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ANALYTICAL METHODS FOR SOLVING STRONGLY NON-LINEAR DIFFERENTIAL EQUATIONS

L. Cveticanin

Faculty of Technical Sciences, *Trg D. Obradovica* 6, 21000 *Novi Sad*, *Yugoslavia*

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In this paper various approximate analytical methods for obtaining solutions for strongly non-linear differential equations in a complex function are developed. The methods are based on the solution of the generating differential equation with a cubic complex term. The method of harmonic balance, the method of Krylov–Bogoliubov and the elliptic perturbation method are adopted for solving strongly non-linear differential equations in a complex function. Three examples are analyzed. They describe the vibrations of a rotor with non-linearity.

1. INTRODUCTION

The motion of a plane one-mass system with two degrees of freedom, is usually described with a complex function $z = x + iy$ where $i = \sqrt{1}$ is the imaginary unit and x and y are time variable co-ordinates of the system [1]. The forces which act in such a system also admit this simplified way of motion description. Rotors, which are the fundamental working elements of many machines, are mechanical systems in which the vibrations occur in one plane and are conveniently formulated by using the complex function z. The dynamics of rotors is mainly described with differential equations with complex deflection functions (see references [2–4]). One of the general forms of such a differential equation of motion is

$$\ddot{z} + c_1 z + c_3 z^3 = \varepsilon f(z, \dot{z}, cc), \tag{1}$$

where c_1 is the coefficient of the linear term, c_3 is the coefficient of the non-linear term, ε is a small parameter, f is the non-linear function and cc are complex conjugate functions. To find the solution of the equation some approximate analytical methods have been developed. Usually the solution is obtained as a function of Jacobi elliptic functions. The method of Bogoliubov–Mitropolski developed for systems with small non-linearity has been extended for the systems with strong non-linearity [5, 6]. The same case can be dealt with by the method of Krylov–Bogoliubov [7, 8]. Very often, the linear term in equation (1) is very small or negligible. Then, the differential equation has the form

$$\ddot{z} + c_3 z^3 = \varepsilon f(z, \dot{z}, cc). \tag{2}$$

The aim in this paper is to present an approximate solution for this special differential equation.

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In recent years much attention has been paid to solving non-linear differential equations which describe the motion of one-degree-of-freedom systems:

$$\ddot{x} + x^3 = \varepsilon f(x, \dot{x}). \tag{3}$$

The analytical solution has the form of Jacobian elliptic function [9]. As the most convenient analytical asymptotic methods, the harmonic balance method and the method of slowly variable amplitude and phase have been applied [10, 11]. Recently, a new elliptic perturbation method has been developed for solving a special class of differential equations which describe the limit cycle motion [12].

In this paper the analytical approximate methods of equation (2) based on the results obtained by analyzing equations (1) and (2) are developed. The method of harmonic balance, the method of slowly variable amplitude and phase and the elliptic perturbation method are extended for solving the non-linear differential equation (2). At the end, several examples are shown. The results obtained by the methods mentioned are compared with numerical ones.

2. THE SOLUTION OF THE GENERATING EQUATION

Two types of generating equations of equation (1) are evident depending on the sign of the coefficient of the non-linear term.

Consider the generating equation with a positive coefficient of the non-linear term, $c_3 > 0$:

$$\ddot{z} + c_1 z + c_3 z^3 = 0. \tag{4}$$

The equivalent system of non-linear differential equations is then

$$\ddot{x} + c_1 x + c_3 (x^3 - 3xy^2) = 0,$$

$$\ddot{y} + c_1 y + c_3 (3x^2 y - y^3) = 0.$$

The solution of equation (4) is assumed in the complex form which satisfies also the aforementioned system of non-linear differential equations:

$$z = x + \mathrm{i}y = A[\mathrm{cn}\,(\Omega t, k^2) + \mathrm{i}\,\mathrm{sn}\,(\Omega t, k^2)] \equiv A(\mathrm{cn} + \mathrm{i}\,\mathrm{sn}). \tag{5}$$

Here cn and sn are Jacobian elliptic functions [13], A is the amplitude of vibrations, Ω is the angular frequency, t is time and k is the modulus of the Jacobi elliptic function. The first and the second time derivatives of equation (5) are

$$\dot{z} = A\mathrm{i}\Omega\,\mathrm{dn}\,(\mathrm{cn} + \mathrm{i}\,\mathrm{sn}), \tag{6}$$

$$\ddot{z} = -A\Omega^2(\mathrm{dn}^2 + \mathrm{i}k^2\,\mathrm{sn}\,\mathrm{cn})\,(\mathrm{cn} + \mathrm{i}\,\mathrm{sn}), \tag{7}$$

where dn is a Jacobian elliptic function. Substituting the assumed solution (5) and its time derivatives (6) into equation (7) and separating the real and imaginary parts yields

$$\Omega^2 = c_1 + c_3 A^2, \qquad k^2 = 2c_3 A^2/(c_1 + c_3 A^2). \tag{8, 9}$$

It is evident that the frequency of vibrations Ω and the parameter of the Jacobian elliptic function k^2 depend on the amplitude of vibration A.

Another particular solution of equation (4) is

$$z = A(\mathrm{cn} - \mathrm{i}\,\mathrm{sn}). \tag{10}$$

Substituting equation (10) and the time derivatives

$$\dot{z} = -A\mathrm{i}\Omega\,\mathrm{dn}\,(\mathrm{cn} - \mathrm{i}\,\mathrm{sn}), \tag{11}$$

$$\ddot{z} = -A\Omega^2(\mathrm{dn}^2 - \mathrm{i}k^2 \,\mathrm{sn}\,\mathrm{cn})\,(\mathrm{cn} - \mathrm{i}\,\mathrm{sn}), \tag{12}$$

into equation (4) yields the frequency and the parameter of the elliptic function in the same forms (8) and (9) as for the previous solution.

For the case when the non-linearity has the opposite sign to the previous one ($c_3 < 0$) the differential equation of motion is

$$\ddot{z} + c_1 z - c_3 z^3 = 0. \tag{13}$$

The solutions which satisfy equation (13) are

$$z = A[\mathrm{sn}\,(\Omega t, k^2) + \mathrm{i}\,\mathrm{cn}\,(\Omega t, k^2)] \equiv A(\mathrm{sn} + \mathrm{i}\,\mathrm{cn}), \tag{14}$$

and

$$z = A(\mathrm{sn} - \mathrm{i}\,\mathrm{cn}), \tag{15}$$

where the frequency Ω and the parameter k^2 are described by equations (8) and (9), respectively.

