Journal of Sound and Vibration (1998) **214**(2), 389–393
Article No. sv981542

FUNDAMENTAL FREQUENCIES OF A MEMBRANE STRIP WITH PERIODIC BOUNDARY CONSTRAINTS

C. Y. Wang

Departments of Mathematics and Mechanical Engineering, Michigan State University, East Lansing, MI 48824, *U.S.A.*

(*Received* 4 *February* 1998)

1. INTRODUCTION

The vibration of membranes is a basic topic in the theory of sound [1]. The behavior of vibrating membranes is also related to that of vibrating plates [2] and electromagnetic waveguides [3]. Many solutions to the governing Helmholtz equation for various geometries have been found [4]. Recently the vibration of a membrane strip with small boundary corrugations was studied [5]. Using perturbation theory, it was found the amplitude and the phase difference of the corrugation have decisive effects on the frequency.

The present note studies the effect of a large boundary perturbation, i.e., a long membrane strip with additional periodic constraints at the boundaries. These constraints may be due to membrane mounting or structural stiffening or used to increase the fundamental frequency. Such a problem can be treated by several methods, such as finite differences, variational methods, and Green's functions. The method of eigenfunction expansion and point match is used, which seems to be the simplest.

2. FORMULATION

Figure 1 shows the two types of constraints considered. All lengths have been normalized by the half width L of the membrane strip. The governing equation is

$$w_{xx} + w_{yy} + k^2 w = 0, \tag{1}$$

where $k = (\text{frequency})L\sqrt{[(\text{tension per length})/(\text{mass per area})]}$. The boundary condition is that $w = 0$ on all boundaries including the constraints. One considers first the in-phase case Figure 1(a). For the fundamental frequency, the entire membrane rises and falls in unison. Due to symmetry one needs to consider only the cell $|x| \leqslant a$, $|y| \leqslant 1$. The boundary conditions are

$$w(x, \pm 1) = 0, \qquad w(\pm a, y) = 0, \quad 1 - b < |y| \leqslant 1 \tag{2, 3}$$

$$(\partial w/\partial x)(\pm a, y) = 0, \qquad 0 \leqslant |y| < 1 - b. \tag{4}$$

The solution satisfying equations (1) and (2) is

$$w(x, y) = \sum_{n=1}^{\infty} A_n \cos(\alpha_n y) F_n(x), \tag{5}$$

where $\alpha_n = (n - \frac{1}{2})\pi$, $\qquad \lambda_n = \sqrt{|\alpha_n^2 - k^2|}$ and

$$F_n(x) = \begin{Bmatrix} \cos(\lambda_n x), & \alpha_n < k \\ e^{\lambda_n (x-a)} + e^{-\lambda_n (x+a)}, & \alpha_n > k \end{Bmatrix}. \tag{6}$$

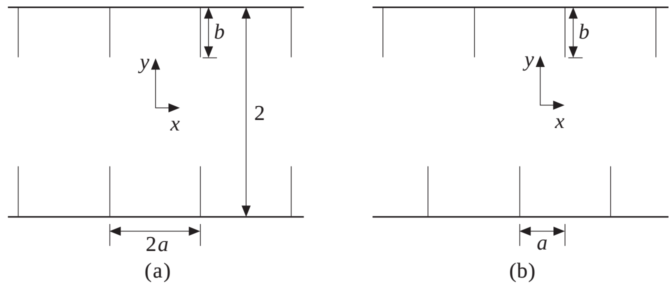

Figure 1. The membrane strip with constraints (a) in-phase, (b) staggered.

By truncating the series to N terms and satisfying equations (3) and (4) by point match at N equally-spaced points:

$$\sum_{n=1}^{N} A_n \cos(\alpha_n y_j) F_n(a) = 0, \qquad 1 - b < y_j \leqslant 1, \tag{7}$$

$$\sum_{n=1}^{N} A_n \cos(\alpha_n y_j) F_n'(a) = 0, \qquad 0 \leqslant y_j < 1 - b, \tag{8}$$

where

$$y_j = (j - \tfrac{1}{2})/N, \qquad j = 1, 2, \ldots, N. \tag{9}$$

For non-trivial A_n, the determinant of the coefficients of equations (7) and (8) is set to zero. A simple root search gives the eigenvalue k. Accuracy is ascertained by increasing N. Usually $N = 30$ is sufficient for three significant figures in k.

