Journal of Sound and Vibration (1998) **214**(2), 377–382
Article No. sv971477

LETTERS TO THE EDITOR

FREQUENCY-LOCKED MOTION AND QUASI-PERIODIC MOTION OF A PIECEWISE-LINEAR SYSTEM SUBJECTED TO EXTERNALLY NON-SYNCHRONOUS EXCITATIONS

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(*Received* 29 *May* 1997, *and in final form* 25 *November* 1997)

1. introduction

Piecewise-linear systems exhibit non-linear behavior due to clearance or backlash between components. It is well known that such systems subjected to two excitations with non-multiple frequencies are commonly used in many engineering fields. Due to strong non-linearity, many phenomena such as superharmonic, subharmonic, frequency-locked, quasi-periodic and chaotic motions may exist or coexist.

Since the harmonic balance method is based on Fourier expansions of the state variables and non-linear terms, a generalized form would suffer in its effectiveness for obtaining steady state responses of a SDOF non-linear system which is subjected to external forces with more than two frequencies.

Choi and Noah [1] applied the fixed point algorithm (FPA) and Kim [2] proposed a modified FPA to perform the stability and bifurcation analyses of SDOF non-linear systems with multi-input frequencies. Although the fixed point algorithm appeared to be better than the harmonic balanced method, the patterns of bifurcation and Poincaré maps related to the chaotic responses and quasi-periodic motions that they obtained were not very accurate.

Oks *et al*. [3] investigated the suppression phenomena of resonant oscillations in these strong non-linear SDOF systems subjected to parametric or forced excitations with two incommensurate frequencies. The resonance and non-resonance regions were determined by using the numerical integration method; however, the details of the periodic, subharmonic, and chaotic motions were not observed.

In the present study, a strongly non-linear SDOF system modelled by a piecewise-linear stiffness and subjected to two forced excitations is considered. This system is simulated by the fourth order Runge–Kutta method for various initial conditions and the solutions are analyzed by the J integral due to a proposition by Kang *et al*. [4]. Furthermore, the bifurcation diagrams of the J integral are constructed to illustrate the jump phenomenon, frequency-locked motion, quasi-periodic motion, and chaos with the assistance of Poincaré maps and frequency spectra.

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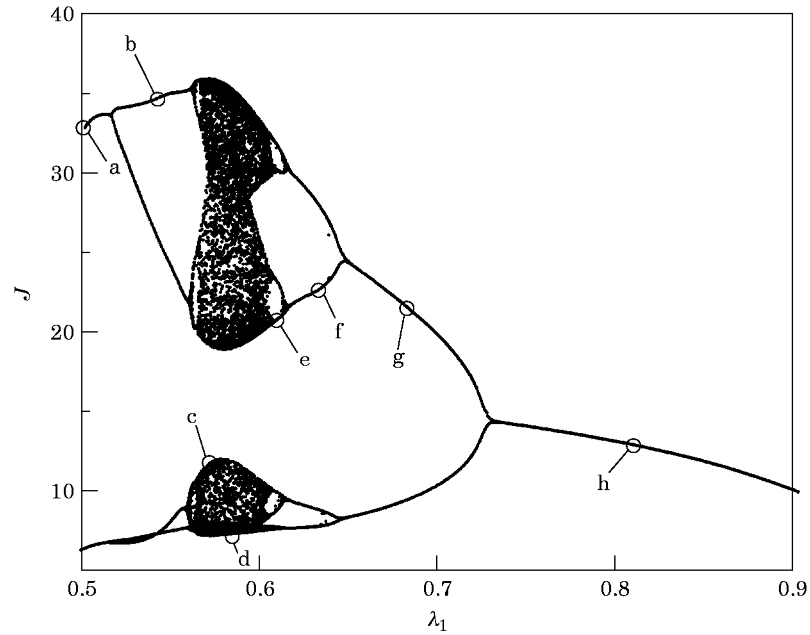

Figure 1. Bifurcation of J integral versus amplitude λ_1; $\nu_1 = 1{\cdot}28$, $\nu_2 = 1{\cdot}0$, $k = 2{\cdot}5$, $\lambda_2 = 0{\cdot}5$, $\gamma = 0{\cdot}08$.