Consider the special case when $c_1 = 0$. The differential equation is

$$\ddot{z} + c_3 z^3 = 0. \tag{16}$$

The solution has the form (5) where

$$\Omega^2 = c_3 A^2, \qquad k^2 = 2. \tag{17, 18}$$

As the parameter $k > 1$, the Jacobian elliptic functions can be transformed to corresponding functions with parameter $k < 1$. Introducing the connections between the functions (see references [9] and [10])

$$\mathrm{sn}\,(\Omega t, 2) = (1/\sqrt{2})\,\mathrm{sn}\,(\sqrt{2}\Omega t, k^2 = 1/2),$$
$$\mathrm{cn}\,(\Omega t, 2) = \mathrm{dn}\,(\sqrt{2}\Omega t, 1/2), \qquad \mathrm{dn}\,(\Omega t, 2) = \mathrm{cn}\,(\sqrt{2}\Omega t, 1/2), \tag{19}$$

yields the solution of equation (16) as

$$z = A[\mathrm{cn}\,(\Omega t, 2) + \mathrm{i}\,\mathrm{sn}\,(\Omega t, 2)] = A[\mathrm{dn}\,(\sqrt{2}\Omega t, 1/2) + \mathrm{i}(1/\sqrt{2})\,\mathrm{sn}\,(\sqrt{2}\Omega t, 1/2)]. \tag{20}$$

The corresponding time derivatives are

$$\dot{z} = A\mathrm{i}\Omega\,\mathrm{cn}\,(\mathrm{dn} + \mathrm{i}(1/\sqrt{2})\,\mathrm{sn}), \tag{21}$$

$$\ddot{z} = -A\Omega^2(\mathrm{cn}^2 + \mathrm{i}\sqrt{2}\,\mathrm{sn}\,\mathrm{dn})\,(\mathrm{dn} + \mathrm{i}(1/\sqrt{2})\,\mathrm{sn}). \tag{22}$$

and the frequency is

$$\Omega = A\sqrt{c_3}. \tag{23}$$

By using the solution of the generating equations (16) the methods for obtaining approximate solutions of (2) are developed.

3. HARMONIC BALANCE METHOD

For the method of harmonic balance one can assume the solution of equation (2) to be of the form (20). One finds the first and the second time derivatives of equation (20) and substitutes them into equation (16). Then one can develop the Jacobian elliptic functions in Fourier series [14]:

$$\operatorname{sn}(\sqrt{2}\Omega t, \tfrac{1}{2}) = \frac{2\pi\sqrt{2}}{K} \sum_{m=0}^{\infty} \frac{q^{m+1/2}}{1-q^{2m+1}} \sin\left[(2m+1)\frac{\pi\sqrt{2}\Omega t}{2K}\right],$$

$$\operatorname{cn}(\sqrt{2}\Omega t, \tfrac{1}{2}) = \frac{2\pi\sqrt{2}}{K} \sum_{m=0}^{\infty} \frac{q^{m+1/2}}{1+q^{2m+1}} \cos\left[(2m+1)\frac{\pi\sqrt{2}\Omega t}{2K}\right],$$

$$\operatorname{dn}(\sqrt{2}\Omega t, \tfrac{1}{2}) = \frac{\pi}{2K} + \frac{2\pi}{K} \sum_{m=0}^{\infty} \frac{q^{m+1}}{1+q^{2m+1}} \cos\left[(m+1)\frac{\pi\sqrt{2}\Omega t}{K}\right]. \tag{24}$$

Here $K \equiv K(k = 1/\sqrt{2}) = 1{\cdot}854075$ is the complete elliptic integral of the first kind, $K' \equiv K'(k = 1/\sqrt{2}) = 1{\cdot}854075$ is the associated complete elliptic integral of the first kind, $q = e^{-\pi K'/K} = 0{\cdot}043214$. Substituting equations (20) and (24) into equation (2) and separating the terms without and with circular functions one obtains

$$F_1(A, \Omega, \varepsilon, \alpha) + \mathrm{i}F_2(A, \Omega, \varepsilon, \alpha) + (\text{terms with harmonic functions}) = 0, \tag{25}$$

where α represents all other parameters of the system. From the equations

$$F_1 = 0, \qquad F_2 = 0, \tag{26}$$

one obtains the solution $A(\alpha, \varepsilon)$ and $\Omega(\alpha, \varepsilon)$. This solution describes the possible steady state motion.

4. KRYLOV–BOGOLIUBOV METHOD

This method is also based on the solution of the generating equation (16). It is assumed that the amplitude and phase are time dependent and the solution of equation (2) is

$$z(t) = A(t)\{\operatorname{dn}[\psi(t), 1/2] + \mathrm{i}(1/(\sqrt{2})\operatorname{sn}[\psi(t), 1/2]\} = A(t)(\operatorname{dn} + \mathrm{i}(1/\sqrt{2})\operatorname{sn}), \tag{27}$$

where

$$\psi(t) = \sqrt{2}\int_0^t \Omega(\tau)\,\mathrm{d}\tau + \phi(t), \qquad \Omega(t) = A(t)\sqrt{c_3}. \tag{28, 29}$$

Two constraints to solution (27) are introduced: (i) equation (27) must be a solution of equation (2); (ii) the time derivative of equation (27) must have the same form as the time derivative of the generating solution, i.e.,

$$\dot{z}(t) = A(t)\mathrm{i}\Omega(t)\operatorname{cn}(\operatorname{dn} + \mathrm{i}(1/\sqrt{2})\operatorname{sn}), \tag{30}$$

when

$$\dot{A}(t) + A(t)\mathrm{i}\dot{\phi}(t)\operatorname{cn}[\psi(t), 1/2] = 0. \tag{31}$$

Differentiating equation (27) twice in time yields

$$\begin{aligned}\ddot{z} = {} & [\dot{A}(t)\Omega(t) + A(t)\dot{\Omega}(t)]\mathrm{i}\operatorname{cn}(\operatorname{dn} + \mathrm{i}(1/\sqrt{2})\operatorname{sn}) \\ & - A(t)\Omega(t)\mathrm{i}\operatorname{sn}\operatorname{dn}[\sqrt{2}\Omega(t) + \dot{\phi}(t)](\operatorname{dn} + \mathrm{i}(1/\sqrt{2})\operatorname{sn}) \\ & - (1/\sqrt{2})A(t)\Omega(t)\operatorname{cn}^2[\sqrt{2}\Omega(t) + \dot{\phi}(t)](\operatorname{dn} + \mathrm{i}(1/\sqrt{2})\operatorname{sn}).\end{aligned} \tag{32}$$

