be made. Since p, q, r, $p^2 - 4q$, α and $\underline{g}(\omega_1)$ are even functions in the variables s_1 and s_2, the region D should be symmetric with respect to both s_1 and s_2 axes. In what follows, the study is confined to the first quadrant of (s_1, s_2) plane.

For $s_1 \geqslant 0$ and $s_2 \geqslant 0$, one can readily verify that

$$\partial p/\partial s_1 = 0, \qquad \partial q/\partial s_1 = -2s_1 \leqslant 0, \qquad (\partial/\partial s_1)(4q - p^2) = -8s_1 \leqslant 0,$$

$$\partial p/\partial s_2 = -2s_2 \leqslant 0, \qquad \partial q/\partial s_2 = 0, \qquad (\partial/\partial s_2)(4q - p^2) = 4ps_1 \leqslant 0, \qquad (p < 0) \tag{36}$$

$$\frac{\partial}{\partial s_1}\underline{g}(\omega_1) = \underline{g}'(\omega_1)\frac{\partial \omega_1}{\partial s_1} + \omega_1^2\frac{\partial p}{\partial s_1} + \frac{\partial q}{\partial s_1} + \omega_1\frac{\partial r}{\partial s_1} = -2(s_1 + s_2\omega_1) \leqslant 0,$$

$$\frac{\partial}{\partial s_2}\underline{g}(\omega_1) = \underline{g}'(\omega_1)\frac{\partial \omega_1}{\partial s_2} + \omega_1^2\frac{\partial p}{\partial s_2} + \frac{\partial q}{\partial s_2} + \omega_1\frac{\partial r}{\partial s_2} = -2\omega_1(s_1 + s_2\omega_1) \leqslant 0. \tag{37}$$

These inequalities imply that if a given system with ζ, s_{10} and s_{20} is delay-independent stable, so is the system with ζ, s_{20} and $0 \leqslant s_1 \leqslant s_{10}$ or with ζ, s_{10}, and $0 \leqslant s_2 \leqslant s_{20}$.

Example 3.1. As known from example 2.1, the system with $\zeta = 0{\cdot}1$, $s_1 = 0{\cdot}1$ and $s_2 = 0{\cdot}05$ is delay-independent stable. According to the analysis above, the system with $\zeta = 0{\cdot}1$, $s_2 = 0{\cdot}05$ and $0 \leqslant s_1 \leqslant 0{\cdot}1$, or with $\zeta = 0{\cdot}1$, $s_1 = 0{\cdot}1$ and $0 \leqslant s_2 \leqslant 0{\cdot}05$ is delay-independent stable, too.

Moreover, equation (37) leads to

$$\mathrm{d}s_2/\mathrm{d}s_1|_{\underline{g}(\omega_1)=0} = -(\partial/\partial s_1)\underline{g}(\omega_1)/(\partial/\partial s_2)\underline{g}(\omega_1) = -1/\omega_1 < 0. \tag{38}$$

Hence, the boundary defined by $\underline{g}(\omega_1) = 0$ in the first quadrant of (s_1, s_2) plane is a simple curve. Along the boundary, s_2 decreases with the increase of s_1.

According to the above analysis of D, one obtains another stability criterion independent of time delays.

