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STABILITY ESTIMATION OF HIGH DIMENSIONAL VIBRATING SYSTEMS UNDER STATE DELAY FEEDBACK CONTROL

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The paper presents a method of assessing the stability of high dimensional vibrating systems under state feedback control with a short time delay. It is first proved that if the time delay is sufficiently short, an n-degree-of-freedom system with feedback delay maintains $2n$ eigenvalues near those of the corresponding system without feedback delay. A perturbation approach is then proposed to determine the first order variation of an arbitrary eigenvalue and corresponding eigenvector of the system with feedback delay by solving a set of linear algebraic equations only. The computation in this approach can be simplified to a matrix multiplication provided that the product of the time delay and the modulus of the eigenvalue is much less than 1. Furthermore, the approach is directly related to the Newton–Raphson iteration in the continuation of eigenvalues for long time delay. The efficacy of the approach is demonstrated via a number of case studies on two feedback delay systems of two degrees of freedom and ten degrees of freedom respectively.

1. INTRODUCTION

Feedback controllers have been widely implemented in the active control of vibration. One of the most important effects that limit the performance of feedback controllers in practice is the unavoidable time delays in controllers and actuators, for they may result in unexpected instability of the controlled system. The time delays are particularly prevalent when digital controllers, analogue anti-aliasing and reconstruction filters and hydraulic actuators are used [1].

Very often a vibrating system under the feedback control with time delays can be modelled by a set of linear differential–difference equations. The controlled system is asymptotically stable if all the roots of the corresponding characteristic equation have negative real parts. As the characteristic equation is transcendental, it is not possible to solve for the infinite number of characteristic roots in general, nor is it straightforward to determine the approximate values of these roots if the system dimension is high. The stability of linear, high dimensional differential–difference equations, therefore, has become a much studied problem in the mathematical literature and in various engineering fields for decades. Recent monographs [2, 3] and papers [4–7] have presented various stability criteria and numerical approaches.

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From the viewpoint of vibration control, many practical problems remain open. For instance, the feedback gains of controllers are usually designed according to well-developed control strategies, say optimal control, neglecting the time delays in the controllers and actuators. One may wonder whether the controlled system is stable if a short time delay appears in the feedback, whether the system stability is robust with respect to small variation of the feedback gains and so forth. When the system has a single degree of freedom, these questions can be answered analytically [8]. However, tremendous computational efforts have to be made when the system dimension increases. As a result, the numerical examples to testify the current approaches [6, 7] were confined to the systems of a few degrees of freedom.

The primary aim of this paper is to propose a numerical technique to estimate efficiently the stability of high dimensional systems under retarded feedback control. The paper is organized as follows. After a brief description of the system of concern in section 2, the effect of a short time delay on the number and the distribution of the characteristic roots is analyzed in section 3. Then in section 4, the small variation of an arbitrary eigenvalue owing to the time delay is studied and three forms of a numerical perturbation approach are suggested for predicting the system instability caused by the time delay. Finally, a number of case studies are discussed in section 5 on two feedback delay systems of two degrees of freedom and ten degrees of freedom.

2. DESCRIPTION OF THE SYSTEM WITH DELAYED FEEDBACK

Consider a linear, time-invariant system under the state feedback control with a bounded time delay, $0 \leqslant \tau \leqslant \rho$. The motion of the system yields

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{f}(t) + \mathbf{U}\mathbf{x}(t-\tau) + \mathbf{V}\dot{\mathbf{x}}(t-\tau), \tag{1}$$

where $\mathbf{x} \in R^n$ is the displacement, $\mathbf{M} \in R^{n\times n}$, $\mathbf{C} \in R^{n\times n}$, $\mathbf{K} \in R^{n\times n}$ are the matrices of mass, damping and stiffness in the usual sense, $\mathbf{U} \in R^{n\times n}$ and $\mathbf{V} \in R^{n\times n}$ are the feedback gain matrices for the displacement and the velocity respectively. In general, these matrices, especially those of feedback gains, are not necessarily symmetric. In contrast to the Hamiltonian description, i.e., the state description, of controlled systems in most publications, the Lagrangian description will enable one to simplify computations in practice and to gain insight into the system dynamics as well.

To analyze the stability of a steady state motion $\mathbf{x}(t)$ of the system, one can study the equation that governs the small variation $\Delta\mathbf{x}(t)$ around $\mathbf{x}(t)$

$$\mathbf{M}\Delta\ddot{\mathbf{x}}(t) + \mathbf{C}\Delta\dot{\mathbf{x}}(t) + \mathbf{K}\Delta\mathbf{x}(t) = \mathbf{U}\Delta\mathbf{x}(t-\tau) + \mathbf{V}\Delta\dot{\mathbf{x}}(t-\tau). \tag{2}$$

Substituting the candidate solution $\Delta\mathbf{x}(t) = \mathbf{a}\,\mathrm{e}^{\lambda t}$ into equation (2) yields a transcendental eigenproblem

$$\mathbf{D}(\lambda, \tau)\mathbf{a} \equiv [\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K} - \mathrm{e}^{-\lambda\tau}(\mathbf{U} + \lambda\mathbf{V})]\mathbf{a} = 0, \tag{3a}$$

where $\lambda \in C^1$, $\mathbf{a} \in C^n$ and $\mathbf{D}(\lambda, \tau) \in C^{n\times n}$. The steady state motion $\mathbf{x}(t)$ is asymptotically stable provided that all the eigenvalues of equation (3a) have negative real parts. In this case, the system (1) is said to be stable for short.

Meanwhile, one has the adjoint eigenproblem of equation (3a):

$$\mathbf{b}^*\mathbf{D}(\lambda, \tau) \equiv \bar{\mathbf{b}}^{\mathrm{T}}\mathbf{D}(\lambda, \tau) = 0, \qquad \mathbf{b} \in C^n. \tag{3b}$$

Even though equation (3b) does not give new information on the system dynamics, it will help one to simplify the algebraic manipulation in subsection 4.2.

3. NUMBER AND DISTRIBUTION OF EIGENVALUES FOR A SHORT DELAY SYSTEM

Consider the characteristic equation of equation (3)

$$P(\lambda, \tau) \equiv \det \mathbf{D}(\lambda, \tau) = \det [\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K} - e^{-\lambda\tau}(\mathbf{U} + \lambda\mathbf{V})] = 0. \tag{4}$$

A controlled system is usually designed to be stable when the time delay in the state feedback vanishes. Henceforth, one assumes that all the $2n$ roots of $P(\lambda, 0)$ have negative real parts throughout this paper. Let $B_L < 0$ and $B_R < 0$ be the smallest real part and the largest real part of these roots respectively, and $B_I > 0$ be the bound of all imaginary parts of the roots in absolute value. Moreover, two bounds for later use are defined as:

$$\tilde{B}_L = B_L - \epsilon < 0, \qquad \tilde{B}_R = B_R + \epsilon < 0, \tag{5}$$

where ϵ is a small positive number. In what follows, the effect of a short time delay on the number and the distribution of the roots of equation (4) in the complex plane spanned by $\mathrm{Re}(\lambda)$ and $\mathrm{Im}(\lambda)$ is studied. It will be shown that equation (4) has only $2n$ roots near those of $P(\lambda, 0)$ if the time delay is short enough.