Substituting equations (27) and (32) into equation (2), applying the relation (31) and separating the real and imaginary parts yields

$$A^2(t)\dot{\phi}(t)\sqrt{c_3}\,\mathrm{cn}^2 = -\varepsilon\sqrt{2}(f_1\,\mathrm{dn} + (1/\sqrt{2})f_2\,\mathrm{sn}), \tag{33}$$

$$2\dot{A}(t)A(t)\sqrt{c_3}\,\mathrm{cn} = \varepsilon(f_2\,\mathrm{dn} - (1/\sqrt{2})f_1\,\mathrm{sn}). \tag{34}$$

To find the solution in closed form of equations (33) and (34) is not an easy task. To simplify the calculation the averaging procedure is introduced. The averaging is over the Jacobian elliptic function period $4K$. Then the averaged equations (33) and (34) are

$$A^2(t)\dot{\phi}(t) = -\frac{\varepsilon\sqrt{2}}{4(2E-K)\sqrt{c_3}}\int_0^{4K}\left(f_1\,\mathrm{dn} + \frac{1}{\sqrt{2}}f_2\,\mathrm{sn}\right)\mathrm{d}\psi, \tag{35}$$

$$A(t)\dot{A}(t) = \frac{\varepsilon}{8\sqrt{c_3}\,(2E-K)}\int_0^{4K}\left(f_2\,\mathrm{dn}\,\mathrm{cn} - \frac{1}{\sqrt{2}}f_1\,\mathrm{sn}\,\mathrm{cn}\right)\mathrm{d}\psi, \tag{36}$$

where

$$\langle \mathrm{cn}^2[\psi(t), \tfrac{1}{2}]\rangle = \int_0^{4K}\mathrm{cn}^2[\psi(t), \tfrac{1}{2}]\,\mathrm{d}\psi = 8\left(E - \frac{K}{2}\right) = 3{\cdot}3889, \tag{37}$$

and $E \equiv E(k^2 = 1/2) = 1{\cdot}350644$ is the complete elliptic integral of the second kind.

The main disadvantage of the previous methods is that they give the solutions only in the first approximation. The method suggested next gives a possibility to obtain the solutions in a higher approximation.

5. THE ELLIPTIC PERTURBATION METHOD

Suppose that the amplitude of vibrations and the angular frequency of vibrations are not the same in the x and y directions and the solution of equation (2) has the form

$$z = A\,\mathrm{dn}\,(\tau_1, 1/2) + A^*\mathrm{i}\,\mathrm{sn}\,(\tau_1^*, 1/2), \tag{38}$$

where

$$\mathrm{d}\tau_1/\mathrm{d}t = \Omega_0\sqrt{2} + \varepsilon\Omega_1(\tau) + \varepsilon^2\Omega_2(\tau) + \cdots, \tag{39}$$

$$\mathrm{d}\tau_1^*/\mathrm{d}t = \Omega_0\sqrt{2} + \varepsilon\Omega_1^*(\tau) + \varepsilon^2\Omega_2^*(\tau) + \cdots, \tag{40}$$

$$A = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots, \qquad A^* = (A_0/\sqrt{2}) + \varepsilon A_1^* + \varepsilon^2 A_2^* + \cdots, \qquad \tau = \Omega_0 t. \tag{41–43}$$

The solution (38) can be rewritten in the form of series as

$$z = z_0 + \varepsilon z_1 + \varepsilon^2 z_1 + \cdots: \tag{44}$$

i.e.,

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots, \qquad y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots. \tag{45, 46}$$

The solution (44) is periodic with period $4K$. This means that $x_0(0) = x_0(4K)$, $y_0(0) = y_0(4K)$, $x_1(0) = x_1(4K)$, $y_1(0) = y_1(4K), \ldots,$ $x_0'(0) = x_0'(4K)$, $y_0'(0) = y_0'(4K)$, $x_1'(0) = x_1'(4K)$, $y_1'(0) = y_1'(4K), \ldots.$

By substituting equation (44) and the corresponding time derivatives into equation (2) and separating the terms with the same order of the small parameter ε, a system of differential equations is obtained:

ε^0:

$$2\Omega_0^2 x_0'' + c_3\, x_0\,(x_0^2 - 3y_0^2) = 0, \tag{47}$$

$$2\Omega_0^2 y_0'' - c_3\, y_0\,(y_0^2 - 3x_0^2) = 0; \tag{48}$$

ε^1:

$$2\Omega_0^2 x_1'' + 2\Omega_0\,\Omega_1\sqrt{2}x_0'' + \Omega_0\,\Omega_1'\,x_0' + c_3\,(3x_1\,x_0^2 - 3x_1\,y_0^2 + 6x_0\,y_0\,y_1) = f_1, \tag{49}$$

$$2\Omega_0^2 y_1'' + 2\Omega_0\,\Omega_1^*\sqrt{2}y_0'' + \Omega_0\,\Omega_1^{*\prime}\,y_0' - c_3\,(3y_1\,y_0^2 - 3y_1\,x_0^2 + 6y_0\,x_0\,x_1) = f_2\,; \tag{50}$$

