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WAVE DISPERSION IN DEEP MULTILAYERED DOUBLY CURVED VISCOELASTIC SHELLS

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This study is concerned with the dispersion of axisymmetric and asymmetric plane waves in viscoelastic cylindrical shells. The originality of the approach lies in the use of a refined laminated shell theory that allows one to satisfy exactly the boundary conditions for displacements and transverse shear stresses, while, at the same time, refinements of membrane and shear terms are considered. By comparison with previous theories, light is shed upon the advantage of using such a refined model to determine the dispersive behaviour of structures. The shell model is then applied to a viscoelastic cylinder, for which frequency and phase velocity spectra are presented. In order to point out the influence of viscoelasticity, especially as concerns phase velocities, comparison is made with the equivalent elastic case.

1. INTRODUCTION

Deep shells of laminated composite materials are being increasingly used in structural applications. Yet, despite the modelling of such structures having been the object of numerous studies over the past few years, there are not many theories that simultaneously satisfy compatibility conditions for displacements and shear stresses at layer interfaces, and on the bounding surfaces of the shell.

This paper presents a new approach for developing a simple and refined theory for deep, doubly curved laminated shells, which allows one to satisfy exactly the continuity of transverse shear stresses and displacements at layer interfaces, while, at the same time, the membrane and shear terms are refined. The theory contains the same independent generalized displacements as in the shear deformation theory, and is based on a new assumed displacement field in which refined transverse shear and membrane deformations are represented by trigonometric functions. This is justified from a three-dimensional point of view in plates. Moreover, the introduction of trigonometric functions in the adopted form of the displacement field allows one to recover previous theories by developing the sine and cosine functions to various orders. Mindlin's [1], Naghdi's [2] and Koiter's [3] theories can thus be obtained. The objective of this research, which extends previous works by Touratier [4–6], and Touratier and Béakou [7, 8] is to develop efficient (i.e., simple and accurate) tools to model composite structures. It is proved, by comparison with previous theories ([5, 9–11]), that the model yields accurate results without the use of transverse shear deformation correction factors.

The model is then applied to the exploration of the dispersive behaviour of a viscoelastic cylinder. An analysis of the axisymmetric and asymmetric modes is made, in which the effects of viscoelasticity are pointed out by comparison with the equivalent elastic case.

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Finally, viscoelastic dispersion is exhibited, and it is shown that torsional waves are weakly dispersive.

2. THE MULTILAYERED SHELL MODEL WITH INTERLAYER CONTINUITY

2.1. GEOMETRIC SHELL CONSIDERATIONS

Let us consider an undeformed laminated shell of constant thickness h, consisting of a finite number N of orthotropic layers in a curvilinear coordinate system (x_1, x_2, x_3); see Figure 1. The space occupied by the shell will be denoted V. The boundary of the shell is the union of the upper surface Ω_h, the lower surface Ω_0, and the edge faces A.

The interface between the ith and $(i+1)$th layer is denoted by Ω_i, the distance between Ω_0 and Ω_i, $x_{3_{(i)}}$.

The *reference surface*, defined by $x_3 = 0$, coincides with the *bottom surface of the shell* Ω_0.

In this paper, the Einsteinian summation convention applies to repeated indices, where Latin indices range from 1 to 3 while Greek indices range from 1 to 2.

A point M outside of the reference surface Ω_0 being given, let P denote the point of the reference surface Ω_0 closest to M. Covariant base vectors $(\vec{a}_i)$, $(\vec{g}_i)$ and contravariant base vectors $(\vec{a}^i)$, $(\vec{g}^i)$ in the undeformed state of the shell are introduced:

$$\vec{a}_\alpha = P_{,\alpha}, \qquad \vec{a}_3 = \vec{a}_1 \wedge \vec{a}_2 / \|\vec{a}_1 \wedge \vec{a}_2\|, \qquad (\vec{a}_1 \wedge \vec{a}_2) \cdot \vec{a}_3 > 0;$$
$$\vec{g}_i = M_{,i}, \qquad (\vec{g}_1 \wedge \vec{g}_2) \cdot \vec{g}_3 > 0, \qquad \vec{a}^\alpha \cdot \vec{a}_\beta = \delta^\alpha_\beta,$$
$$\vec{a}^3 = \vec{a}_3, \qquad \vec{g}^\alpha \cdot \vec{g}_\beta = \delta^\alpha_\beta, \qquad \vec{g}^3 = \vec{g}_3. \tag{1}$$

Here differentiation with respect to x_i is denoted by $\langle\langle_{,i}\rangle\rangle$.