The first step of analysis is to exclude the roots of equation (4) from the shaded region in Figure 1. Equation (4) can be written as

$$P(\lambda, \tau) = p_0 (\lambda^{2n} + p_1 \lambda^{2n-1} + \cdots + p_{2n}) = 0, \quad p_0 \neq 0. \tag{6}$$

where p_j, $j = 0, 1, \ldots, 2n$ are the polynomials in terms of the entries of matrices $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$, $\mathbf{U}$ and $\mathbf{V}$, as well as $e^{-\lambda\tau}$. It is easy to see that in the right half plane $\mathrm{Re}(\lambda) > \tilde{B}_L$, the following inequality holds

$$|e^{-\lambda\tau}| = e^{-\mathrm{Re}(\lambda)\tau} < e^{-\tilde{B}_L \rho} \qquad \text{for} \qquad 0 \leqslant \tau \leqslant \rho. \tag{7}$$

Thus, p_j, $j = 0, 1, \ldots, 2n$ are bounded in absolute value. This enables one to define two bounds

$$B_1 \equiv \max_{1 \leqslant j \leqslant 2n} |p_j| \qquad \text{for} \qquad \mathrm{Re}(\lambda) > \tilde{B}_L, \qquad 0 \leqslant \tau \leqslant \rho, \tag{8}$$

$$B_2 \equiv \max \{1, B_1, (2n+1)B_1\}. \tag{9}$$

There follows the inequality

$$|P(\lambda, \tau)| = |p_0||\lambda^{2n} + p_1 \lambda^{2n-1} + \cdots + p_{2n}| \geqslant |p_0||\lambda|^{2n}[1 - |p_1|/|\lambda| - \cdots - |p_{2n}|/|\lambda^{2n}|]$$
$$\geqslant |p_0| B_2^{2n} (1 - 2nB_1/B_2) > |p_0| B_2^{2n} [1 - 2n/(2n+1)] > 0. \tag{10}$$

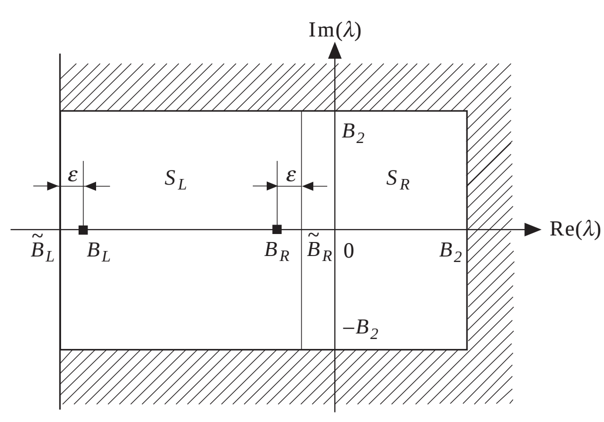

Figure 1. Existence region of the roots of $P(\lambda, \tau)$ on the complex plane.

This implies that none of the roots of equation (4) exists in the shaded region in Figure 1 if $0 \leqslant \tau \leqslant \rho$.

Next, the possibility of the roots of $P(\lambda, \tau)$ falling into the right closed rectangle $S_R = \{\lambda | \tilde{B}_R \leqslant \mathrm{Re}\,(\lambda) \leqslant B_2, \mathrm{Im}\,(\lambda) \leqslant B_2\}$ is analyzed. Rewriting $P(\lambda, \tau)$ in S_R as

$$P(\lambda, \tau) = P(\lambda, 0) + Q(\lambda, \tau), \quad Q(\lambda, 0) \equiv 0, \tag{11}$$

one can see that there exist no roots of $P(\lambda, 0)$ in the closed rectangle S_R, thereby

$$B_3 \equiv \min_{\lambda \in S_R} |P(\lambda, 0)| > 0. \tag{12}$$

From the continuity of $P(\lambda, \tau)$ with respect to τ, there exists a small, positive number $\delta_R = \delta(\epsilon) < \rho$ such that

$$\max_{\lambda \in S_R} |Q(\lambda, \tau)| < B_3, \qquad 0 \leqslant \tau < \delta_R. \tag{13}$$

Consequently, one has

$$|P(\lambda, \tau)| > |P(\lambda, 0)| - |Q(\lambda, \tau)| > B_3 - \max_{\lambda \in S_R} |Q(\lambda, \tau)| > 0, \qquad 0 \leqslant \tau < \delta_R, \tag{14}$$

which excludes the roots of $P(\lambda, \tau)$ from the closed rectangle S_R.

Finally, the number of roots of $P(\lambda, \tau)$ in the left closed rectangular region $S_L = \{\lambda | \tilde{B}_L \leqslant \mathrm{Re}\,(\lambda) \leqslant \tilde{B}_R, \mathrm{Im}\,(\lambda) \leqslant B_2\}$ is studied, where $P(\lambda, \tau)$ can be written as in equation (11) again. The definitions of bounds $\tilde{B}_L$, $\tilde{B}_R$ and B_2 ensure that there is no root of $P(\lambda, 0)$ on the boundary Γ of the closed region S_L, namely

$$B_4 \equiv \min_{\lambda \in \Gamma} |P(\lambda, 0)| > 0. \tag{15}$$

Also from the continuity of $P(\lambda, \tau)$ with respect to τ, there exists a small, positive number $\delta_L = \delta(\epsilon) < \rho$ such that

$$\max_{\lambda \in \Gamma} |Q(\lambda, \tau)| < B_4, \qquad 0 \leqslant \tau < \delta_L. \tag{16}$$

According to Rouché's theorem in complex analysis [9], the number of roots of $P(\lambda, \tau)$ in the closed rectangle S_L is the same as that of $P(\lambda, 0)$ in S_L as long as $0 \leqslant \tau < \delta_L$. This completes the present analysis.

In summary, given $\epsilon > 0$, there exists a delay bound $\delta(\epsilon) = \min\,(\delta_L\,(\epsilon), \delta_R\,(\epsilon))$ such that $P(\lambda, \tau)$ continues to have $2n$ roots in the closed rectangle S_L if $0 < \tau < \delta(\epsilon)$. However, the region of distribution of these roots in the complex plane may become slightly larger.

Without loss of generality, one can assume that the roots with the largest real part are a pair of complex roots of $P(\lambda, 0)$ and distinct from the other roots of $P(\lambda, 0)$. If one lets $\tilde{B}_L$ be greater than the second largest real part of the roots, one can similarly prove that $P(\lambda, \tau)$ has a pair of complex roots only in the narrow strip S_L when $0 < \tau < \delta(\epsilon)$. In this case, the real part of this pair of complex roots is bounded within $[\tilde{B}_L, B_R + \epsilon]$. One can therefore estimate the stability of the system with feedback delay simply from the variation of a single pair of roots of $P(\lambda, 0)$ provided the time delay is sufficiently short. This pair of roots will be referred to as "the most dangerous eigenvalues" hereafter for simplicity.