ε^2:

$$\begin{aligned}
&2\Omega_0^2 x_2'' + 2\Omega_0\,\Omega_2\sqrt{2}x_0'' + 2\Omega_0\,\Omega_1\sqrt{2}x_1'' + \Omega_1^2 x_0'' + \Omega_0\,\Omega_2'\,x_0' + \Omega_0\,\Omega_1'\,x_1'\\
&\quad + c_3\,[3x_2\,x_0^2 + 3x_0\,x_1^2 - 3(x_0\,y_1^2 + 2x_0\,y_0\,y_2 + 2x_1\,y_0\,y_1 + x_2\,y_0^2)]\\
&= \frac{\partial f_1}{\Delta x'}(\Omega_0\sqrt{2}x_1' + \Omega_1\,x_0') + \frac{\partial f_1}{\partial x}x_1 + \frac{\partial f_1}{\partial y'}(\Omega_0\sqrt{2}y_1' + \Omega_1\,y_0') + \frac{\partial f_1}{\partial y}y_1,
\end{aligned} \tag{51}$$

$$\begin{aligned}
&2\Omega_0^2 y_2'' + 2\Omega_0\,\Omega_2^*\sqrt{2}y_0'' + 2\Omega_0\,\Omega_1^*\sqrt{2}y_1'' + \Omega_1^{*2}\,y_0'' + \Omega_0\,\Omega_2^{*\prime}\,y_0' + \Omega_0\,\Omega_1^{*\prime}\,y_1'\\
&\quad + c_3\,[-3y_0\,(y_0\,y_2 + 3y_1^2) + 3(y_0\,x_1^2 + 2x_0\,x_1\,y_1 + 2x_0\,x_2\,y_0 + y_2\,x_0^2)]\\
&= \frac{\partial f_2}{\Delta x'}(\Omega_0\sqrt{2}x_1' + \Omega_1\,x_0') + \frac{\partial f_2}{\partial x}x_1 + \frac{\partial f_2}{\partial y'}(\Omega_0\sqrt{2}y_1' + \Omega_1\,y_0') + \frac{\partial f_2}{\partial y}y_1.
\end{aligned} \tag{52}$$

Here

$$f_1 \equiv f_1\,(x_0, y_0, \sqrt{2}\Omega_0\,x_0', \sqrt{2}\Omega_0\,y_0'), \qquad f_2 \equiv f_2\,(x_0, y_0, \sqrt{2}\Omega_0\,x_0', \sqrt{2}\Omega_0\,y_0'). \tag{53, 54}$$

The solutions of equations (47) and (48) can be summarized as

$$z_0 = x_0 + \mathrm{i}y_0 = A_0\,[\mathrm{dn}\,(\tau, 1/2) + (\mathrm{i}/\sqrt{2})\,\mathrm{sn}\,(\tau, 1/2)], \tag{55}$$

where

$$\Omega_0 = A_0\sqrt{c_3}.$$

One can now multiply equation (49) by x_0' and equation (50) by y_0' and integrate them, to obtain

$$\int_0^\tau f_1\,x_0'\,\mathrm{d}\tau = 2\Omega_0^2 x_1'\,x_0'\,|_0^\tau + \Omega_0\,\Omega_1\,x_0'^2\,(1 + \sqrt{2})\,|_0^\tau + c_3\,(x_0^3 x_1 - 3x_0\,x_1\,y_0^2 - 3x_0^2 y_0\,y_1\,|_0^\tau, \tag{56}$$

$$\int_0^\tau f_2\,y_0'\,\mathrm{d}\tau = 2\Omega_0^2 y_1'\,y_0'\,|_0^\tau + \Omega_0\,\Omega_1^*\,y_0'^2\,(1 + \sqrt{2})\,|_0^\tau - c_3\,(y_0^3 y_1 - 3y_0\,y_1\,x_0^2 - 3y_0^2 x_0\,x_1\,|_0^\tau. \tag{57}$$

Next one can integrate equations (56) and (57) over the period $4K$. Due to the periodic properties of the solution (44) one finds

$$\int_0^{4K} f_1\,x_0'\,\mathrm{d}\tau = 0, \qquad \int_0^{4K} f_2\,y_0'\,\mathrm{d}\tau = 0. \tag{58, 59}$$

From equation (58) or (59) the amplitude A_0 is obtained. This fact gives a constraint to the method: it is applicable only for the case when the motion has a limit cycle. Otherwise, the amplitude A_0 is zero, and the method is without meaning.

Integrating equations (51) and (52) yields

$$\int_0^{\tau}\left[\frac{\partial f_1}{\partial x}x_1\,x_0' + \frac{\partial f_1}{\partial x'}x_0'\,(x_1'\sqrt{2}\Omega_0 + x_0'\,\Omega_1) + \frac{\partial f_1}{\partial y}x_0'\,y_1 + \frac{\partial f_1}{\partial y'}x_0'\,(y_1'\sqrt{2}\Omega_0 + y_0'\,\Omega_1^*)\right]\mathrm{d}\tau$$

$$= [\Omega_0\,\Omega_2\,(\sqrt{2}+1) + (\Omega_1^2/2)]x_0'^2\,|_0^{\tau} + \Omega_0\,\Omega_1\,(2\sqrt{2}+1)x_0'\,x_1'\,|_0^{\tau} + 2\Omega_0^2 x_0'\,x_2'\,|_0^{\tau}$$
$$+ c_3\,[x_0^3 x_2 + \tfrac{3}{2}\,x_0^2 x_1^2 - 3(\tfrac{1}{2}\,x_0^2 y_1^2 + x_0^2 y_0\,y_2 + 2x_0\,x_1\,y_0\,y_1 + x_0\,x_2\,y_0^2)], \tag{60}$$

$$\int_0^{\tau}\left[\frac{\partial f_2}{\partial x}x_1\,y_0' + \frac{\partial f_2}{\partial x'}y_0'\,(x_1'\sqrt{2}\Omega_0 + x_0'\,\Omega_1) + \frac{\partial f_2}{\partial y}y_0'\,y_1 + \frac{\partial f_2}{\partial y'}y_0'\,(y_1'\sqrt{2}\Omega_0 + y_0'\,\Omega_1^*)\right]\mathrm{d}\tau$$

$$= [\Omega_0\,\Omega_2^*\,(\sqrt{2}+1) + \Omega_1^2/2]y_0'^2\,|_0^{\tau} + \Omega_0\,\Omega_1^*\,(2\sqrt{2}+1)y_0'\,y_1'\,|_0^{\tau} + 2\Omega_0^2 y_0'\,y_2'\,|_0^{\tau}$$
$$- c_3\,[\,y_0^3 y_2 + \tfrac{3}{2}\,y_0^2 y_1^2 - 3(\tfrac{1}{2}\,y_0^2 x_1^2 + x_0^2 y_0\,y_2 + 2x_0\,x_1\,y_0\,y_1 + x_0\,x_2\,y_0^2)]. \tag{61}$$