It is recalled that

$$M = P + x_3 \vec{a}^3. \tag{2}$$

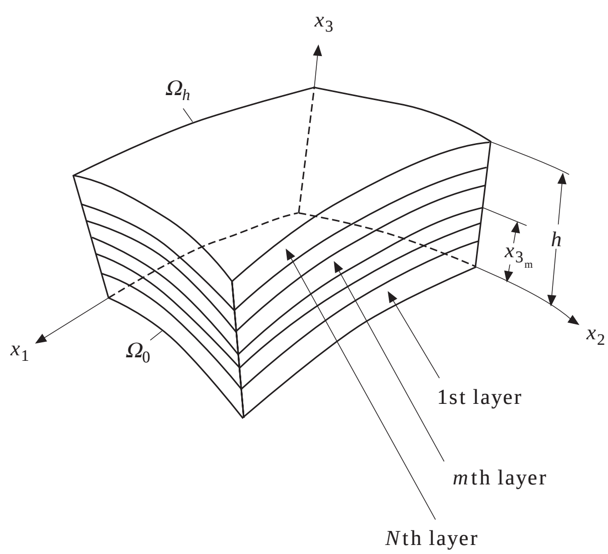

Figure 1. The geometry of the laminated shell.

The above equations ensure the following relations, due to Naghdi [2]:

$$\vec{g}_\alpha = \mu_\alpha^\beta \vec{a}_\beta, \qquad \vec{g}_3 = \vec{a}_3, \qquad \vec{g}^\alpha = -\mu_\beta^{\alpha-1} \vec{a}^\beta, \qquad \vec{g}^3 = \vec{a}^3,$$
$$\vec{g}_\alpha = g_{\alpha\beta}\vec{g}^\beta, \qquad \vec{g}^\alpha = g^{\alpha\beta}\vec{g}_\beta, \qquad \vec{a}_\alpha = a_{\alpha\beta}\vec{a}^\beta, \qquad \vec{a}^\alpha = a^{\alpha\beta}\vec{a}_\beta. \tag{3}$$

The components of the shifter tensor are denoted by

$$\mu_\beta^\alpha = \delta_\beta^\alpha - b_\beta^\alpha x_3, \tag{4}$$

those of the curvature tensor by

$$b_{\alpha\beta} = \vec{a}_{\alpha,\beta} \cdot \vec{a}^3 \tag{5}$$

and the curvilinear mixed terms by

$$b_\beta^\alpha = -\vec{a}_{3,\beta} \cdot \vec{a}^\alpha. \tag{6}$$

The surface metrics α_1 and α_2 are related to the $a_{\alpha\beta}$ coefficients via

$$\alpha_l^2 = a_{ll}. \tag{7}$$

In the following, the curvilinear coordinates (or *shell coordinates*) are supposed to be orthogonal, and are such that the x_1- and x_2-curves are lines of curvature on the reference surface $x_3 = 0$; x_3-curves are straight lines perpendicular to the surface $x_3 = 0$. The values of the principal radii of curvature of the reference surface are denoted by R_1 and R_2.

The distance ds between two points P $(x_1, x_2, 0)$, P' $(x_1 + \mathrm{d}x_1, x_2 + \mathrm{d}x_2, 0)$ of the reference surface Ω_0 of the shell is given by

$$(\mathrm{d}s)^2 = \alpha_1^2(\mathrm{d}x_1)^2 + \alpha_2^2(\mathrm{d}x_2)^2, \tag{8}$$

where α_1 and α_2 are the surface metrics

$$\alpha_l^2 = (\partial P/\partial x_l)(\partial P/\partial x_l). \tag{9}$$

The distance dS between two points M (x_1, x_2, x_3) and $M'(x_1 + \mathrm{d}x_1, x_2 + \mathrm{d}x_2, x_3 + \mathrm{d}x_3)$ outside of the reference surface is given by

$$(\mathrm{d}S)^2 = L_1^2(\mathrm{d}x_1)^2 + L_2^2(\mathrm{d}x_2)^2 + L_3^2(\mathrm{d}x_3)^2, \tag{10}$$

where L_1, L_2 and L_3 are the Lamé coefficients:

$$L_1 = \alpha_1(1 + x_3/R_1), \qquad L_2 = \alpha_2(1 + x_3/R_2), \qquad L_3 = 1. \tag{11}$$

2.2. KINEMATIC ASSUMPTIONS

Geometrically linear shells are considered including elastic and viscoelastic linear behaviour for laminates.