4. ESTIMATION OF EIGENVALUES OF A SHORT DELAY SYSTEM

Should there exist no time delay in the state feedback, equations (3a) and (3b) would become a pair of adjoint, quadratic eigenproblems, the solutions of which yield

$$\mathbf{D}(\lambda_r, 0)\mathbf{a}_r \equiv [\lambda_r^2\,\mathbf{M} + \lambda_r\,(\mathbf{C} - \mathbf{V}) + (\mathbf{K} - \mathbf{U})]\mathbf{a}_r = 0,$$

$$\mathbf{b}_r^*\,\mathbf{D}(\lambda_r, 0) = \mathbf{b}_r^*\,[\lambda_r^2\,\mathbf{M} + \lambda_r\,(\mathbf{C} - \mathbf{V}) + (\mathbf{K} - \mathbf{U})] = 0, \qquad r = 1, 2, \ldots, 2n, \tag{17}$$

where $\lambda_r \in C^1$ and $\lambda_{n+r} = \bar{\lambda}_r \in C^1$, $r = 1, 2, \ldots, n$ are n pairs of conjugate complex eigenvalues, $\mathbf{a}_r \in C^n$, $\mathbf{a}_{r+n} = \bar{\mathbf{a}}_r \in C^n$, $\mathbf{b}_r \in C^n$ and $\mathbf{b}_{r+n} = \bar{\mathbf{b}}_r \in C^n$, $r = 1, 2, \ldots n$ are the corresponding eigenvectors. Specifically, all the eigenvectors are scaled to

$$\mathbf{a}_r^* \, \mathbf{a}_r = \mathbf{b}_r^* \, \mathbf{b}_r = 1, \qquad r = 1, 2, \ldots, 2n. \tag{18}$$

When the feedback control involves a short time delay, there exists an eigenvalue $\tilde{\lambda}_r$ near the eigenvalue λ_r. Similarly there is a corresponding eigenvector $\tilde{\mathbf{a}}_r$ near $\mathbf{a}_r$. In this section, one studies how to determine $\tilde{\lambda}_r$ and $\tilde{\mathbf{a}}_r$ for a specific time delay τ when λ_r and $\mathbf{a}_r$ are given.

4.1. APPROACH BASED ON TRUNCATION PERTURBATION OF THE EIGENVALUE

One can write

$$\tilde{\lambda}_r = \lambda_r + \Delta\lambda_r, \qquad \tilde{\mathbf{a}}_r = \mathbf{a}_r + \Delta\mathbf{a}_r, \qquad \tilde{\mathbf{a}}_r^* \, \mathbf{a}_r = 1. \tag{19}$$

Substituting the first two equations in equation (19) into equation (3a) and dropping the higher order terms of $\Delta\lambda_r$, $\Delta\lambda_r \, \Delta\mathbf{a}_r$, etc., one has

$$\mathbf{D}(\lambda_r, \tau) \, (\mathbf{a}_r + \Delta\mathbf{a}_r) - \Delta\lambda_r \, \mathbf{E}(\lambda_r, \tau)\mathbf{a}_r = 0, \tag{20}$$

where

$$\mathbf{E}(\lambda_r, \tau) \equiv -\frac{\mathrm{d}}{\mathrm{d}\lambda} \mathbf{D}(\lambda, \tau)|_{\lambda=\lambda_r} = -\{2\lambda_r \mathbf{M} + \mathbf{C} + \mathrm{e}^{-\lambda_r \tau}[(\mathbf{U} + \lambda_r \mathbf{V})\tau - \mathbf{V}]\} \in C^{n \times n}. \tag{21}$$

To solve equation (20) for $\Delta\lambda_r$ and $\Delta\mathbf{a}_r$, a set of linear equations with the unknown complex vector $\mathbf{p}_r$ is constructed:

$$\mathbf{D}(\lambda_r, \tau)\mathbf{p}_r = \mathbf{E}(\lambda_r, \tau)\mathbf{a}_r. \tag{22}$$

Since λ_r is not the eigenvalue of equation (3a) when $\tau > 0$, the matrix $\mathbf{D}(\lambda_r, \tau)$ in equation (22) is invertable. Besides, $\mathbf{E}(\lambda_r, \tau)\mathbf{a}_r$ must be a non-zero vector. Otherwise equation (20) implies that λ_r is the eigenvalue of equation (3a). The solution of equation (22) thus is a unique non-zero vector $\mathbf{p}_r$. Comparing equation (22) with equation (20) yields

$$\tilde{\mathbf{a}}_r = \mathbf{a}_r + \Delta\mathbf{a}_r = \Delta\lambda_r \, \mathbf{p}_r, \tag{23}$$

namely $\mathbf{p}_r$ is an eigenvector associated with the eigenvalue $\tilde{\lambda}_r$ of equation (3a).

Following the idea of the Rayleigh quotient, one writes out

$$\frac{\mathbf{p}_r^* \, \mathbf{D}(\lambda_r, \tau)\mathbf{p}_r}{\mathbf{p}_r^* \, \mathbf{E}(\lambda_r, \tau)\mathbf{p}_r} = \frac{\tilde{\mathbf{a}}_r^* \, \mathbf{E}(\lambda_r, \tau)\mathbf{a}_r \, / \Delta\bar{\lambda}_\mu}{\tilde{\mathbf{a}}_r^* \, \mathbf{E}(\lambda_r, \tau) \, (\mathbf{a}_r + \Delta\mathbf{a}_r)/\Delta\lambda_r \, \Delta\bar{\lambda}_\mu} = \Delta\lambda_r \left[1 - \frac{\tilde{\mathbf{a}}_r^* \, \mathbf{E}(\lambda_r, \tau)\Delta\mathbf{a}_r}{\tilde{\mathbf{a}}_r^* \, \mathbf{E}(\lambda_r, \tau)\mathbf{a}_r} + \cdots \right]. \tag{24}$$

There follows an explicit expression for $\Delta\lambda_r$

$$\Delta\lambda_r \approx \frac{\mathbf{p}_r^* \, \mathbf{D}(\lambda_r, \tau)\mathbf{p}_r}{\mathbf{p}_r^* \, \mathbf{E}(\lambda_r, \tau)\mathbf{p}_r} = \frac{\mathbf{p}_r^* \, [\lambda_r^2 \, \mathbf{M} + \lambda_r \, \mathbf{C} + \mathbf{K} - \mathrm{e}^{-\lambda_r t}(\mathbf{U} + \lambda_r \, \mathbf{V})]\mathbf{p}_r}{\mathbf{p}_r^* \, \{2\lambda_r \, \mathbf{M} + \mathbf{C} + \mathrm{e}^{-\lambda_r \tau}[(\mathbf{U} + \lambda_r \, \mathbf{V})\tau - \mathbf{V}]\}\mathbf{p}_r}. \tag{25}$$

Substituting equation (25) and equation (23) into equation (20), one has the new eigenvalue and eigenvector.