Integrating equations (60) and (61) over the time period $4K$ yields

$$\int_0^{4K}\left[\frac{\partial f_1}{\partial x}x_1\,x_0' + \frac{\partial f_1}{\partial x'}x_0'\,(x_1'\sqrt{2}\Omega_0 + x_0'\,\Omega_1) + \frac{\partial f_1}{\partial y}x_0'\,y_1 + \frac{\partial f_1}{\partial y'}x_0'\,(y_1'\sqrt{2}\Omega_0 + y_0'\,\Omega_1^*)\right]\mathrm{d}\tau = 0, \tag{62}$$

$$\int_0^{4K}\left[\frac{\partial f_2}{\partial x}x_1\,y_0' + \frac{\partial f_2}{\partial x'}y_0'\,(x_1'\sqrt{2}\Omega_0 + x_0'\,\Omega_1) + \frac{\partial f_2}{\partial y}y_0'\,y_1 + \frac{\partial f_2}{\partial y'}y_0'\,(y_1'\sqrt{2}\Omega_0 + y_0'\,\Omega_1^*)\right]\mathrm{d}\tau = 0. \tag{63}$$

Separating the variables Ω_1, Ω_1^*, A_1, A_1^* from equations (56), (57), (62) and (63) yields

$$\Omega_1 = W_0 + A_1\,W_1(\tau) + A_1^*\,W_2(\tau), \tag{64}$$

$$\Omega_1^* = W_0^* + A_1\,W_1^*(\tau) + A_1^*\,W_2^*(\tau), \tag{65}$$

$$A_1^* = \frac{N_0\,M_1 - N_1\,M_0}{M_1^*\,N_1 - M_1\,N_1^*}, \qquad A_1 = \frac{N_0\,M_1^* - M_0\,N_1^*}{M_1\,N_1^* - N_1\,M_1^*}, \tag{66, 67}$$

where

$$M_1 = \int_0^{4K}\left\{\frac{\partial f_1}{\partial x'}x_0'^2\left[\frac{\sqrt{2}\Omega_0}{A_0} + W_1(\tau)\right] + \frac{\partial f_1}{\partial x}\frac{1}{A_0}x_0\,x_0' + \frac{\partial f_1}{\partial y'}x_0'\,y_0'\,W_1^*(\tau)\right\}\mathrm{d}\tau, \tag{68}$$

$$M_1^* = \int_0^{4K}\left\{\frac{\partial f_1}{\partial x'}x_0'^2\,W_2(\tau) + \frac{\partial f_1}{\partial y}\frac{1}{A_0}y_0\,x_0' + \frac{\partial f_1}{\partial y'}x_0'\,y_0'\left[W_2^*(\tau) + \frac{\Omega_0\sqrt{2}}{A_0}\right]\right\}\mathrm{d}\tau, \tag{69}$$

$$N_1 = \int_0^{4K}\left\{\frac{\partial f_2}{\partial x'}x_0'\,y_0'\left[\frac{\sqrt{2}\Omega_0}{A_0} + W_1(\tau)\right] + \frac{\partial f_2}{\partial x}\frac{1}{A_0}x_0\,y_0' + \frac{\partial f_2}{\partial y'}y_0'^2\,W_1^*(\tau)\right\}\mathrm{d}\tau, \tag{70}$$

$$N_1^* = \int_0^{4K}\left\{\frac{\partial f_2}{\partial x'}x_0'\,y_0'\,W_2(\tau) + \frac{\partial f_2}{\partial y}\frac{1}{A_0}y_0\,y_0' + \frac{\partial f_2}{\partial y'}y_0'^2\left[W_2^*(\tau) + \frac{\Omega_0\sqrt{2}}{A_0}\right]\right\}\mathrm{d}\tau, \tag{71}$$

$$M_0 = \int_0^{4K}\frac{\partial f_1}{\partial x'}x_0'^2\,W_0(\tau)\,\mathrm{d}\tau + \int_0^{4K}\frac{\partial f_1}{\partial y'}x_0'\,y_0'\,W_0^*(\tau)\,\mathrm{d}\tau, \tag{72}$$

$$N_0 = \int_0^{4K} \frac{\partial f_2}{\partial x'} x_0' \, y_0' \, W_0(\tau) \, \mathrm{d}\tau + \int_0^{4K} \frac{\partial f_2}{\partial y'} y_0'^2 \, W_0^*(\tau) \, \mathrm{d}\tau, \tag{73}$$

$$W_1(\tau) = -\frac{2\Omega_0}{A_0(1+\sqrt{2})} - \frac{c_3(x_0^3 + 3x_0 y_0^2)}{x_0'^2(1+\sqrt{2})\Omega_0 A_0}, \qquad W_2(\tau) = \frac{3c_3 x_0^2 y_0}{A_0 x_0'^2 (1+\sqrt{2})\Omega_0}, \tag{74, 75}$$

$$W_0(\tau) = \frac{\int_0^t f_1 x_0' \, \mathrm{d}\tau}{x_0'^2(1+\sqrt{2})\Omega_0}, \qquad W_1^*(\tau) = -\frac{3c_3 x_0 y_0^2}{A_0 \Omega_0 y_0'^2 (1+\sqrt{2})}, \tag{76, 77}$$

$$W_2^*(\tau) = \frac{2\Omega_0}{A_0(1+\sqrt{2})} - \frac{c_3(y_0^3 + 3y_0 x_0^2)}{y_0'^2(1+\sqrt{2})\Omega_0 A_0}, \qquad W_0^*(\tau) = \frac{\int_0^t f_2 y_0' \, \mathrm{d}\tau}{y_0'^2(1+\sqrt{2})\Omega_0}. \tag{78, 79}$$

The solution of equation (2) in the first approximation is

$$z = (A_0 + \varepsilon A_1) \, \mathrm{dn}(\tau, 1/2) + \mathrm{i}(A_0/\sqrt{2}) + \varepsilon A_1^*) \, \mathrm{sn}(\tau, 1/2), \tag{80}$$

$$\begin{aligned} \dot{z} = &-\tfrac{1}{2}(A_0 + \varepsilon A_1)(\Omega_0 \sqrt{2} + \varepsilon \Omega_1) \, \mathrm{sn}(\tau, 1/2) \, \mathrm{cn}(\tau, 1/2) \\ &+ \mathrm{i}(A_0/\sqrt{2}) + \varepsilon A_1^*)(\Omega_0 \sqrt{2} + \varepsilon \Omega_1^*) \, \mathrm{cn}(\tau, 1/2) \, \mathrm{dn}(\tau, 1/2). \end{aligned} \tag{81}$$