The components of the displacement field of any point M (x_1, x_2, x_3) of the volume occupied by the shell (V), expressed in the contravariant base $(\vec{g}^\alpha, \vec{g}^3)$, are assumed in the following form:

$$U_\alpha = u_\alpha + x_3\eta_\alpha + f(x_3)\gamma_\alpha^0 + g(x_3)\varphi_\alpha + \sum_{m=1}^{N-1} u_{(m)_\alpha}(x_3 - x_{3_{(m)}})\mathrm{H}(x_3 - \mathrm{x}_{3_{(m)}}),$$
$$U_3 = \mathrm{w}. \tag{12}$$

Here

$$f(x_3) = (h/\pi)\sin(\pi x_3/h), \qquad g(x_3) = (h/\pi)\cos(\pi x_3/h), \tag{13}$$

and H denotes the Heaviside step function, defined by

$$\mathrm{H}(x_3 - x_{3_{(m)}}) = \begin{Bmatrix} 1 & \text{for } x_3 \geqslant x_{3_{(m)}} \\ 0 & \text{for } x_3 < x_{3_{(m)}} \end{Bmatrix}. \tag{14}$$

This step function has been previously used, among others by Di Sciuva [12] and He [13] to analyze laminated shells in statics. The present work extends the use of the step function to dynamics.

In the proposed form of the displacement field, u_α are membrane displacements, γ_α^0 are the transverse shear strains at $x_3 = 0$, and w is the transverse deflection of the shell. The $g(x_3)\ \varphi_\alpha$ terms are refinements of membrane displacements, the η_α and φ_α being functions to determine by exploiting the boundary conditions of the transverse shear stresses at the top and bottom surfaces of the shell. The $u_{(m)_\alpha}$, which represent the generalized displacements ⟪*per layer*⟫, allow one to satisfy automatically the continuity of the displacements at layer interfaces from the Heaviside step function. They are to be determined by satisfying the continuity conditions on the transverse shear stresses at the interfaces.

From a three-dimensional point of view, the kinematics proposed in equations (12), with the introduction of the sine and cosine functions f and g can be justified on the basis of the work of Cheng [14], who proposed a method for solving Navier's equations, in the case of thick plates. Cheng showed that there exist three distinct types of fundamental solutions, which will be denoted by $\vec{U}^B$, $\vec{U}^S$, $\vec{U}^T$, each one being the solution of a specific differential equation (here without any loading, for the sake of simplicity):

$\vec{U}^B$, which is the solution of the well-known biharmonic equation

$$\boldsymbol{\nabla}^2\boldsymbol{\nabla}^2\vec{U}^B = \vec{0}; \tag{15}$$

$\vec{U}^S$, the solution of a so-called shear equation, defined by

$$(\boldsymbol{\nabla}^2 - (2p+1)^2\,\pi^2/h^2)s(x_1, x_2) = 0; \qquad (p \text{ being an integer}); \tag{16}$$

and given by

$$U_1^S = \sin\left((2p+1)\,\frac{\pi x_3}{h}\right)s_{,2}, \qquad U_2^S = -\sin\left((2p+1)\,\frac{\pi x_3}{h}\right)s_{,1}; \tag{17}$$

$\vec{U}^T$, the solution of the transcendental equation

$$(1/\boldsymbol{\nabla}^2)(1 - \sin(h\boldsymbol{\nabla})/h\boldsymbol{\nabla})H(x_1, x_2) = 0 \qquad (H \text{ being a stress function}). \tag{18}$$

Hence, the final solution, in terms of displacements, is obtained via

$$\vec{U} = \vec{U}^B + \vec{U}^S + \vec{U}^T. \tag{19}$$

The shear term in the present theory for plates is therefore naturally obtained by imposing

$$(h/\pi)\gamma_1^0 = +s_{,2}, \qquad (h/\pi)\gamma_2^0 = -s_{,1}, \qquad p = 0. \tag{20}$$

Moreover, the advantage of keeping γ_α^0 functions in equations (12) allows one to find Mindlin's theory [1] by developing the sine at the first order.

In order to reduce the number of unknowns in the displacement field, the conditions on the transverse shear stresses at layer interfaces and on the bounding surfaces will be used in the following way. The transverse normal stress is ignored. It is assumed that no tangential tractions are exerted on the upper and lower surfaces of the shell.

2.3. THE LINEAR CONSTITUTIVE LAW

It is recalled that the coefficients of the constitutive law for each layer are generally given in a co-ordinate system related to the material of the layers (that shall be qualified as ⟪*material co-ordinates*⟫), whereas the authors presently deal with shell co-ordinates. In what follows, recall the link between the two coordinate systems.