4.2. SIMPLIFIED APROACH BASED ON TRUNCATION OF A VERY SHORT DELAY

If the time delay τ is so short that the delay phase $|\lambda_r \tau| \ll 1$, the matrices $\mathbf{D}(\lambda_r, \tau)$ and $\mathbf{E}(\lambda_r, \tau)$ can be expanded in a Taylor series at λ_r with respect to $\lambda_r \tau$, then

$$\mathbf{D}(\lambda_r, \tau) \approx \mathbf{D}(\lambda_r, 0) + \lambda_r \, \tau(\mathbf{U} + \lambda_r \, \mathbf{V}), \qquad \mathbf{E}(\lambda_r, \tau) \approx \mathbf{E}(\lambda_r, 0) = -(2\lambda_r \, \mathbf{M} + \mathbf{C} - \mathbf{V}). \tag{26}$$

Substituting equation (26) into equation (20) yields

$$\mathbf{D}(\lambda_r, 0)\Delta\mathbf{a}_r + \lambda_r\tau(\mathbf{U} + \lambda_r\mathbf{V})\mathbf{a}_r - \Delta\lambda_r\mathbf{E}(\lambda_r, 0)\mathbf{a}_r = 0. \tag{27}$$

Premultiplying equation (27) by the left eigenvector $\mathbf{b}_r^*$ associated with eigenvalue λ_r, one has

$$\lambda_r\tau\mathbf{b}_r^*(\mathbf{U} + \lambda_r\mathbf{V})\mathbf{a}_r - \Delta\lambda_r\mathbf{b}_r^*\mathbf{E}(\lambda_r, 0)\mathbf{a}_r = 0. \tag{28}$$

As proved in the Appendix, one has

$$\mathbf{b}_r^*\mathbf{E}(\lambda_r, \tau)\mathbf{a}_r = -\mathbf{b}_r^*(2\lambda_r\mathbf{M} + \mathbf{C} - \mathbf{V})\mathbf{a}_r \neq 0. \tag{29}$$

The simplified explicit expression for $\Delta\lambda_r$, therefore, reads

$$\Delta\lambda_r = \frac{\lambda_r\mathbf{b}_r^*(\mathbf{U} + \lambda_r\mathbf{V})\mathbf{a}_r}{\mathbf{b}_r^*\mathbf{E}(\lambda_r, 0)\mathbf{a}_r}\tau = -\frac{\lambda_r\mathbf{b}_r^*(\mathbf{U} + \lambda_r\mathbf{V})\mathbf{a}_r}{\mathbf{b}_r^*(2\lambda_r\mathbf{M} + \mathbf{C} - \mathbf{V})\mathbf{a}_r}\tau. \tag{30}$$

It is easy to verify that equation (30) is identical to the result obtained by the first order perturbation with respect to the small parameter τ.

From equation (30), one can define the sensitivity of the eigenvalue module with respect to the time delay

$$\mu(\lambda_r) \equiv |\Delta\lambda_r/\lambda_r\tau| = |\mathbf{b}_r^*(\mathbf{U} + \lambda_r\mathbf{V})\mathbf{a}_r/\mathbf{b}_r^*(2\lambda_r\mathbf{M} + \mathbf{C} - \mathbf{V})\mathbf{a}_r|, \tag{31}$$

and thus one has,

$$\lim_{\lambda_r\to 0}\mu(\lambda_r) = |\mathbf{b}_r^*\mathbf{U}\mathbf{a}_r/\mathbf{b}_r^*(\mathbf{C} - \mathbf{V})\mathbf{a}_r|, \qquad \lim_{\lambda_r\to\infty}\mu(\lambda_r) = |\mathbf{b}_r^*\mathbf{V}\mathbf{a}_r/2\mathbf{b}_r^*\mathbf{M}\mathbf{a}_r|. \tag{32}$$

It is worth noting that the sensitivity is independent of the system stiffness matrix $\mathbf{K}$. Keeping these relations in mind, one can estimate the relative change of the eigenvalues due to a very short time delay.

4.3. DISCUSSIONS

Presented above are two forms of the new approach for estimating an eigenvalue of the system with feedback delay. The difference between these forms is the truncation of higher order terms, which consequently effects accuracy and computational effort. Provided the time delay is very short, i.e., $|\lambda_r\tau| \ll 1$, equation (30) gives an accurate and efficient estimate. If this inequality does not hold, yet $|\Delta\lambda_r|/|\lambda_r|$ is still a small quantity, one can estimate the eigenvalue from equation (25), where the eigenvector $\mathbf{p}_r$ has to be determined from a set of n-dimensional, complex, linear equations in advance.

Even though λ_r is denoted as the eigenvalue of the delay-free system in subsection 4.1, none of the eigenvalue properties of the delay-free system are used during the analysis. Thus, one can take λ_r as an initial estimate of the eigenvalue of the delay system and repeatedly use equation (25) as a Newton–Raphson iteration if $|\Delta\lambda_r|/|\lambda_r|$ is not small. If the time delay τ is considered as a parameter, equation (25) can repeatedly be used as a continuation technique to trace the variation of an eigenvalue with increase in time delay τ. This is the third form of the present approach to the case of long time delay and will be demonstrated in the next section.

In addition, it is interesting to apply these estimates to an underdamped, single-degree-of-freedom system with feedback delay. Now equation (25) reads.

$$\Delta\lambda_1 = \frac{D(\lambda_1, \tau)}{E(\lambda_1, \tau)} = -\frac{\lambda_1^2 m + \lambda_1 c + k - e^{-\lambda_1 t}(u + \lambda_1 v)]}{2\lambda_1 m + c + e^{-\lambda_1\tau}[(u + \lambda_1 v)\tau - v]}, \tag{33}$$

where m, c, k, u and v are the scalar parameters corresponding to the matrices in equation (1), and

$$\lambda_1 = [(v-c) + \mathrm{j}\sqrt{4m(k-u)-(c-v)^2}]/2m, \qquad \mathrm{j} = \sqrt{-1}. \tag{34}$$

Similarly, equation (30) in this case becomes

$$\Delta\lambda_1 = D(\lambda_1, \tau)/E(\lambda_1, \tau) = -\tau(\lambda_1 u + \lambda_1^2 v)/(2\lambda_1 m + c - v). \tag{35}$$

Substituting equation (35) into equation (34), one obtains

$$\mathrm{Re}\,(\Delta\lambda_1) = -\tau(mu + v^2 - cv)/2m^2. \tag{36}$$

It is worth noting again that the variation of the real part of the eigenvalue is independent of the system stiffness.

5. ILLUSTRATIVE EXAMPLES

5.1. A TWO-DOF SYSTEM WITH DELAYED STATE FEEDBACK

To demonstrate the relative merits of the above approaches, the stability of the steady state motion of a dual-mass system with feedback delay as shown in Figure 2 is considered. The motion of the system yields equation (1), where

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 0{\cdot}2 & -0{\cdot}1 \\ -0{\cdot}1 & 0{\cdot}1 \end{bmatrix}, \qquad \mathbf{K} = \begin{bmatrix} k & -1 \\ -1 & 1 \end{bmatrix}, \tag{37}$$

whereas the stiffness coefficient k and the feedback gain matrices $\mathbf{U}$ and $\mathbf{V}$ will be variously specified in different cases. As a base comparison, the eigenvalues for a given time delay τ in each case were first determined from the intersections of the curves $\mathrm{Re}\,(P(\lambda, \tau)) = 0$ and $\mathrm{Im}\,(P(\lambda, \tau)) = 0$ plotted numerically in the complex plane of λ by using MAPLE® V 3.0. These eigenvalues shall be taken as the exact numerical results in what follows. For the sake of simplicity, the terms ST, DT and NR will be used hereafter for the approach based on single truncation of eigenvalues in subsection 4.1, the approach based on double truncations of both eigenvalues and the time delay in subsection 4.2, and the Newton–Raphson iteration on the basis of ST, respectively, Also, τ_r will be used to denote the shortest time delay when the rth order mode of the delay-feee system goes unstable and is referred to as the rth critical time delay for short.