2. EQUATIONS OF MOTION AND THE ANALYSIS

Consider this system to be described by the following non-dimensional equation

$$\ddot{x} + \gamma\dot{x} + x + f(x) = \lambda_1 \sin \nu_1\tau + \lambda_2 \sin \nu_2\tau, \tag{1a}$$

where

$$f(x) = \begin{Bmatrix} 0 \\ k(x - \operatorname{sgn} x) \end{Bmatrix} \begin{matrix} \text{for } |x| \leqslant 1 \\ \text{for } |x| > 1, \end{matrix} \tag{1b}$$

and the superscript dot denotes the differentiation with respect to τ.

The steady state solutions of equations (1a, 1b) are obtained from numerical time integration by using a fourth order Runge–Kutta method for various values of the system parameters. An integral monitored on the computer may be utilized as

$$J = \int_0^{\text{one forcing period}} [\dot{x}(\tau)]^2 \, \mathrm{d}\tau, \tag{2}$$

which has been proposed by Kang *et al.* [4].

The initial conditions of the non-dimensional displacement are from $-2{\cdot}5$–$2{\cdot}5$ and the non-dimensional velocity from $-5{\cdot}0$–$5{\cdot}0$. $N = 600$ is adopted for a time interval $\Delta\tau = 2\pi/(\nu_1 N)$ in numerical integration. Before the performance of monitoring long terms were not recorded for the avoidance of transient solutions. When values of the J integral are computed for this system with various initial conditions, period-one motions (P1) correspond to a single integral value, period-two motions (P2) correspond to two integral values, . . . , and chaotic behaviors correspond to many J integrals. Therefore, multiple solutions and response types are identified and classified by the J integral very efficiently.

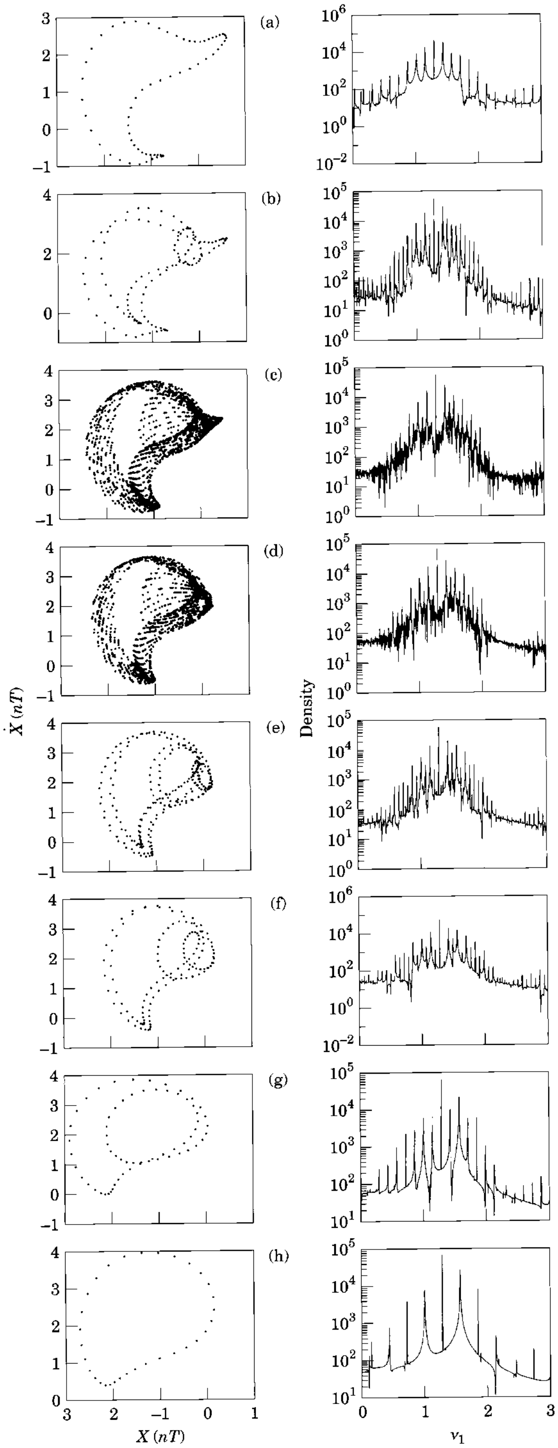

Figure 2. Poincaré map (left) and frequency spectrum (right) for points in Figure 1; (a) P2 motion at point a ($\lambda_1 = 0{\cdot}5$); (b) P4 motion at point b ($\lambda_1 = 0{\cdot}54$); (c) Chaotic motion at point c ($\lambda_1 = 0{\cdot}57$); (d) Chaotic motion at point d ($\lambda_1 = 0{\cdot}59$); (e) P8 motion at point e ($\lambda_1 = 0{\cdot}61$); (f) P4 motion at point f ($\lambda_1 = 0{\cdot}63$); (g) P2 motion at point g ($\lambda_1 = 0{\cdot}7$); (h) P1 motion at point h ($\lambda_1 = 0{\cdot}8$).