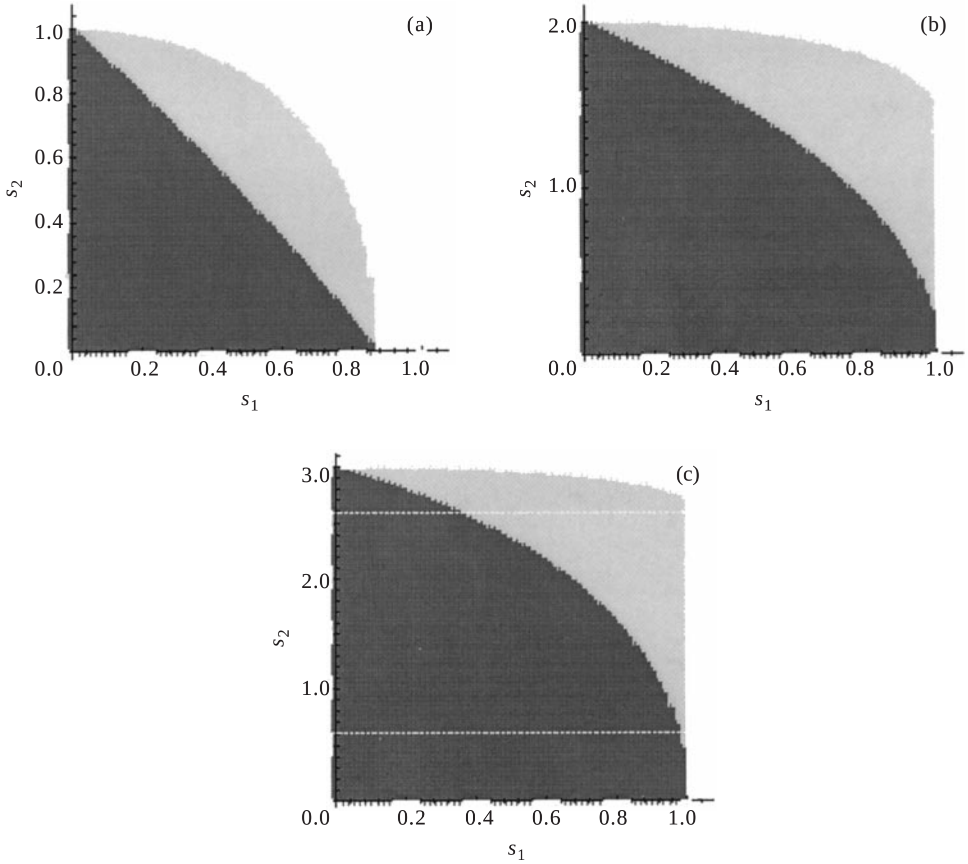

Figure 3. Regions of delay-independent stability in the first quadrant of the (s_1, s_2) plane for different damping ratios when $\tau_1 \neq \tau_2$.

Criterion 2

Let $U = \{(s_1, s_2) \,||s_1| \leqslant a_1, |s_2| \leqslant a_2\}$. It is sufficient to check $(a_1, a_2) \in D$ in order to make sure $U \subset D$.

Example 3.2. Consider a system with $\zeta = 0{\cdot}5$. From Criterion 2 and Example 2.4, it is obvious that $[-0{\cdot}6, 0{\cdot}6] \times [-0{\cdot}3, 0{\cdot}3] \subset D$. This rectangular can be broadened in D by further numerical tries. For instance, one can first fix $s_1 = 0{\cdot}6$ and choose a larger s_2, say $s_2 = 0{\cdot}5$. A direct computation gives $p = -1{\cdot}250$, $q = 0{\cdot}64$, $p^2 - 4q = -0{\cdot}9975$ and $\bar{g}(\omega_1) = -0{\cdot}0629$. This means that the system is not delay-independent stable and such a s_2 is too large. Thus, one can choose a less larger s_2, say $s_2 = 0{\cdot}4$ as the second try. In this case, $p = -1{\cdot}160$, $q = 0{\cdot}6400$, $p^2 - 4q = -1{\cdot}214$, $\bar{g}(\omega_1) = 0{\cdot}0421$. Hence, a larger rectangular $[-0{\cdot}6, 0{\cdot}6] \times [-0{\cdot}4, 0{\cdot}4] \subset D$ is obtained.

For a given system, the region of delay-independent stability can be determined by a simple and short subroutine. The typical regions of delay-independent stability for underdamped, critically damped and over-damped systems are shown in Figure 3, where the grey regions and the dark regions correspond to the cases of equal time delays and unequal time delays respectively.

4. STABILITY FOR FINITE TIME DELAYS

As analyzed in section 3, the region of delay-independent stability is always bounded in the rectangular $\{(s_1, s_2)\,||s_1| \leqslant 1, |s_2| \leqslant 2\zeta\}$. Since most mechanical systems are lightly damped, it is too restrictive for them to be delay-independent stable. In practice, the stability of many controlled systems is only required for bounded time delays, especially for short time delays in a bounded interval.