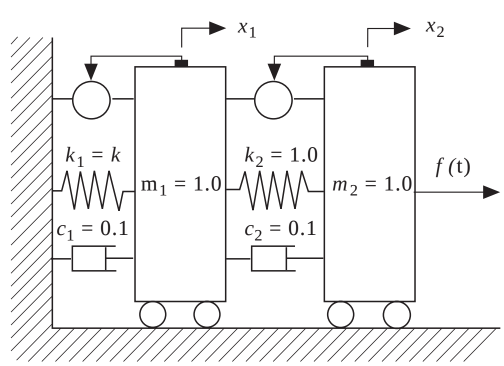

Figure 2. Dual-mass system under the state feedback with time delay.

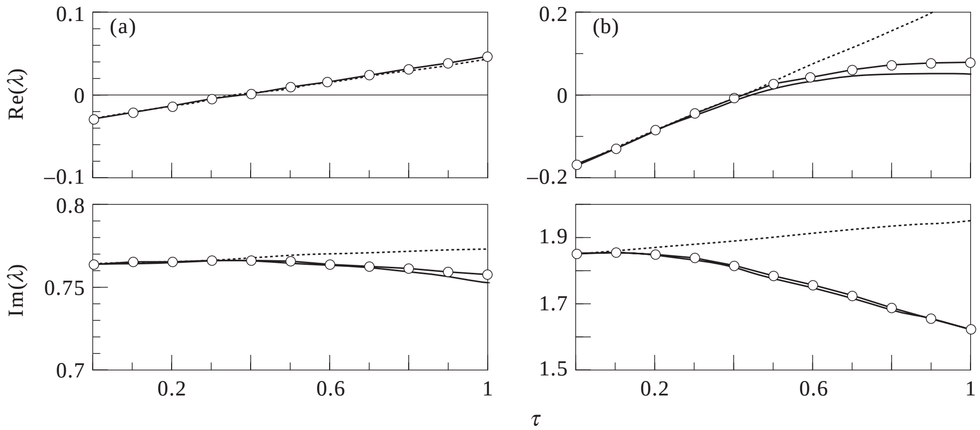

Figure 3. Variation of the eigenvalues with an increase of time delay in Case 1. (a) First eigenvalue; (b) second eigenvalue. Key: –○–, NR; —, ST; . . . . , DT.

5.1.1. *Case* 1

As the first and the simplest case, a state feedback was introduced to the system from the right mass to the connection only, so that

$$k = 2{\cdot}0, \qquad \mathbf{U} = \begin{bmatrix} 0{\cdot}0 & 1{\cdot}0 \\ 0{\cdot}0 & -1{\cdot}0 \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} 0{\cdot}0 & 0{\cdot}1 \\ 0{\cdot}0 & -0{\cdot}1 \end{bmatrix}. \tag{38}$$

The variation of the real and imaginary parts of two eigenvalues with an increase of time delay τ is shown in Figure 3, where the real parts of two pairs of conjugate eigenvalues went to zero when the time delay reached the critical values $\tau_1 \approx 0{\cdot}396$ and $\tau_2 \approx 0{\cdot}418$ respectively. The results of NR in Figure 3 were identical to the exact results represented by circles. Both ST and DT gave good estimates of τ_r. The relative errors were $-0{\cdot}1\%$ and $1{\cdot}5\%$ for the first pair of conjugate eigenvalues, and $3{\cdot}1\%$ and $-0{\cdot}96\%$ for the second, respectively. Since DT provides a linear relationship between an eigenvalue and the time delay, the estimation error, especially that of the second pair of conjugate eigenvalues, became unacceptable when the time delay was longer.

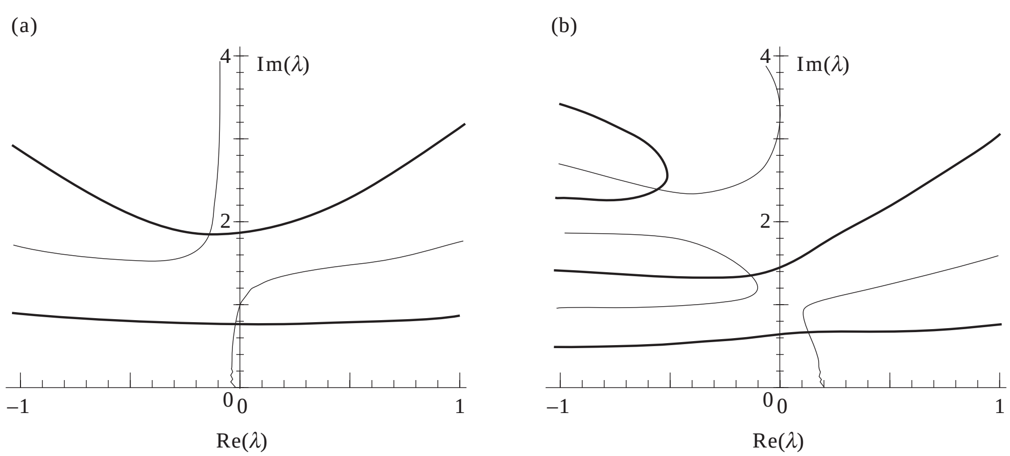

Figure 4. Distribution of eigenvalues of equation (3) for Case 1. (a) $\tau = 0{\cdot}1$; (b) $\tau = 2{\cdot}5$.

Shown in Figure 4 are the curves of Re $(P(\lambda, \tau)) = 0$ (thick) and Im $(P(\lambda, \tau)) = 0$ (thin) on the upper half complex plane for two specific time delays $\tau = 0{\cdot}1$ and $\tau = 2{\cdot}5$, corresponding to a stable status and an unstable status respectively. Each intersection point of these curves indicates an eigenvalue of equation (3) on the complex plane. The new eigenvalues emerged in the figure only when the time delay was long enough. It is this fact that enables one to analyze the system stability according to the evolution of eigenvalues of the delay-free system.