Additionally, Poincaré maps are constructed by sampling the steady state responses over 2000 points $(x, \dot{x})$ for one forcing period ($T = 2\pi/\nu_1$) and frequency spectra are determined by using 200 forced periods.

3. FREQUENCY-LOCKED OSCILLATIONS

When a system governed by equations (1a, 1b) has a rational ratio between two frequencies, the frequency-locked oscillation can be obtained.

For example, a system has parameters at $\nu_1 = 1{\cdot}28$, $\nu_2 = 1$, $k = 2{\cdot}5$, $\lambda_2 = 0{\cdot}5$, $\gamma = 0{\cdot}08$ and $0{\cdot}5 < \lambda_1 < 0{\cdot}9$ with various initial conditions. Figure 1 shows the bifurcation of the J integral versus amplitude λ_1 to illustrate that a complete period-doubling cascade leads to chaos by two inverted ways as the amplitude increases and decreases.

A period-n (Pn) motion has the nth subharmonic order which can be monitored by n values of the J integral. Also, chaos may be distinguished from these frequency-locked motions by the occurrence of an immense amount of the J integral. Thus, P1 motion is observed at point h, P2 motions at points a and g, P4 motions at points b and f, P8 motion at point e, and chaotic motions at points c and d, respectively.

Poincaré maps and frequency spectra corresponding to points denoted from a–h in Figure 1 are shown in Figure 2. Because of the frequency ratio $\nu_2/\nu_1 = 25/32$, the Poincaré sections of the frequency-locked motions are composed of a multiple number of 32 points.

4. QUASI-PERIODIC OSCILLATION

For another example, this system has an irrational ratio between two frequencies, which exhibits quasi-periodic motions, When the parameters of equations (1a, 1b) are $\nu_1 = \sqrt{2}$, $\nu_2 = 1$, $\lambda_1 = 0{\cdot}7$, $\lambda_2 = 0{\cdot}5$, and $\gamma = 0{\cdot}06$ with various initial conditions, the bifurcation of the J integral versus stiffness k is shown in Figure 3. A period-doubling cascade leading to chaotic motions as the stiffness increases is observed, and a sudden transition from a

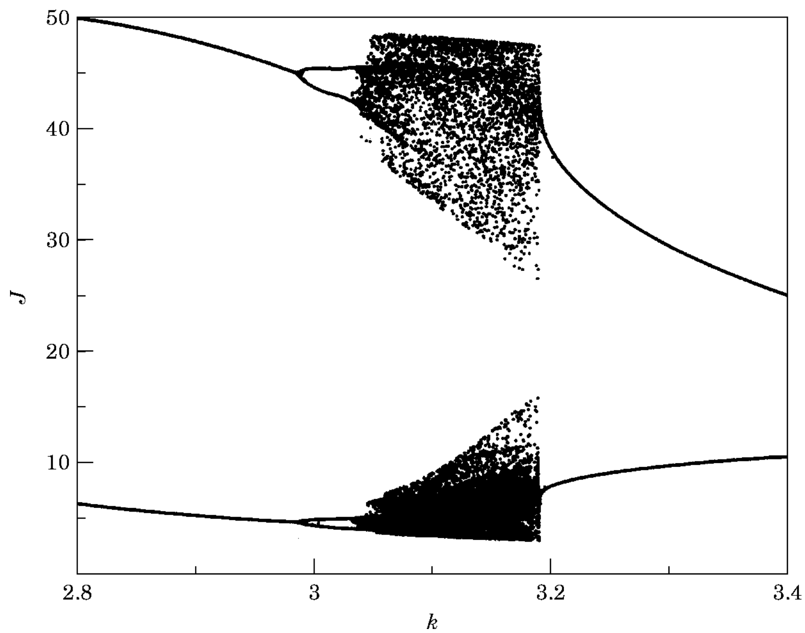

Figure 3. Bifurcation of J integral versus stiffness k; $\nu_1 = \sqrt{2}$, $\nu_2 = 1{\cdot}0$, $\lambda_1 = 0{\cdot}7$, $\lambda_2 = 0{\cdot}5$, $\gamma = 0{\cdot}06$,