Now, the system stability when two finite time delays are given is studied. By separating the real part and imaginary part of the critical condition $D(\mathrm{i}\omega, \tau_1, \tau_2) = 0$, one has

$$\mathrm{Re}\,[D(\mathrm{i}\omega, \tau_1, \tau_2)] \equiv 1 - \omega^2 - s_1 \cos \omega\tau_1 - s_2\omega \sin \omega\tau_2 = 0,$$
$$\mathrm{Im}\,[D(\mathrm{i}\omega, \tau_1, \tau_2)] \equiv 2\zeta\omega + s_1 \sin \omega\tau_1 - s_2\omega \cos \omega\tau_2 = 0. \quad (39)$$

Solving equation (39) for s_1 and s_2 yields

$$s_1 = \frac{(1-\omega^2)\cos\omega\tau_2 - 2\zeta\omega\sin\omega\tau_2}{\cos[\omega(\tau_1-\tau_2)]}, \qquad s_2 = \frac{(1-\omega^2)\sin\omega\tau_1 + 2\zeta\omega\cos\omega\tau_1}{\omega\cos[\omega(\tau_1-\tau_2)]}. \quad (40)$$

In the plane of (s_1, s_2), equation (40) gives the transition set where at least one characteristic root of equation (3) changes the sign of the real part. As both s_1 and s_2 are even functions in frequency ω, the transition set will be discussed in the semi-infinite interval $\omega \in [0, +\infty)$.

4.1. EQUAL TIME DELAYS

If $\tau_1 = \tau_2 = \tau$, equation (40) becomes

$$s_1 = [(1-\omega^2)\cos\omega\tau - 2\zeta\omega\sin\omega\tau], \qquad s_2 = [(1-\omega^2)\sin\omega\tau + 2\zeta\omega\cos\omega\tau]/\omega. \quad (41)$$

Thus, the transition set is a continuous curve C_τ in the plane of (s_1, s_2) when the parameter ω varies in $[0, +\infty)$. It is easy to find from equation (41) that the curve C_τ starts from the point $A = (1, 2\zeta + \tau)$ in the plane of (s_1, s_2). As shown in Figure 4, the curve C_τ becomes complicated and intersects itself if the time delay is long enough.

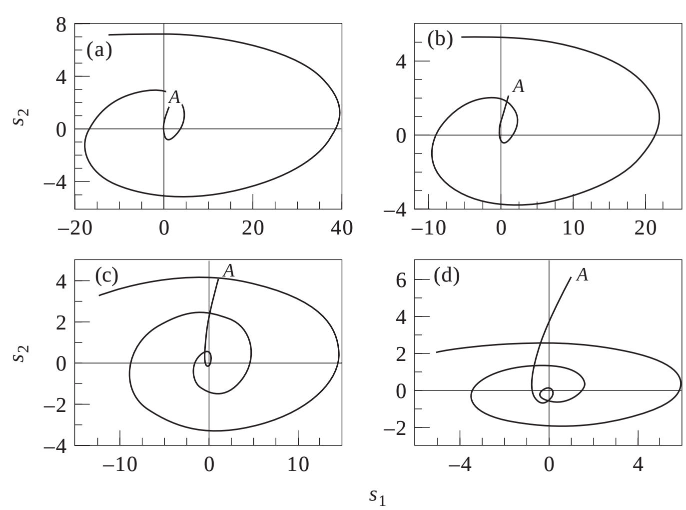

Figure 4. Transition sets in the (s_1, s_2) plane for various equal delays ($\zeta = 0{\cdot}05$). τ values: (a) 1·5; (b) 2·0; (c) 4·0; (d) 6·0.