5.1.2. *Case* 2

The type of feedback in this case was kept the same as that in Case 1 and only the velocity feedback gains were increased from $\pm 0{\cdot}1$ in Case 1 to $\pm 1{\cdot}0$ here, i.e.,

$$k = 2{\cdot}0, \qquad \mathbf{U} = \begin{bmatrix} 0{\cdot}0 & 1{\cdot}0 \\ 0{\cdot}0 & -1{\cdot}0 \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} 0{\cdot}0 & 1{\cdot}0 \\ 0{\cdot}0 & -1{\cdot}0 \end{bmatrix}. \tag{39}$$

The negative velocity feedback reduced the real part of the eigenvalues of the delay-free system, and hence increased the critical time delays. Intuitively speaking, it would appear more difficult to estimate eigenvalues in this case. In Figure 5 are shown the variations of the real and imaginary parts of the two pairs of conjugate eigenvalues with increase in time delay, which reached the critical values respectively at $\tau_1 = 1{\cdot}135$ and $\tau_2 = 0{\cdot}644$, much longer than those in Case 1. Here again the results of NR were the same as the exact results. As shown in Figure 5, both ST and DT offered good estimations of the critical time delay τ_1 with relative errors of $-1{\cdot}76\%$ and $0{\cdot}44\%$ respectively. For the estimation of the second critical time delay τ_2, ST gave an under-estimation $\tau_2 = 0{\cdot}51$. However, DT totally failed because of the non-monotonic trend of Re (λ_2) with an increase of the time delay. In this case, NR is the more appealing approach even though it required a few iterations. It is important to note that even though λ_1 was the "most dangerous eigenvalue" when the system did not involve time delay, Re (λ_2) became positive earlier then Re (λ_1) when the time delay increased. Hence, the "most dangerous eigenvalue" can change for a sufficiently long time delay.

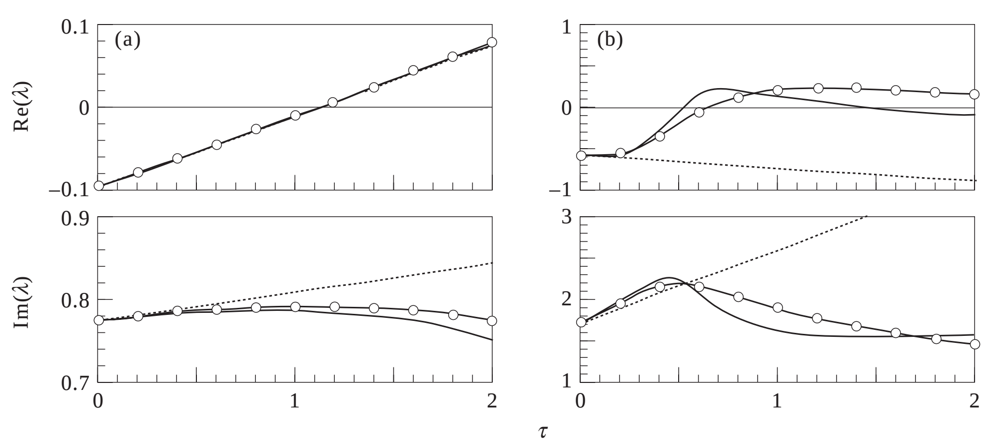

Figure 5. Variation of the eigenvalues with an increase of time delay in Case 2. (a) First eigenvalue; (b) second eigenvalue. Key: –○–, NR; ——, ST; . . . . , DT.

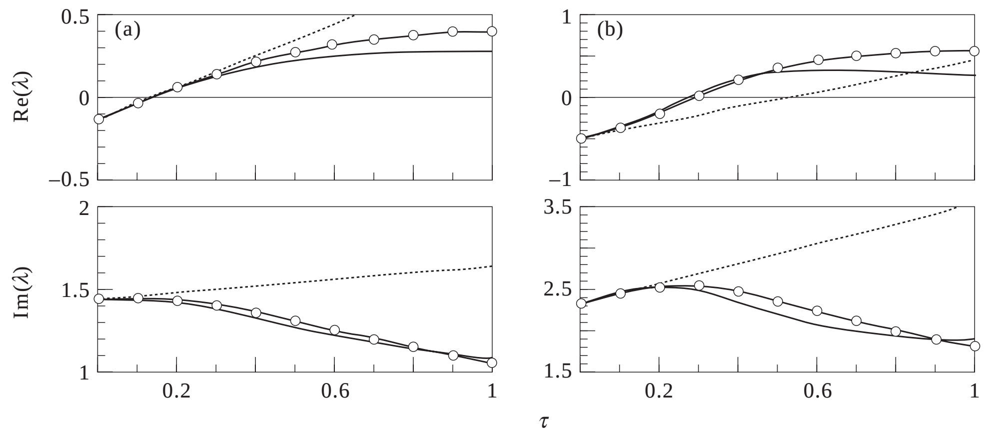

Figure 6. Variation of the eigenvalues with an increase of time delay in Case 3. (a) First eigenvalue; (b) second eigenvalue. Key: –○–, NR; ——, ST; . . . . , DT.

5.1.3. *Case* 3

To test the efficacy of the approach, the system was intentionally designed to be more complicated by introducing a stronger displacement feedback from both masses, namely

$$k = 2{\cdot}0, \qquad \mathbf{U} = \begin{bmatrix} -2{\cdot}0 & 3{\cdot}0 \\ 0{\cdot}0 & -3{\cdot}0 \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} 0{\cdot}0 & 1{\cdot}0 \\ 0{\cdot}0 & -1{\cdot}0 \end{bmatrix}. \tag{40}$$

As shown in Figure 6, the real parts of the first and the second eigenvalues vanished at $\tau_1 = 0{\cdot}139$ and $\tau_2 = 0{\cdot}298$ respectively. Both ST and DT again gave good estimations for the critical time delay τ_1 with relative errors of $-0{\cdot}07\%$ and $0{\cdot}94\%$ respectively. For more difficult estimation of the second critical time delay τ_2, the relative errors of ST and DT were $-6{\cdot}7\%$ and 76%, respectively.

5.1.4. *Case* 4

Compared with Case 3, the system was rendered even more complicated by adding the velocity feedback from the left mass to itself, i.e.,

$$k = 2{\cdot}0, \qquad \mathbf{U} = \begin{bmatrix} -2{\cdot}0 & 3{\cdot}0 \\ 0{\cdot}0 & -3{\cdot}0 \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} -1{\cdot}0 & 1{\cdot}0 \\ 0{\cdot}0 & -1{\cdot}0 \end{bmatrix}. \tag{41}$$

As shown in Figure 7, even though DT failed to estimate accurately the critical time delays for either eigenvaue in this case, both NR and ST worked successfully. For the two critical time delays $\tau_1 = 0{\cdot}383$ and $\tau_2 = 0{\cdot}365$, the relative errors of ST were respectively 9·1% and 13·8%. Here again the real part of the "norminally less dangerous" eigenvalue λ_2 of the delay-free system became positive a little bit earlier than that of the "most dangerous" one when the time delay increased.