By taking into account the zero condition on the transverse normal stress σ_{33}, the orthotropic constitutive law ⟪*per layer*⟫ can be written as follows in material co-ordinates, for the ith layer:

$$\sigma_{\alpha\alpha}^{(i)^{mat}} = C_{\alpha\alpha\beta\beta}^{(i)^{mat}} e_{\beta\beta}^{mat}, \qquad \sigma_{\alpha\beta}^{(i)^{mat}} = C_{\alpha\beta\alpha\beta}^{(i)^{mat}} e_{\alpha\beta}^{mat} \qquad (\beta \neq \alpha),$$
$$\sigma_{\alpha 3}^{(i)^{mat}} = C_{\alpha 3\alpha 3}^{(i)^{mat}} e_{\alpha 3}^{mat}; \tag{21}$$

or, in a matrix form,

$$\{\sigma^{(i)^{mat}}\} = [\mathbf{C}^{(i)^{mat}}]\{\mathbf{e}^{mat}\}. \tag{22}$$

Here:

$$\{\sigma^{(i)^{mat}}\} = \{\sigma_{11}^{(i)^{mat}}, \sigma_{22}^{(i)^{mat}}, \sigma_{31}^{(i)^{mat}}, \sigma_{12}^{(i)^{mat}}, \sigma_{32}^{(i)^{mat}}\}, \tag{23}$$

$$\{\mathbf{e}^{mat}\} = \{e_{11}^{mat}, e_{22}^{mat}, e_{31}^{mat}, e_{12}^{mat}, e_{32}^{mat}\}, \tag{24}$$

are respectively the stress and strain vectors.

The (i) exponent refers to the (i)th layer, and the ⟪^{mat}⟫ exponent to the material co-ordinates.

The $C_{\alpha\beta\gamma\delta}^{(i)}$ coefficients are *two-dimensional* coefficients, related to *three-dimensional* $C_{\alpha\beta\gamma\delta}^{(i)^{3D_{mat}}}$ ones via

$$C_{\alpha\alpha\beta\beta}^{(i)^{mat}} = C_{\alpha\alpha\beta\beta}^{(i)^{3D_{mat}}} - \frac{C_{\alpha\alpha 33}^{(i)^{3D_{mat}}} C_{\beta\beta 33}^{(i)^{3D_{mat}}}}{C_{3333}^{(i)^{3D_{mat}}}},$$

$$C_{\alpha\beta\alpha\beta}^{(i)^{mat}} = C_{\alpha\beta\alpha\beta}^{(i)^{3D_{mat}}} \quad (\beta \neq \alpha), \qquad C_{\alpha\alpha 33}^{(i)^{mat}} = C_{\alpha\alpha 33}^{(i)^{3D_{mat}}}. \tag{25}$$

Let $(\vec{E}_1, \vec{E}_2, \vec{E}_3)$ denote the spatial Cartesian system. A point M of the structure will be located by its co-ordinates (X_1, X_2, X_3) in this system, which are functions of the curvilinear co-ordinates x_α.

The covariant and Cartesian vectors of the shell are related by

$$\vec{g}_\alpha = \partial\vec{M}/\partial x_\alpha = X_{\beta,\alpha}\vec{E}_\beta, \qquad \vec{g}_3 = \vec{E}_3, \tag{26}$$

or

$$\begin{bmatrix} \vec{g}_1 \\ \vec{g}_2 \\ \vec{g}_3 \end{bmatrix} = [\mathbf{T}] \begin{bmatrix} \vec{E}_1 \\ \vec{E}_2 \\ \vec{E}_3 \end{bmatrix}, \tag{27}$$

where the coefficients of the rotation matrix $[\mathbf{T}]$ are given by

$$T_{\alpha\beta} = X_{\alpha,\beta}, \qquad T_{i3} = \delta_{i3}. \tag{28}$$

Denoting

$$\{\sigma^{(i)^{mat}}\}_{plane} = \begin{Bmatrix} \sigma_{11}^{(i)^{mat}} \\ \sigma_{22}^{(i)^{mat}} \\ \sigma_{12}^{(i)^{mat}} \end{Bmatrix}, \qquad \{\sigma^{(i)}\}_{plane} = \begin{Bmatrix} \sigma_{11}^{(i)} \\ \sigma_{22}^{(i)} \\ \sigma_{12}^{(i)} \end{Bmatrix},$$

$$\{\mathbf{e}^{mat}\}_{plane} = \begin{Bmatrix} e_{11}^{mat} \\ e_{22}^{mat} \\ e_{12}^{mat} \end{Bmatrix}, \qquad \{\mathbf{e}\}_{plane} = \begin{Bmatrix} e_{11} \\ e_{22} \\ e_{12} \end{Bmatrix}, \tag{29}$$