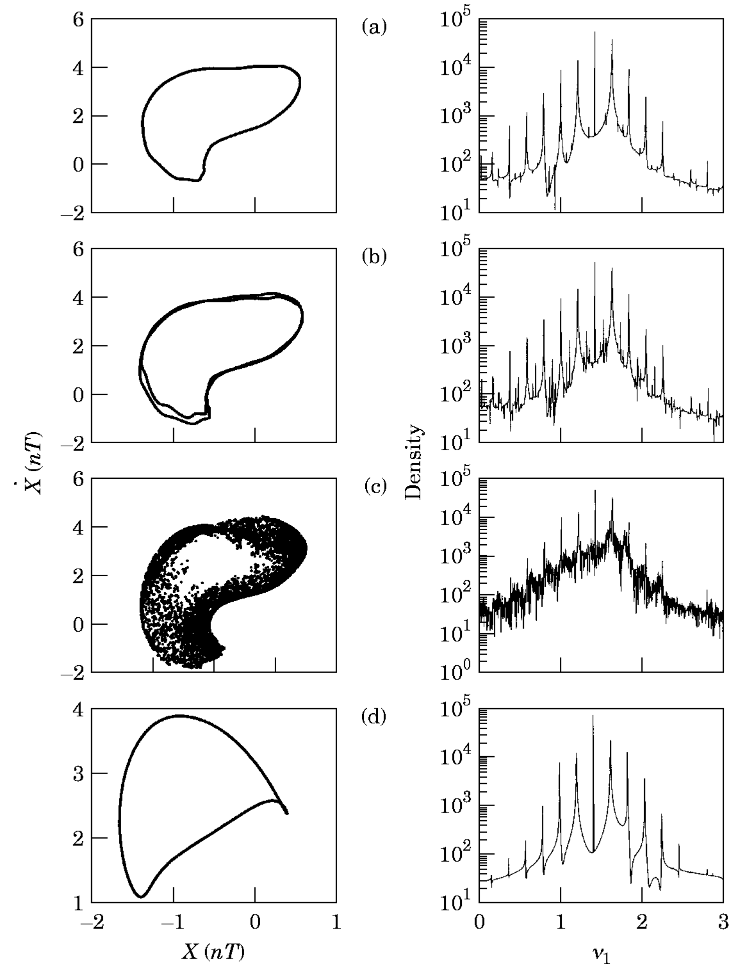

Figure 4. Poincaré map (left) and frequency spectrum (right) for points in Figure 3; (a) P2 motion at $k = 2{\cdot}9$; (b) P4 motion at $k = 3{\cdot}0$; (c) chaotic motion at $k = 3{\cdot}1$; (d) P2 motion at $k = 3{\cdot}3$.

P2 motion to chaos occurs at about $k = 3{\cdot}191$ as stiffness decreases. P2, P4 and chaotic behaviors monitored by 2, 4, and huge numbers of J integral values can be observed in Figure 3.

Frequency spectra and Poincaré maps are shown in Figure 4 for four values of stiffness in Figure. 3. Because the frequency ratio is irrational, Poincaré sections of quasi-periodic responses are constructed by continuous curves because the motion never exactly repeats itself. These motions are composed of two periodic components. Thus, the frequency spectra of the motions contain the nth harmonics $|v_2 - v_1|/n$ as shown in Figure 4(a), (b), and (d). As the chaotic behavior occurs, the frequency spectrum and a corresponding Poincaré map become random-like structure, as shown in Figure 4(c).

5. CONCLUSION

Oscillations of a SDOF piecewise-linear system subjected to forced excitations with two harmonic frequencies has been investigated. A J integral was applied to determine the response types and to construct the bifurcation diagrams. Phenomena including jump behavior, subharmonics of various orders, period-doubling, and intermittency to chaos were observed by the J bifurcation. Also, frequency-locked oscillations, quasi-periodic oscillations, and chaotic motions have been distinguished.

ACKNOWLEDGMENT

This study was supported by the National Science Council of the Republic of China under grant number NSC 85-2212-E033-007.

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