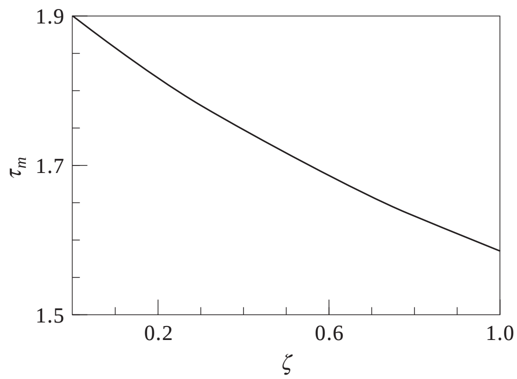

Figure 5. Critical time delay versus damping ratio.

From the viewpoint of an engineer, one may want to know the critical time delay τ_m, with which the curve C_τ just intersects itself at $(1, 2\zeta, +\tau)$ in (s_1, s_2) plane. According to equation (41), such a time delay yields

$$(1-\omega^2)\cos\omega\tau_m - 2\zeta\omega\sin\omega\tau_m = 1, \qquad (1-\omega^2)\sin\omega\tau_m + 2\zeta\omega\cos\omega\tau_m = (2\zeta+\tau_m)\omega. \tag{42}$$

This condition can be written as

$$\tan(\omega\tau_m + \varphi) = (2\zeta+\tau_m)\omega, \qquad \varphi = \tan^{-1}(2\zeta\omega/(1-\omega^2)). \tag{43}$$

In addition, the self-intersection of curve C_τ at $(1, 2\zeta+\tau)$ implies that equation (10) has two different positive roots when $s_1 = 1$ and $s_2 = 2\zeta + \tau_m$. From equations (11) and (12), these two roots are

$$\omega_1 = 0, \qquad \omega_2 = \sqrt{-p} = \sqrt{\tau_m^2 + 4\zeta\tau_m + 2}. \tag{44}$$

Substituting ω in equation (43) with ω_2 yields

$$\tan\left(\tau_m\sqrt{\tau_m^2+4\zeta\tau_m+2} + \varphi\right) = (2\zeta+\tau_m)\sqrt{\tau_m^2+4\zeta\tau_m+2},$$

$$\varphi = \tan^{-1}\frac{2\zeta\sqrt{\tau_m^2+4\zeta\tau_m+2}}{1-(\tau_m^2+4\zeta\tau_m+2)}. \tag{45}$$

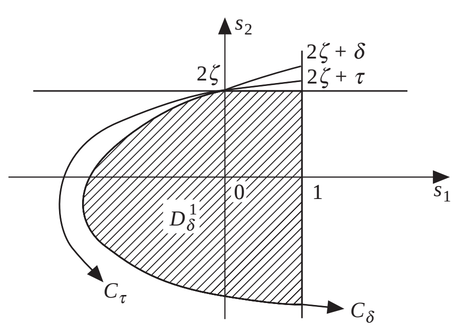

Figure 6. Stability region D_δ^1 in the (s_1, s_2) plane for equal time delays $\tau = \tau_2 = \tau \leqslant \delta$.

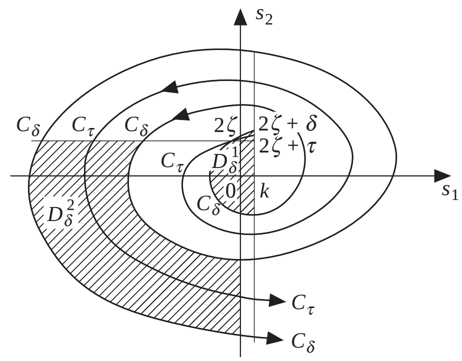

Figure 7. Transition sets C_δ and C_τ in the (s_1, s_2) plane for equal time delays.

Solving equation (45) numerically for the minimal positive root, one obtains the critical time delay for given damping ratio. As shown in Figure 5, the critical time delay decreases with increase of the damping ratio.

Given a time delay $\delta < \tau_m$, the transition set C_δ shown in Figure 6 does not intersect itself. This curve, together with the lines of $s_1 = 1$ and $s_2 = 2\zeta$, surrounds a shaded region denoted by D_δ^1 in Figure 6. According to $(0, 0) \subset D_\delta^1$, all combinations of $(s_1, s_2) \in D_\delta^1$ guarantee the system stability for equal time delays $\tau_1 = \tau_2 = \delta$. This region will be called the stability region hereafter for short.