$$[\mathbf{C}^{(i)^{mat}}]_{plane} = \begin{bmatrix} C_{1111}^{(i)^{mat}} & C_{1122}^{(i)^{mat}} & 0 \\ C_{2211}^{(i)^{mat}} & C_{2222}^{(i)^{mat}} & 0 \\ 0 & 0 & 2C_{1212}^{(i)^{mat}} \end{bmatrix},$$

$$[C^{(i)^{mat}}]_{shear} = \begin{bmatrix} 2C_{1313}^{(i)^{mat}} & 2C_{1323}^{(i)^{mat}} \\ 2C_{1323}^{(i)^{mat}} & 2C_{2323}^{(i)^{mat}} \end{bmatrix}, \tag{30}$$

one has then, for each layer, in shell co-ordinates,

$$\{\sigma^{(i)}\}_{plane} = [\mathbf{T}]\{\sigma^{(i)^{mat}}\}_{plane} = [\mathbf{T}][\mathbf{C}^{(i)^{mat}}]_{plane}[\mathbf{T}]^{-1}\{e\}_{plane}.$$

$$\begin{bmatrix} \sigma_{13}^{(i)} \\ \sigma_{23}^{(i)} \end{bmatrix} = [\mathbf{T}_{\alpha\beta}]\begin{bmatrix} \sigma_{13}^{(i)^{mat}} \\ \sigma_{23}^{(i)^{mat}} \end{bmatrix} = [\mathbf{T}_{\alpha\beta}][C^{(i)^{mat}}]_{shear}[\mathbf{T}_{\alpha\beta}]^{-1}\begin{bmatrix} e_{13}^{(i)} \\ e_{23}^{(i)} \end{bmatrix}, \tag{31}$$

or

$$\{\sigma^{(i)}\}_{plane} = [\mathbf{C}^{(i)}]_{plane}\{e\}_{plane}, \qquad \begin{bmatrix} \sigma_{13}^{(i)} \\ \sigma_{23}^{(i)} \end{bmatrix} = [\mathbf{C}^{(i)}]_{shear}\begin{bmatrix} e_{13}^{(i)} \\ e_{23}^{(i)} \end{bmatrix}, \tag{32}$$

where

$$[\mathbf{C}^{(i)}]_{plane} = [\mathbf{T}][\mathbf{C}^{(i)^{mat}}]_{plane}[\mathbf{T}]^{-1},$$
$$[\mathbf{C}^{(i)}]_{shear} = [\mathbf{T}_{\alpha\beta}][\mathbf{C}^{(i)^{mat}}]_{shear}[\mathbf{T}_{\alpha\beta}]^{-1}. \tag{33}$$

2.4. BOUNDARY CONDITIONS FOR TRANSVERSE SHEAR STRESSES

The transverse shear strain components of the shell can be obtained by the formula (see, for instance, reference [2])

$$e_{\alpha 3} = \tfrac{1}{2}[U_{\alpha,3} + U_{3|\alpha} + b_\alpha^\beta(U_\beta - x_3 U_{\beta,3})] \tag{34}$$

where the covariant derivative on the reference surface Ω_0 with respect to x_α is denoted by $\langle\langle_{|\alpha}\rangle\rangle$.

One thus has

$$e_{\alpha 3} = \tfrac{1}{2}[\eta_\alpha + [\delta_\alpha^\beta f' + b_\alpha^\beta(f - x_3 f')]\gamma_\beta^0 + [\delta_\alpha^\beta g' + b_\alpha^\beta(g - x_3 g')]\varphi_\beta$$
$$+ w_{|\alpha} + b_\alpha^\beta u_\beta + \sum_{m=1}^{N-1}[\delta_\alpha^\beta - x_{3_{(m)}} b_\alpha^\beta]u_{(m)_\beta}\mathrm{H}(x_3 - x_{3_{(m)}})]. \tag{35}$$

2.4.1. *Free traction conditions for transverse shear stresses on the top and bottom surfaces of the shell*

The traction-free boundary conditions on the top and bottom surfaces of the shell can be written as follows, according to equations (31):

$$e_{\alpha 3}(x_3 = 0) = 0, \qquad e_{\alpha 3}(x_3 = h) = 0. \tag{36}$$

This yields

$$\eta_\alpha + \gamma_\alpha^0 + w_{|\alpha} + b_\alpha^\beta[u_\beta + (h/\pi)\varphi_\beta] = 0 \tag{37a}$$

and

$$\eta_\alpha + [-\delta_\alpha^\beta + hb_\alpha^\beta]\gamma_\beta^0 - b_\alpha^\beta\varphi_\beta + w_{|\alpha} + b_\alpha^\beta u_\beta + \sum_{m=1}^{N-1}[\delta_\alpha^\beta - x_{3_{(m)}}b_\alpha^\beta]u_{(m)\beta} = 0, \tag{37b}$$