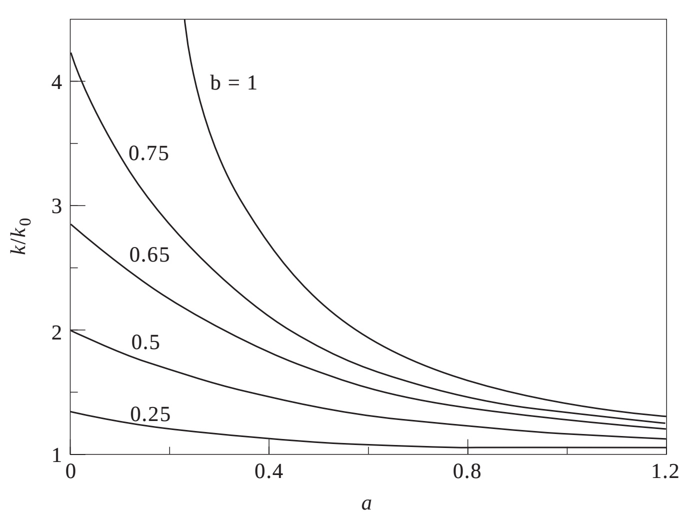

Figure 2. Fundamental frequencies for the in-phase case.

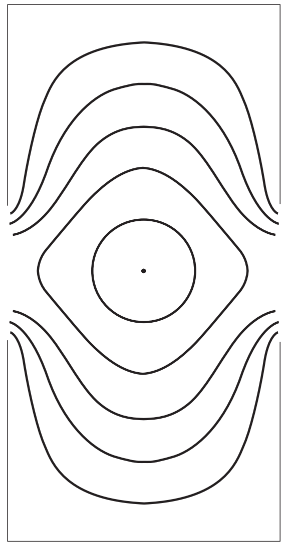

Figure 3. Typical displacements for the in-phase case ($a = 0{\cdot}5$, $b = 0{\cdot}75$).

If the constraints are staggered as in Figure 1(b), the co-ordinates are centered such that a polar symmetry exists:

$$w(x, y) = w(-x, -y). \tag{10}$$

The boundary conditions are equation (2) and

$$w(a/2, y) = 0, \qquad 1 - b < y \leqslant 1, \tag{11}$$

$$(\partial w/\partial x)(a/2, y) = 0, \qquad -1 \leqslant y < 1 - b. \tag{12}$$

The general solution satisfying equations (1), (2) and (10) is

$$w(x, y) = \sum_{1}^{N} [A_n \cos(\alpha_n y) F_n(x) + B_n \sin(\beta_n y) G_n(x)], \tag{13}$$

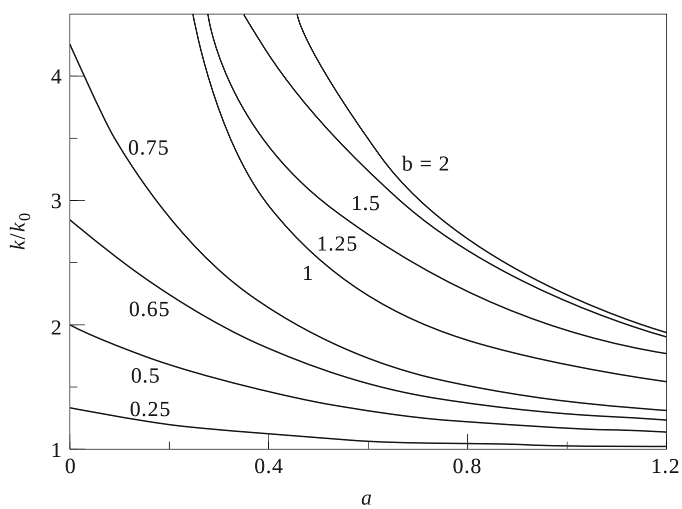

Figure 4. Fundamental frequencies for the staggered case.