5.1.5. *Case* 5

In the final case, the right spring in the system was replaced with an extremely stiff one, while the state feedback was kept the same as that in Case 3, namely

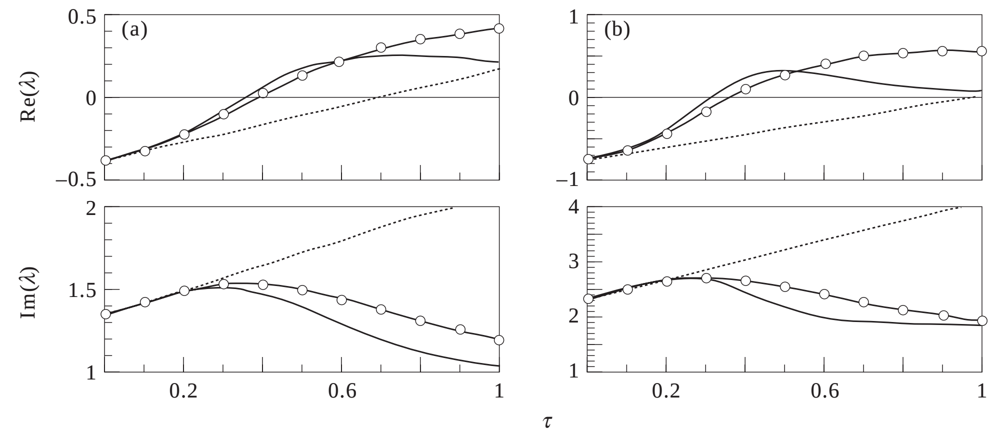

Figure 7. Variation of the eigenvalues with an increase of time delay in Case 4. (a) First eigenvalue; (b) second eigenvalue. Key: –○–, NR; ——, ST; . . . . , DT.

$$k = 399{\cdot}0, \qquad \mathbf{U} = \begin{bmatrix} -2{\cdot}0 & 3{\cdot}0 \\ 0{\cdot}0 & -3{\cdot}0 \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} 0{\cdot}0 & 1{\cdot}0 \\ 0{\cdot}0 & -1{\cdot}0 \end{bmatrix}. \tag{42}$$

Figure 8 shows that the first mode of the system became unstable when the time delay reached $\tau_1 = 0{\cdot}316$, whereas the second mode remained stable no matter how long the time delay became. Not surprisingly, ST predicted the oscillation of the second eigenvalue with respect to time delay as accurately as NR did, but DT gave a totally inaccurate prediction. This demonstrates the premise that ST and DT work within the ranges $|\Delta\lambda_r/\lambda_r| \ll 1$ and $|\lambda_r \tau| \ll 1$ respectively.

5.2. A TEN-DOF SYSTEM WITH DELAYED VELOCITY FEEDBACK

In order to demonstrate the applicability of the new approach to high dimensional systems with delayed feedback, a numerical study was made on an undamped chain system of ten degrees of freedom as shown in Figure 9, where

$$m_r = 1{\cdot}0, \qquad k_r = 1{\cdot}0, \qquad r = 1, 2, \ldots, 10. \tag{43}$$

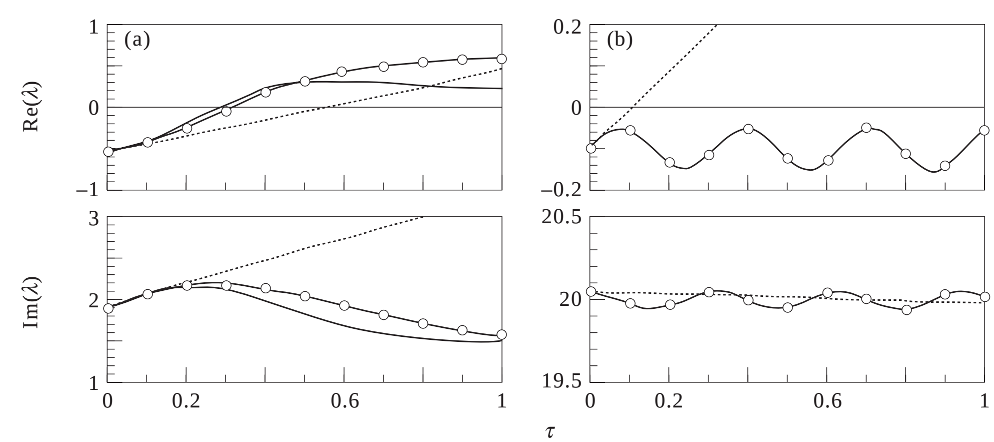

Figure 8. Variation of eigenvalues with an increase of time delay in Case 5. (a) First eigenvalue; (b) second eigenvalue. Key: –○–, NR; ——, ST; . . . . , DT.

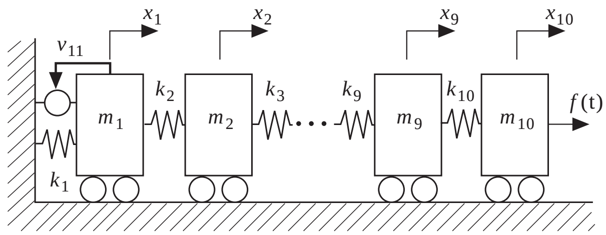

Figure 9. A ten-DOF system with a delayed velocity feedback.

To increase the damping of the system artificially, one channel of velocity feedback was introduced with the feedback gain $v_{11} = -1{\cdot}0$. When there was no time delay in the feedback, the 10 pairs of conjugate eigenvalues of the system could easily be solved by using commercially available eigenproblem codes. The real parts and the imaginary parts of these eigenvalues are listed in Table 1 according to the absolute value of their imaginary parts, from the minimum to the maximum. In this case, the dangerous eigenvalue was λ_1.

When there was a time delay in the feedback, the stability analysis of this high dimensional system became very complicated. For example, the numerical approaches proposed by Su *et al*. [6] and Chen [7] involve very lengthy algebraic manipulations including the decomposition of singular values and so forth. However, the new approach written in a few lines of FORTRAN and incorporated with standard subroutines of linear algebra completed the analysis within 10 s on a PC with a Pentium-MMX166 chip. The critical time delays for all pairs of eigenvalues determined by NR are listed as the last column in Table 1.

Intuitively speaking, the higher a natural frequency, the shorter the critical time delay. So, it was expected that the "most dangerous eigenvalue" should be λ_{10} with an increase of time delay since the real part of λ_{10} was the second smallest when there was no time delay in the feedback. Nevertheless, the "safest eigenvalue" λ_5 became dangerous first with increase of time delay, and vanished at $\tau_5 = 0{\cdot}732$. Figure 10 shows the evolution of eigenvalues λ_1, λ_5 and λ_{10} with increase of the time delay. This example indicates that care must be taken when the feedback of a high dimensional system involves any time delay.