Substituting η_α from the equation (37a) into the above equation (37b) yields:

$$b_\alpha^\beta\varphi_\beta = \frac{\pi}{2h}[-2\delta_\alpha^\beta + hb_\alpha^\beta]\gamma_\beta^0 + \sum_{m=1}^{N-1}\frac{\pi}{2h}[\delta_\alpha^\beta - x_{3_{(m)}}b_\alpha^\beta]u_{(m)\beta}, \tag{38}$$

or

$$\varphi_\alpha = d_\alpha^\beta\gamma_\beta^0 + \sum_{m=1}^{N-1}\frac{\pi}{2h}[\delta_\alpha^\beta - x_{3_{(m)}}b_\alpha^\beta]u_{(m)\beta}, \tag{39}$$

where the tensor $[d_\alpha^\beta]$ is given by

$$[d_\alpha^\beta] = \frac{\pi}{2h}[b_\alpha^\beta]^{-1}[-2\delta_\alpha^\beta + hb_\alpha^\beta] = \frac{\pi}{2h}[-2[b_\alpha^\beta]^{-1} + h[\delta_\alpha^\beta]], \tag{40}$$

the identity tensor being denoted by $[\delta_\alpha^\beta]$.

Let D denote the determinant of the latter system. One has

$$\varphi_\alpha = d_\alpha^\beta\gamma_\beta^0 + \sum_{m=1}^{N-1} f_{(m)\alpha}^{\beta}u_{(m)\beta}, \tag{41}$$

where

$$f_{(m)\alpha}^{\beta} = \frac{\pi}{2hD}\left[\sum_{m=1}^{N-1}(\delta_v^\lambda - x_{3_{(m)}}b_v^\lambda)\right]\Delta_{v\alpha}\epsilon_{\lambda\mu}b_\mu^\beta, \tag{42}$$

The $\epsilon_{\lambda\mu}$ coefficients are defined by

$$\epsilon_{11} = \epsilon_{22} = 0, \qquad \epsilon_{12} = -\epsilon_{21} = 1, \tag{43}$$

and the coefficients $\Delta_{v\alpha}$ by

$$\Delta_{v\alpha} = 1 - \delta_\alpha^v. \tag{44}$$

Thus, the transverse shear strains can be expressed as

$$e_{\alpha 3} = \tfrac{1}{2}[[\delta_\alpha^\beta(f' - 1) + b_\alpha^\beta(f - x_3 f')]\gamma_\beta^0 + \left[\delta_\alpha^\beta g' + b_\alpha^\beta\left(g - \frac{h}{\pi} - x_3 g'\right)\right]\varphi_\beta$$

$$+ \sum_{m=1}^{N-1}[\delta_\alpha^\beta - x_{3_{(m)}} b_\alpha^\beta]u_{(m)_\beta}\mathrm{H}(x_3 - x_{3_{(m)}})]. \tag{45}$$

2.4.2. *Continuity conditions for transverse shear stresses at layer interfaces*

These conditions can be written as

$$\sigma_{\alpha 3}^{(i)}(x_3 = x_{3_{(i)}}) = \sigma_{\alpha 3}^{(i+1)}(x_3 = x_{3_{(i)}}), \qquad \alpha = 1, 2; \quad i = 1, \ldots, N-1, \tag{46}$$

or

$$2C_{\alpha 3\omega 3}^{(i)}\{\lim_{\substack{\epsilon\to 0\\ \epsilon > 0}} e_{\omega 3}(x_{3_{(i)}} - \epsilon)\} = 2C_{\alpha 3\omega 3}^{(i+1)}\{\lim_{\substack{\epsilon\to 0\\ \epsilon > 0}} e_{\omega 3}(x_{3_{(i)}} + \epsilon)\}. \qquad \alpha = 1, 2; \qquad i = 1, \ldots, N-1. \tag{47}$$

Substituting equation (41) into equation (45), and using equation (47) yields

$$(C_{\alpha 3\omega 3}^{(i)} - C_{\alpha 3\omega 3}^{(i+1)})\left[\left[\delta_\omega^\beta(f'(x_{3_{(i)}}) - 1) + b_\omega^\beta(f(x_{3_{(i)}}) - x_{3_{(i)}}f'(x_{3_{(i)}}))\right.\right.$$

$$\left.\left. + d_v^\beta\left[\delta_\omega^v g'(x_{3_{(i)}}) + b_\omega^v\left(g(x_{3_{(i)}}) - \frac{h}{\pi} - x_{3_{(i)}}g'(x_{3_{(i)}})\right)\right]\right]\right]\gamma_\beta^0$$