where $\beta_n = n\pi$, $\gamma_n = \sqrt{|\beta_n^2 - k^2|}$ and

$$G_n(x) = \begin{cases} \sin(\gamma_n x), & \beta_n < k, \\ e^{\gamma_n(x - a/2)} - e^{-\gamma_n(x + a/2)}, & \beta_n > k \end{cases}. \tag{14}$$

Using $y_i = -1 + (i - 1/2)/N$, $i = 1, 2, \ldots, 2N$, equations (11) and (2) give $2N$ linear homogeneous equations. The value of k is determined as before.

3. RESULTS AND DISCUSSION

Figure 2 shows the fundamental frequency for the in-phase case. When $b = 0$, there are no additional constraints and the frequency for the strip is $k = k_0 = \pi/2$. This is also the asymptotic value when $a \to \infty$ or when the contstraints are far apart. When $a \to 0$ the constraints are stacked together, giving an effective width of $2(1 - b)$. Thus

$$k/k_0 = 1/(1 - b). \tag{15}$$

When $b = 1$ the constraints partition the membrane strip into rectangles. The frequency is accordingly [1]

$$k/k_0 = \sqrt{1 + 1/a^2}. \tag{16}$$

Using the value of k found, the eigenfunction $w(x, y)$ can be obtained by setting $A_1 = 1$, deleting the last equation from equations (7) and (8) and solving for the remaining $A_n's$. Figure 3 shows a typical displacement distribution.

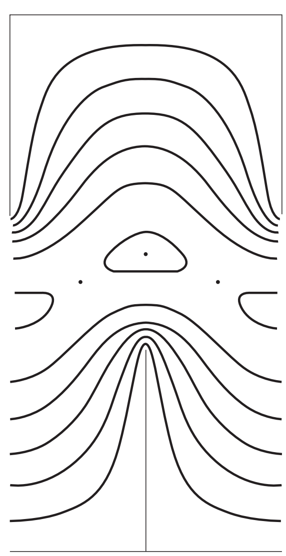

Figure 5. Typical displacements for the staggered case ($a = 0{\cdot}5$, $b = 0{\cdot}75$).

The fundamental frequency for the staggered constraints are shown in Figure 4. Since staggered constraints do not overlap, the length b can be as long as the width of the strip. For $b = 2$ we find

$$k/k_0 = \sqrt{1 + 4/a^2}. \tag{17}$$

Note for large a, the frequency is insensitive to b between 1·5 and 2.

In comparison to the in-phase results, for the same $b(<1)$, the frequency for the staggered case is somewhat higher. The difference diminishes for smaller a or b cases, where interactions are less important. Figure 5 shows the displacement distribution for the staggered case is quite different from that of Figure 3.

The present note confirms a previous wavy strip result [5], that a staggered boundary gives higher frequency, although the stagger affects neither the boundary length nor the membrane area. On the other hand, the frequency of the wavy strip may be increased or decreased with wavelength from the unperturbed frequency while in the present case the frequency is always larger than the non-constrained frequency k_0.

REFERENCES

1. LORD RAYLEIGH 1945 *The Theory of Sound* New York: Dover; second edition **1**, chapter 9.
2. A. W. LEISSA 1969 *NASA SP*-160. Vibration of Plates.
3. R. F. HARRINGTON 1961 *Time-Harmonic Electromagnetic Fields*. New York: McGraw-Hill; chapter 8.
4. F. L. NG 1974 *IEEE Transactions* **MMT 22**, 322–329. Tabulation of methods for the numerical solution of the hollow waveguide problem.
5. C. Y. WANG 1997 *Journal of Sound and Vibration* **207**, 128–131. Vibration frequencies of a membrane strip with wavy boundaries.