6. EXAMPLES

6.1. EXAMPLE 1

Consider the motion of the rotor centre described by the equation

$$\ddot{z} + c_3 z^3 = \varepsilon(1 - z\bar{z})\dot{z}, \tag{82}$$

where $\bar{z}$ is the complex conjugate function. Applying the method of harmonic balance and substituting the solution (20) into equation (82) yields

$$\begin{aligned} &-A\Omega^2\sqrt{2}(\mathrm{cn}^2 + \mathrm{i}\sqrt{2} \, \mathrm{sn} \, \mathrm{dn}) + A^3 c_3(\mathrm{dn}^2 + \mathrm{i}\sqrt{2} \, \mathrm{dn} \, \mathrm{sn} - \tfrac{1}{2}\mathrm{sn}^2) \\ &= \varepsilon(1 - A^2) A \mathrm{i}\Omega \, \mathrm{cn}. \end{aligned} \tag{83}$$

Separating the real and imaginary parts, one has

$$\Omega = A_s \sqrt{c_3} \qquad \text{and} \qquad A_s^2 = 1. \tag{84, 85}$$

Applying the method of time variable amplitude and phase (see section 4) transforms the differential equation (82) into two first order differential equations:

$$\dot{A}(t) = (\varepsilon/2)(1 - A^2)A, \qquad \dot{\phi}(t) = 0. \tag{86, 87}$$

For the initial conditions $t = 0$, $A(0) = A_0$, $\phi(0) = \phi_0$, the solution for transient motion described by equations (86) and (87) is

$$\phi = \phi_0, \qquad \frac{1 - A}{1 - A_0} \frac{1 + A_0}{1 + A} \frac{A_0^2}{A^2} = \mathrm{e}^{-\varepsilon t}. \tag{88}$$

In Figure 1 the amplitude–time diagrams for various initial conditions are plotted. The steady state solution is $A_s = 1$ and is the same as obtained by the harmonic balance method (see relation (85)).

Upon applying the elliptic perturbation method it is evident that the conditions (58) and (59) are satisfied only for $A_0 = 1$. For this value of the amplitude, the values of Ω_1, Ω_1^*, A_1, A_1^* are zero.

In Figure 2 the motion of the rotor centre described by equations (82) is plotted. The initial conditions are $A(0) = A_0 = 1$, $\phi(0) = \phi_0 = 0$, and the parameter of the system is

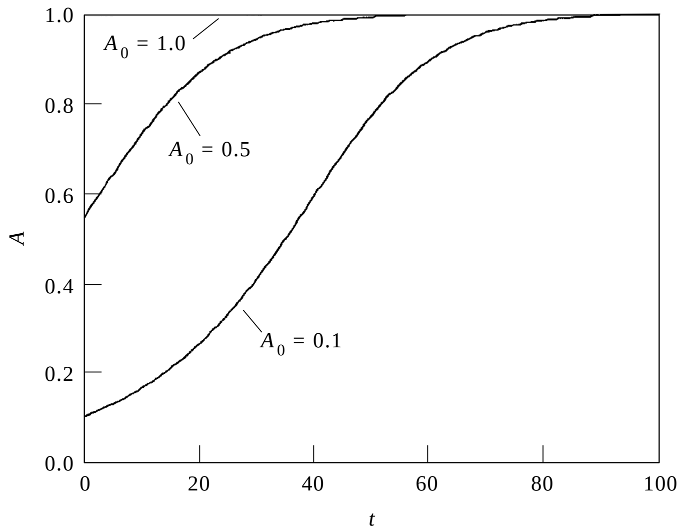

Figure 1. Amplitude–time diagrams for various initial conditions.

$c_3 = 1$. In Figure 2(a) the $x_A - y_A$ and in Figure 2(b) the $x_N - y_N$ diagrams obtained analytically and numerically (by applying the Runge–Kutta procedure), respectively, are plotted. The trajectory of motion in the x–y plane is an arc. It can be concluded that the analytically obtained solution lies on top of the numerical solution.

6.2. EXAMPLE 2

Consider the case when the differential equation of motion is

$$\ddot{z} + c_3 z^3 = \varepsilon(1 - z\bar{z})z, \tag{89}$$

where

$$f_1 = (1 - x_0^2 - y_0^2)x_0, \qquad f_2 = (1 - x_0^2 - y_0^2)y_0. \tag{90, 91}$$

To obtain the solution the elliptic perturbation method is assumed. Substituting the solution (55) into equations (58) and (59) yields

$$(1 - A_0^2)A_0^2 \int_0^{4K} \mathrm{sn}\,\mathrm{cn}\,\mathrm{dn}\,\mathrm{d}\tau = 0. \tag{92}$$

From equation (92) it can be concluded that for $A_0 = 1$ the trajectory of motion is a limit cycle. For the small non-linear terms (90) and (91) the coefficients M_0 and N_0 are zero, and

$$A_1 = 0, \qquad A_1^* = 0. \tag{93}$$

Then

$$\Omega_1 = W_0, \qquad \Omega_1^* = W_0^*, \tag{94}$$

where

$$W_0(\tau) = -\frac{(1 - A_0^2)}{(1 + \sqrt{2})\Omega_0 (2\,\mathrm{dn}^2 - 1)},$$