It should be emphasized that D_δ^1 in Figure 6 is the unique stability region where the feedback gains guarantee the system stability for any equal time delays $\tau_1 = \tau_2 = \tau \leqslant \delta$. What follows is an intuitive proof of this assertion.

By differentiating equation (41) with respect to ω, one obtains the tangent of C_τ at the starting location

$$\frac{\mathrm{d}s_2}{\mathrm{d}s_1} = \frac{\partial s_2}{\partial \omega}\Big/\frac{\partial s_1}{\partial \omega} = \frac{(\tau(1-\omega^2)/\omega)\cos\omega\tau + ((\omega^2-1)/\omega^2 - 2 - 2\zeta\tau)\sin\omega\tau}{-2\omega(1+\zeta\tau)\cos\omega\tau + (\tau\omega^2 - \tau - 2\zeta)\sin\omega\tau}$$

$$\approx \frac{2\tau + 2\zeta\tau^2}{2 + 4\zeta\tau + \tau^2} = \tau + \mathcal{O}(\tau). \tag{46}$$

If $0 \leqslant \tau < \delta$, the curves C_τ and C_δ intersect each other near $(0, c)$ as shown in Figure 6. According to the analysis in section 2, only the parameter combination (s_1, s_2) under the condition $p^2 - 4q \geqslant 0$, $p < 0$, $q \geqslant 0$ enables equation (10) to have two positive roots. The third inequality here implies that C_τ and C_δ intersect each other only in the region of $|s_1| < 1$. Thus, C_τ cannot enter into the stability region D_δ^1 if $s_1 < -1$. As a result, the combination of feedback gains $(s_1, s_2) \subset D_\delta^1$ ensures that the system is stable for any equal time delays $\tau_1 = \tau_2 = \tau \leqslant \delta$.

Then, it is easy to see that the system is not stable for any $0 < \tau \leqslant \delta$ if the combination of feedback gains (s_1, s_2) falls into other shaded regions, say D_δ^2 in Figure 7, where the roots of equation (3) seem to have the negative real parts again with variation of (s_1, s_2). In fact, even for a very short time delay τ, the corresponding spiral C_τ will enter D_δ^2 as long as the frequency ω is high enough. This implies that the system will undergo an instability if the disturbance $\Delta x(t)$ involves any harmonic components of sufficiently high frequency. As a result, the assertion made by Palkovics and Venhovens in [1] that there exist other possible stability regions in (s_1, s_2) plane is not true.

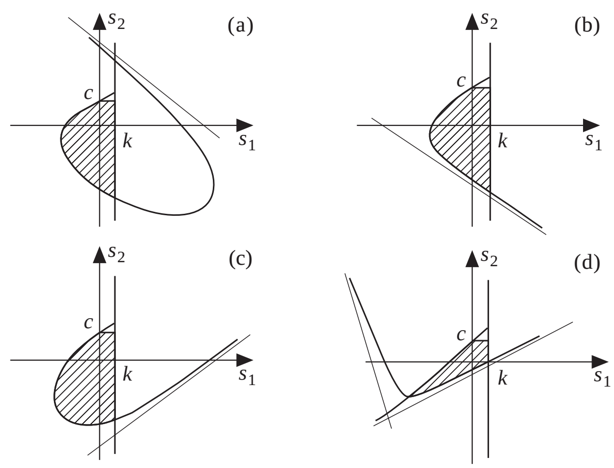

Figure 8. Transition sets and stability regions in the (s_1, s_2) plane for unequal time delays. (a) $\tau_1 < \tau_2$ and small $\Delta\tau$; (b) $\tau_1 < \tau_2$ and large $\Delta\tau$; (c) $\tau_1 > \tau_2$ and small $\Delta\tau$; (d) $\tau_1 > \tau_2$ and large $\Delta\tau$.