$$+ \sum_{m=1}^{i-1}[(\delta_\omega^\beta - x_{3_{(m)}}b_\omega^\beta)]u_{(m)_\beta}$$

$$+ \sum_{m=1}^{N-1}\left[\left[\left(\delta_\omega^v g'(x_{3_{(i)}}) + b_\omega^v\left(g(x_{3_{(i)}}) - \frac{h}{\pi} - x_{3_{(i)}}g'(x_{3_{(i)}})\right)\right)f_{(m)^v}^{\beta}\right]u_{(m)_\beta}\right]$$

$$- C_{\alpha 3\omega 3}^{(i+1)}(\delta_\omega^\beta - x_{3_{(m)}}b_\omega^\beta)u_{(i)_\beta} = 0. \tag{48}$$

This can be regarded as a linear algebraic system of 2(N-1) equations, the 2(N-1) unknowns being the generalized displacements ⟪per layer⟫ $u_{(m)_\alpha}$ ($\alpha = 1, 2$). These latter can thus be expressed as functions of the generalized displacements γ_α^0 as

$$u_{(m)_\alpha} = a_{(m)_\alpha}^\beta \gamma_\beta^0, \tag{49}$$

where the $a_{(m)^\alpha}^\beta$ coefficients depend only on the curvatures and on the material properties of the various layers.

For a given laminated shell, b_α^β are known. All the $a_{(m)\alpha}^\beta$ are therefore constants.

2.5. THE FINAL DISPLACEMENT FIELD

Combining equation (37a) with equations (41) and (49) leads to

$$\eta_\alpha = -b_\alpha^\beta u_\beta + \Lambda_{(\gamma)_\alpha}^{(\eta)^\beta} \gamma_\beta^0 - w_{|\alpha}, \tag{50}$$

where

$$\Lambda_{(\gamma)_\alpha}^{(\eta)^\beta} = -\left[\delta_\alpha^\beta + \frac{h}{\pi} b_\alpha^\lambda d_\lambda^\beta + \sum_{m=1}^{N-1} \frac{h}{\pi} f^{\gamma}_{(m)_\lambda} b_\alpha^\lambda a_{(m)_\gamma}^\beta \right]. \tag{51}$$

Combining equation (39) with equation (49) yields

$$\varphi_\alpha = \Lambda_{(\gamma)_\alpha}^{(\varphi)^\beta} \gamma_\beta^0, \tag{52}$$

where

$$\Lambda_{(\gamma)_\alpha}^{(\eta)^\beta} = -\left[\delta_\alpha^\beta + \frac{h}{\pi} b_\alpha^\lambda d_\lambda^\beta + \sum_{m=1}^{N-1} \frac{h}{\pi} f^{\gamma}_{(m)_\lambda} b_\alpha^\lambda a_{(m)_\gamma}^\beta \right]. \tag{53}$$

The approximate expressions of the displacement components become thus

$$U_\alpha = \mu_\alpha^\beta u_\beta - x_3 w_{|\alpha} + h_\alpha^\beta \gamma_\beta^0, \qquad U_3 = w, \tag{54}$$

where h_α^β are known functions of x_3, defined by

$$h_\alpha^\beta = \delta_\alpha^\beta f(x_3) + x_3 \Lambda_{(\gamma)_\alpha}^{(\eta)^\beta} + g(x_3) \Lambda_{(\gamma)_\alpha}^{(\varphi)^\beta} + \sum_{m=1}^{N-1} \alpha_{(m)_\alpha}^\beta (x_3 - x_{3_{(m)}}) \mathrm{H}(x_3 - x_{3_{(m)}}). \tag{55}$$

2.6. FORMULATION OF THE TWO-DIMENSIONAL BOUNDARY VALUE PROBLEM

The equations of motion and the natural boundary conditions are derived from the Hamilton's principle:

$$\int_0^t \left\{ \int_V \sigma^{ij} \, \delta e_{ij} \, \mathrm{d}V - \int_V \rho \ddot{\vec{U}} \cdot \delta \vec{U} \, \mathrm{d}V + \int_V \vec{f} \cdot \delta \vec{U} \, \mathrm{d}V + \int_A \vec{s} \cdot \delta \vec{U} \, \mathrm{d}A \right.$$

$$\left. + \int_{\Omega_0} (-\mu_{(h)} p_h + p_0) \, \mathrm{d}S \right\} \mathrm{d}t = 0, \tag{56}$$

$\mu_{(h)}$ denotes the value of

$$\mu = \det [\mu_\alpha^\beta] \tag{57}$$

at $x_3 = h$.