$$W_0^* = -\frac{1 - A_0^2}{4\,\mathrm{dn}^2 (1 + \sqrt{2})\Omega_0}.$$

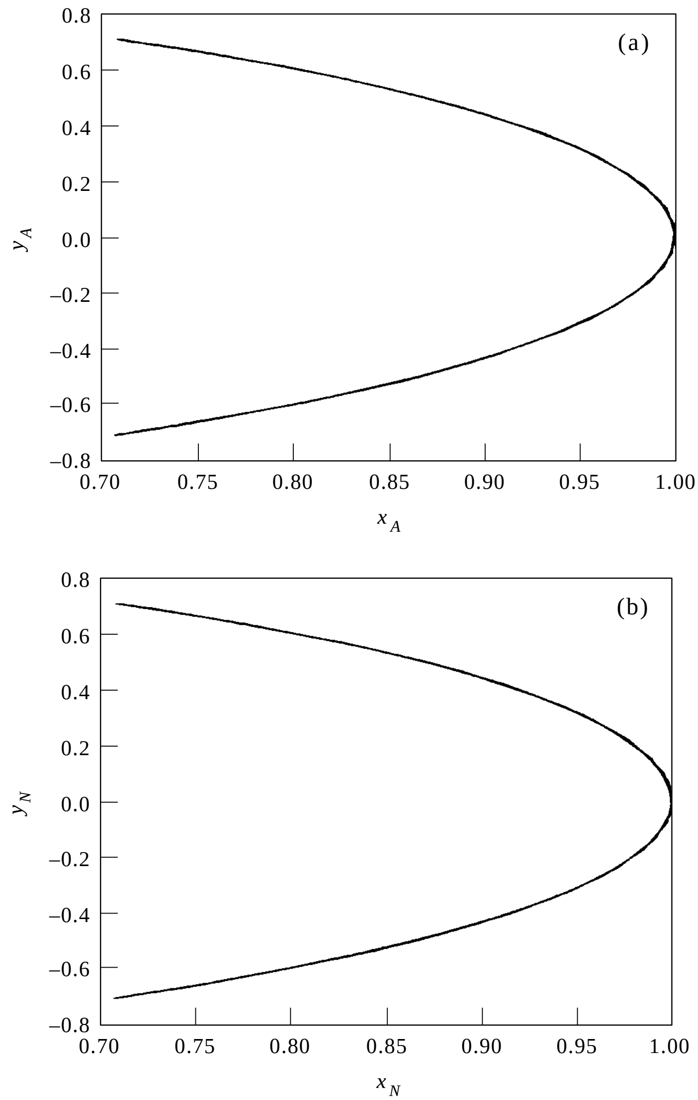

Figure 2. x–y diagrams obtained (a) analytically, (b) numerically.

For $A_0 = 1$ the solution of equation (89) is

$$z = A_0 \,\mathrm{dn}\,(A_0 \sqrt{2c_3} t,\, 1/2) + (A_0 \,\mathrm{i}/\sqrt{2})\,\mathrm{sn}\,(A_0 \sqrt{2c_3}\, t(1/2)). \tag{95}$$

Applying the Krylov–Bogoliubov method transforms equation (89) into a system of two first order differential equations:

$$A^2(t)\dot{\phi}(t)\sqrt{c_3}\,\mathrm{cn}^2 = -\varepsilon[1 - A^2(t)]A(t)\sqrt{2}, \qquad \dot{A}(t) = 0. \tag{96, 97}$$

After averaging of the elliptic function over the period $4K$ one obtains

$$A(t)\dot{\phi}(t) = -\frac{\varepsilon\sqrt{2}[1 - A^2(t)]}{4(2E - K)\sqrt{c_3}}, \qquad \dot{A}(t) = 0.$$

Integrating the equations for the initial conditions $A(0) = A_0$ and $\phi(0) = \phi_0$ yields

$$\phi(t) = \phi_0 - \frac{\varepsilon\sqrt{2}(1 - A_0^2)}{4(2E - K)A_0\sqrt{c_3}}\, t,$$

and the solution is

$$z(t) = A_0\left[\mathrm{dn}\left(\sqrt{2c_3}\, A_0\, t - \frac{\varepsilon\sqrt{2}(1 - A_0^2)t}{4(2E - K)A_0\sqrt{c_3}} + \phi_0, \tfrac{1}{2}\right)\right.$$
$$\left. + \frac{\mathrm{i}}{\sqrt{2}}\,\mathrm{sn}\left(\sqrt{2c_3}\, A_0\, t - \frac{\varepsilon\sqrt{2}(1 - A_0^2)t}{4(2E - K)A_0\sqrt{c_3}} + \phi_0, \tfrac{1}{2}\right)\right]. \tag{98}$$

For the initial conditions $A(0) = A_0 = 1$ and $\phi(0) = \phi_0 = 0$, the solution is the same as that obtained by elliptic perturbation (95). For the aforementioned initial conditions and parameter value $c_3 = 1$, the x–t and y–t diagrams obtained by analytical and numerical solving of (89) are plotted in Figure 3. The solutions show good agreement.

6.3. EXAMPLE 3

Consider the case when the differential equation of motion is

$$\ddot{z} + z^3 = -(\varepsilon/4)\dot{\bar{z}}[(z^2 + \dot{z}^2) - (\bar{z}^2 + \dot{\bar{z}}^2)]; \tag{99}$$

i.e.,

$$\ddot{x} + (x^3 - 3xy^2) = \varepsilon f_1, \qquad \ddot{y} + (3x^2y - y^3) = \varepsilon f_2,$$

where

$$f_1 = -\dot{y}(xy + \dot{x}\dot{y}), \qquad f_2 = -\dot{x}(xy + \dot{x}\dot{y}). \tag{100}$$

By applying the suggested elliptic perturbation procedure the limit cycle motion is obtained. Substituting equations (100) into equation (58) or (59) yields

$$\int_0^{4K} (\mathrm{sn}^2\,\mathrm{cn}^2\,\mathrm{dn}^2 - 2A_0^2\,\mathrm{sn}^2\,\mathrm{cn}^4\,\mathrm{dn}^2)\,\mathrm{d}\tau = 0, \tag{101}$$

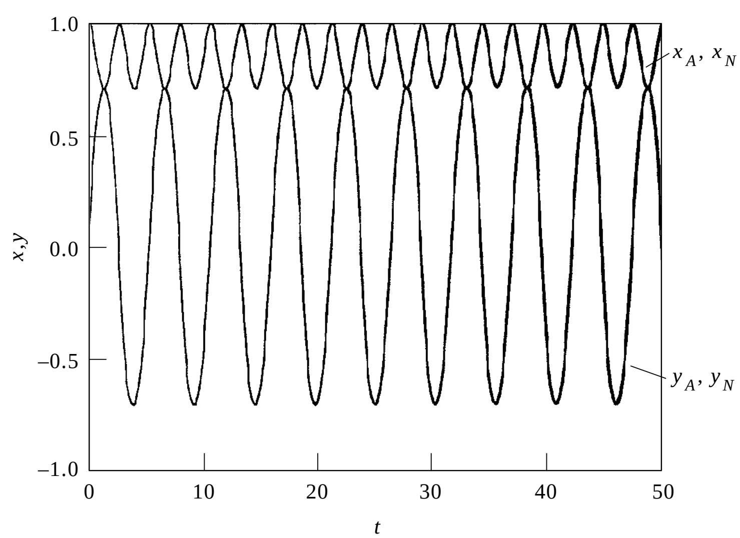

Figure 3. x–t and y–t diagrams obtained analytically and numerically.