4.2. UNEQUAL TIME DELAYS

In the case of $\tau_1 \neq \tau_2$, let $\Delta\tau = |\tau_1 - \tau_2|$. It is obvious that both s_1 and s_2 in equation (40) will approach infinity when $\omega\Delta\tau \to n\pi/2$, $n = 1, 3, 5, \ldots$. Thus, the transition set given by equation (40) in this case is no longer a continuous curve. It consists of an infinite number of curves defined by the parametric equation of ω in the intervals $[0, \pi/2\Delta\tau)$, $(\pi/2\Delta\tau, 3\pi/2\Delta\tau)$, etc. As analyzed in the previous subsection, the boundary of the stability region is a small part of the transition set corresponding to the lower frequency ω. So, one focuses on the transition set in the frequency range $\omega \in [0, \pi/2\Delta\tau)$.

Consider the case of $0 < \tau_1 < \tau_2$ first. The transition set is a curve starting from $(1, 2\zeta + \tau_1)$ in (s_1, s_2) plane and approaching infinity when $\omega(\tau_2 - \tau_1) \to \pi/2$. It is easy to find that the tangent of the asymptotic line reads

$$\frac{s_2}{s_1} \to \frac{(1-\omega^2)\sin\omega\tau_1 + 2\zeta\omega\cos\omega\tau_1}{\omega[(1-\omega^2)\cos\omega(\tau_1+\pi/2) - 2\zeta\omega\sin\omega(\tau_1+\pi/2)]} = -1/\omega = \pi/2(\tau_1-\tau_2) < 0. \tag{47}$$

Obviously, if $\Delta\tau$ is small, the curve will spiral one or more rounds and go to infinity in the second quadrant or the fourth quadrant. Otherwise, it goes to infinity in the fourth quadrant. These two cases are shown in Figures 8a and 8b respectively.

If $0 < \tau_2 < \tau_1$, one can similarly find the tangent of the asymptotic line

$$\frac{s_2}{s_1} \to \frac{(1-\omega^2)\sin\omega(\tau_2+\pi/2) + 2\zeta\omega\cos\omega(\tau_2+\pi/2)}{\omega[(1-\omega^2)\cos\omega\tau_2 - 2\zeta\omega\sin\omega\tau_2]} = -1/\omega = \pi/2(\tau_1-\tau_2) > 0 \tag{48}$$

when $\omega(\tau_1 - \tau_2) \to \pi/2$. The transition set will approach infinity in the first quadrant or the third quadrant if $\Delta\tau$ is small. Otherwise, it goes to infinity in the third quadrant. These two cases are shown in Figures 8c and 8d respectively.

Noting the stability conditions (5) for a system without time delays in feedback, one can determine the stability region shown as the shaded one in Figure 8. Figure 8d shows that the stability region shrinks to a very small size if $0 < \tau_2 \ll \tau_1$. This is the most dangerous case and should be avoided in practice.

5. CONCLUSIONS

(a) It is found in this paper that there exists a sufficient and necessary algebraic criterion of the delay-independent stability for linear, SDOF vibrating systems with two arbitrary time delays in displacement feedback and velocity feedback respectively. With the help of this criterion, it is quite straightforward to check the stability of a given system.

(b) The feedback gains s_1 and s_2, with which the system is delay-independent stable, form a region D in the plane of (s_1, s_2). The region D is bounded in the rectangular $\{(s_1, s_2) \mid |s_1\| \leqslant 1, |s_2| \leqslant 2\zeta\}$ and is symmetric with respect to both the s_1- and s_2-axes. When the two time delays are identical, the boundary of the region can be determined analytically. Otherwise, it is a simple curve without explicit expression. An arbitrary point (a_1, a_2) in region D can ensure that the rectangle $U = \{(s_1, s_2) \mid |s_1| \leqslant a_1, |s_2| \leqslant a_2\}$ falls into D. This property enables one to compute region D very efficiently.

(c) Given short time delays τ_1 and τ_2, the feedback gains s_1 and s_2 that guarantee the system stability fall into a simple region D in the plane of (s_1, s_2). In most cases, the boundary of region D consists of a spiral and two straight segments defined by $s_1 = 1$ and $s_2 = 2\zeta > 0$ respectively. However, if $0 < \tau_2 \ll \tau_1$, the boundary becomes complicated and the region shrinks to a very small size.

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