TABLE 1

Real and imaginary parts of eigenvalues of the delay-free system and corresponding critical time delays

r	Re (λ_r)	Im (λ_r)	τ_r
1	−0·0021	0·1497	10·375
2	−0·0165	0·4506	3·425
3	−0·0383	0·7500	2·064
4	−0·0660	1·0416	1·501
5	−0·2136	1·3013	0·732
6	−0·0877	1·3555	1·201
7	−0·0431	1·5927	1·023
8	−0·0215	1·7688	0·909
9	−0·0089	1·8964	0·839
10	−0·0022	1·9740	0·799

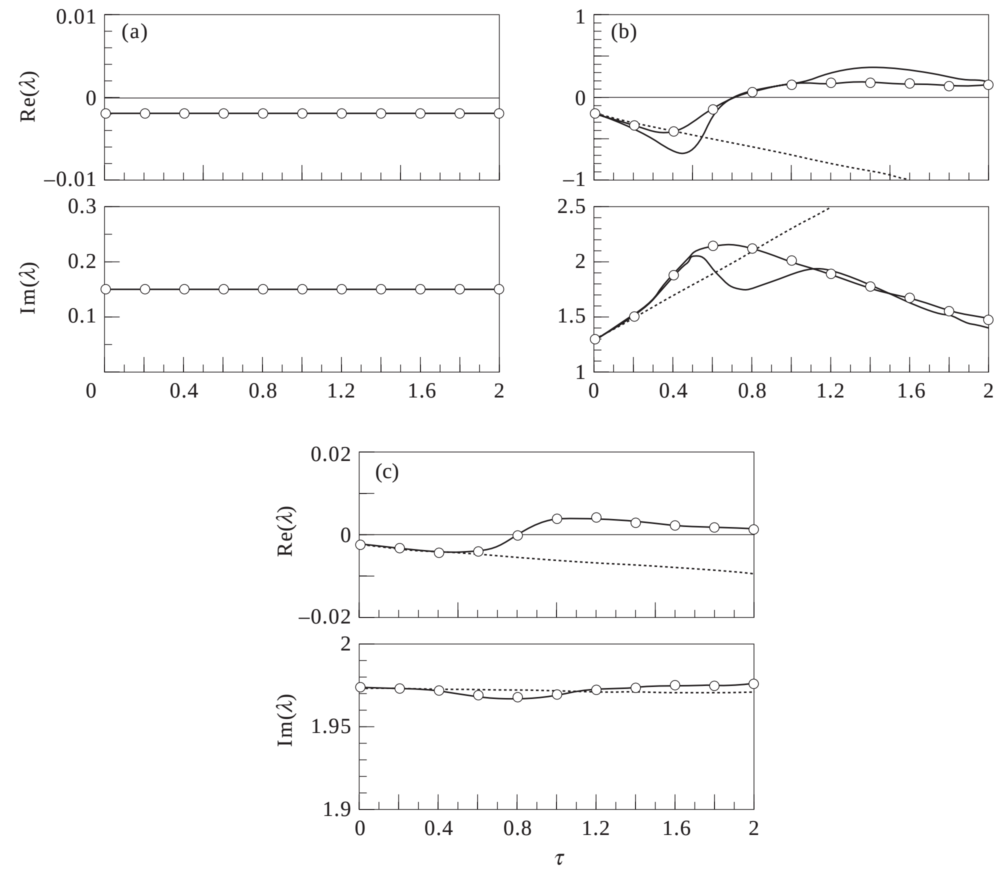

Figure 10. Variation of the eigenvalues with an increase of time delay: (a) λ_1; (b) λ_5; (c) λ_{10}. Key: –○–, NR; ——, ST; . . . . , DT.

6. CONCLUDING REMARKS

Provided the time delay in the state feedback is sufficiently short, the stability of a linear n-degree-of-freedom system with single feedback delay is governed by the evolution of the $2n$ eigenvalues of the delay-free system with increase in time delay. To study the stability of the system involving feedback delay, a perturbation approach is proposed so as to estimate efficiently the evolution of these eigenvalues. The approach can be used in three forms according to the length of time delay. If the time delay τ is so short that the eigenvalue λ_r of concern yields $|\lambda_r \tau| \ll 1$, the simplest form of the approach gives an expression, similar to the Rayleigh quotient, for the variation of λ_r proportional to τ. When the time delay is not so short, two alternative forms of the approach enable one to trace the variation of λ_r by solving a set of linear algebraic equations or by using Newton–Raphson iteration. The later form gives the exact numerical envolution of the eigenvalues with increase of time delay. The efficacy of these forms of the approach was well supported by the case studies on two feedback delay systems of two degrees of freedom and ten degrees of freedom, respectively.

Finally it is worthy of mention that it is straightforward to generalize the analysis and the assertions in section 3 for the following n-degree-of-freedom system with asynchronous time delays in different feedback channels

$$\sum_{j=1}^{n} [m_{ij}\, \ddot{x}_j(t) + c_{ij}\, \dot{x}_j(t) + k_{ij}\, x_j(t)] = f_j(t) + \sum_{j=1}^{n} [u_{ij}\, x_j(t-\tau_{ij}) + v_{ij}\, \dot{x}_j(t-\eta_{ij})]. \tag{44}$$

As the truncation in equation (20) requires only a small variation of eigenvalues due to the time delay, the approach described here can be directly used to analyze the stability of this kind of system also. However, much more computational effort is required in tracing the evolution of eigenvalues if a system involves many different time delays as in equation (44).

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APPENDIX

Consider the equation of the rth eigenvalue and its eigenvector in the state space

$$(\mathbf{A} - \lambda_r\, \mathbf{I})\mathbf{u}_r \equiv \left(\begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1}(\mathbf{K}-\mathbf{U}) & -\mathbf{M}^{-1}(\mathbf{C}-\mathbf{V}) \end{bmatrix} - \lambda_r\, \mathbf{I} \right) \begin{bmatrix} \mathbf{u}_{r1} \\ \mathbf{u}_{r_2} \end{bmatrix} = 0. \tag{A1}$$

By comparing equation (A1) with the first equation in equation (17), one can readily find

$$\mathbf{u}_{r1} = \mathbf{a}_r, \qquad \mathbf{u}_{r2} = \lambda_r\, \mathbf{a}_r. \tag{A2}$$

The adjoint relation of equation (A1) reads

$$\mathbf{v}_r^* (\mathbf{A} - \lambda_r\, \mathbf{I}) \equiv [\mathbf{v}_{r1}^* \quad \mathbf{v}_{r2}^*] \left(\begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} - \lambda_r\, \mathbf{I} \right) = 0, \tag{A3}$$

whereby one obtains

$$\lambda_r^2 \mathbf{v}_{r2}^* + \lambda_r \mathbf{v}_{r2}^* \mathbf{M}^{-1}\mathbf{C} + \mathbf{v}_{r2}^* \mathbf{M}^{-1}\mathbf{K} = 0, \qquad \mathbf{v}_{r1}^* = \mathbf{v}_{r2}^* (\lambda_r + \mathbf{M}^{-1}\mathbf{C}). \tag{A4}$$

Comparing the first equation in equation (A3) with the second equation in equation (17), one finds

$$\mathbf{v}_{r2}^* = \mathbf{b}_r^* \mathbf{M}, \qquad \mathbf{v}_{r1}^* = \mathbf{b}_r^* (\lambda_r \mathbf{M} + \mathbf{C}). \tag{A5}$$

Noting the orthogonality relation of adjoint eigenvectors

$$\mathbf{v}_r^* \mathbf{u}_r = \mathbf{v}_{r1}^* \mathbf{u}_{r1} + \mathbf{v}_{r2}^* \mathbf{u}_{r2} \neq 0. \tag{A6}$$

and substituting equation (A2) and equation (A5) into equation (A6), one obtains

$$2\lambda_r \mathbf{b}_r^* \mathbf{M}\mathbf{a}_r + \mathbf{b}_r^* (\mathbf{C} - \mathbf{V})\mathbf{a}_r \neq 0. \tag{A7}$$

This completes the proof.