where

$$\mathrm{sn} \equiv \mathrm{sn}\,(\tau, 1/2), \qquad \mathrm{cn} \equiv \mathrm{cn}\,(\tau, 1/2), \qquad \mathrm{dn} \equiv \mathrm{dn}\,(\tau, 1/2).$$

Calculating the relation (101) gives

$$A_0 = 0{\cdot}8544.$$

The correction values are

$$W_0 = \frac{-0{\cdot}44333\tau + 0{\cdot}60857E(\tau) + 0{\cdot}53476\,\mathrm{sn\,cn\,dn} - 0{\cdot}41714\,\mathrm{sn}^3\,\mathrm{cn\,dn} + 0{\cdot}20857\,\mathrm{sn}^5\,\mathrm{cn\,dn}]}{1{\cdot}5031\,\mathrm{sn}^2\,\mathrm{cn}^2}$$

$$W_0^* = \frac{-0{\cdot}44333\tau + 0{\cdot}60857E(\tau) + 0{\cdot}53476\,\mathrm{sn\,cn\,dn} - 0{\cdot}41714\,\mathrm{sn}^3\,\mathrm{cn\,dn} + 0{\cdot}20857\,\mathrm{sn}^5\,\mathrm{cn\,dn}]}{0{\cdot}7515\,\mathrm{dn}^2\,\mathrm{cn}^2},$$

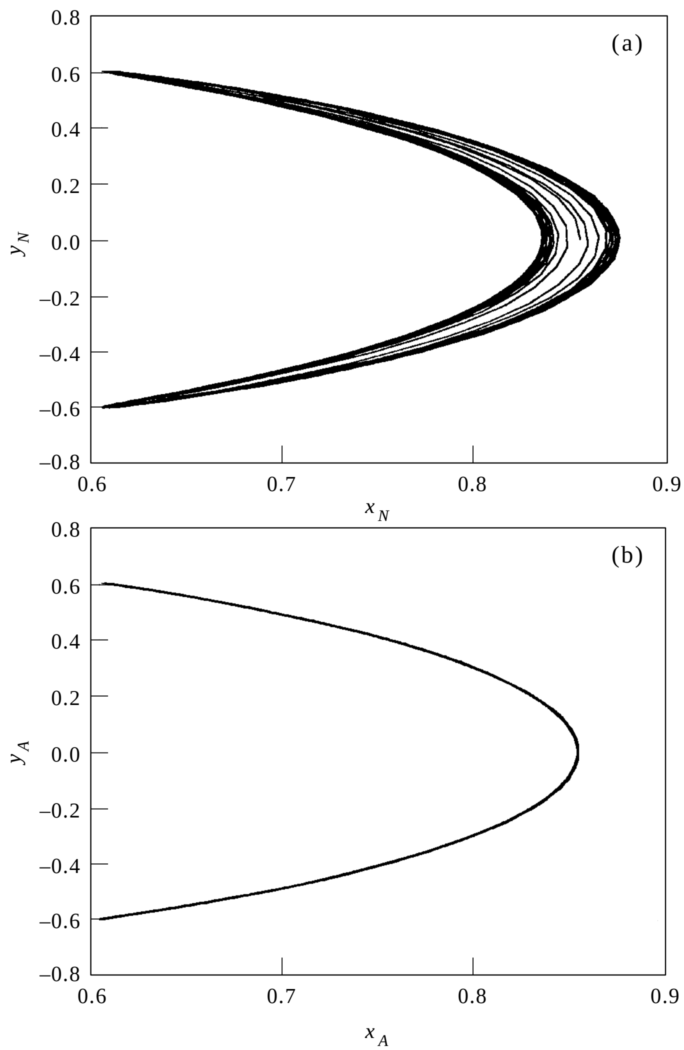

Figure 4. x–y diagrams obtained (a) numerically, (b) analytically.

$$W_1 = -0{\cdot}82988 - 0{\cdot}35452\,\mathrm{dn}^2\,\frac{\mathrm{dn} + 2{\cdot}1213\,\mathrm{sn}}{\mathrm{sn}^2\,\mathrm{cn}^2}, \qquad W_1^* = -1{\cdot}4569\,\frac{\mathrm{sn}^2}{\mathrm{cn}^2\,\mathrm{dn}},$$

$$W_2 = 1{\cdot}0302\,\frac{\mathrm{dn}^2}{\mathrm{sn}\,\mathrm{cn}^2}, \qquad W_2^* = 0{\cdot}82988 - \mathrm{sn}\,\frac{4\,\mathrm{sn}^2 + 3\,\mathrm{dn}^2}{1{\cdot}7041\,\mathrm{cn}^2\,\mathrm{dn}^2}$$

and the solution in the first approximation is

$$z = 0{\cdot}90130\,\mathrm{dn}\left(1{\cdot}2083t + 0{\cdot}1\int_0^\tau \Omega_1\,\mathrm{d}\tau\right) + 0{\cdot}58415\,\mathrm{sn}\left(1{\cdot}2083t + \varepsilon\int_0^\tau \Omega_1^*\,\mathrm{d}\tau\right), \quad (102)$$

where

$$\Omega_1 = W_0 + A_1\,W_1 + A_1^*\,W_2, \qquad \Omega_1^* = W_0^* + A_1\,W_1^* + A_1^*\,W_2^*.$$

The solution of equation (99) is obtained numerically by applying the Runge–Kutta procedure. The parameter of the system is $\varepsilon = 0{\cdot}1$. The initial conditions are $A(0) = A_0 = 0{\cdot}8544$ and $\phi(0) = \phi_0 = 0$. The analytically obtained solution (102) is compared with the numerical one. In Figure 4 the x–y diagrams are plotted. The analytical solution represents the averaged value of the "exact" numerical solution.

7. CONCLUSION

In this paper approximate analytical methods for solving non-linear differential equations with a strong cubic complex term have been presented. The harmonic balance method, the method of Krylov and Bogoliubov and the elliptic perturbation method have been extended for equations in complex function. The solution is based on the generating solution of a strongly non-linear differential equation with a cubic complex term. The solution is described with Jacobi elliptic functions. The method of elliptic perturbation is applicable for the case when the motion has a limit cycle. The solution obtained by the method presented in this paper is a particular solution which describes the limit cycle motion. After applying the methods for solving differential equations which describe the vibrations of the rotor it can be concluded that the suggested methods give solutions which are in good agreement with those obtained numerically, even for high values of non-linearity.

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