Differentiation with respect to time t is denoted by a superposed dot; ρ is the mass density, and δ the variational operator; f^i are components of body forces, s^i the prescribed components of the stress vector per unit area of the undeformed lateral surface of the shell, and p_0 and p_h the prescribed components of the stress vector per unit area of the surfaces Ω_0 and Ω_h.

This principle, which is generally used for elastic behaviour, can be extended to viscoelastic behaviour by using the correspondence theorem, keeping in mind that boundary conditions can be expressed as separable functions of space and time.

By performing numerical integration (Gauss points) through the thickness of the shell, the following equations of motion are obtained:

$$M^{(1)^{\alpha\beta}}_{|\beta} - N^{(1)^{\alpha}} = I^{(1)^{\beta\alpha}}\ddot{u}_{\beta} - I^{(2)^{\alpha\beta}}\ddot{w}_{|\beta} + I^{(3)^{\beta\alpha}}\ddot{\gamma}^{0}_{\beta} - F^{(1)^{\alpha}},$$

$$M^{(2)^{\beta\alpha}}_{|\alpha\beta} + N^{(1)^{3}} = I^{(2)^{\alpha\beta}}_{|\beta}\ddot{u}_{\alpha} + I^{(2)^{\alpha\beta}}\ddot{u}_{\alpha|\beta} + I^{(1)^{33}}\ddot{w} - I^{(4)^{\alpha\beta}}_{|\beta}\ddot{w}_{|\alpha} - I^{(4)^{\alpha\beta}}\ddot{w}_{|\alpha\beta} + I^{(6)^{\alpha\beta}}_{|\beta}\gamma^{0}_{\alpha} + I^{(6)^{\alpha\beta}}\ddot{\gamma}^{0}_{\alpha|\beta} - P^{3} - F^{(1)^{3}} - F^{(2)^{\beta}}_{|\beta},$$

$$M^{(3)^{\alpha\beta}}_{|\beta} - N^{(2)^{\alpha}} - N^{(3)^{\alpha}} = I^{(3)^{\beta\alpha}}\ddot{u}_{\beta} + I^{(5)^{\beta\alpha}}\ddot{\gamma}^{0}_{\beta} - I^{(6)^{\alpha\beta}}\ddot{w}_{|\beta} - F^{(3)^{\alpha}}, \qquad \alpha = 1, 2. \tag{58}$$

Here the generalized stresses are given by

$$[N^{(1)^{\alpha}}, N^{(2)^{\alpha}}] = \int_0^h \sigma^{\lambda\beta}\mu^{v}_{\lambda}[\mu^{\alpha}_{v|\beta}, h^{\alpha}_{v|\beta}]\mu \, \mathrm{d}x_3, \qquad N^{(1)^{3}} = \int_0^h \sigma^{\lambda\beta}\mu^{v}_{\lambda}b_{v\beta}\mu \, \mathrm{d}x_3, \tag{59}$$

$$[M^{(1)^{\alpha\beta}}, M^{(2)^{\alpha\beta}}, M^{(3)^{\alpha\beta}}] = \int_0^h \sigma^{\lambda\beta}\mu^{v}_{\lambda}[\mu^{\alpha}_{v}, x_3\delta^{\alpha}_{v}, h^{\alpha}_{v}]\mu \, \mathrm{d}x_3, \tag{60}$$

$$N^{(3)^{\alpha}} = \int_0^h \sigma^{\lambda 3}[\mu^{v}_{\lambda}h^{\alpha}_{v,3} + b^{v}_{\lambda}h^{\alpha}_{v}]\mu \, \mathrm{d}x_3, \tag{61}$$

and

$$[F^{(1)^{\alpha}}, F^{(2)^{\alpha}}, F^{(3)^{\alpha}}] = \int_0^h f^{v}\mu^{\beta}_{v}[\mu^{\alpha}_{\lambda}, \delta^{\alpha}_{\lambda}x_3, h^{\alpha}_{\lambda}]a^{\lambda\beta}\mu \, \mathrm{d}x_3,$$

$$F^{(1)^{3}} = \int_0^h f^{3}\mu \mathrm{d}x_3, \tag{62}$$

$$[S^{(1)^{\alpha}}, S^{(2)^{\alpha}}, S^{(3)^{\alpha}}] = \int_0^h S^{v\beta}[\mu^{\alpha}_{v}, h^{\alpha}_{v}, x_3\delta^{\alpha}_{v}]n_{\beta}\mu \, \mathrm{d}x_3, \tag{63}$$

$$S^{(1)^{3}} = \int_0^h s^{3\beta}n_{\beta}\mu \, \mathrm{d}x_3, \qquad P^{3} = -\mu_{(h)}p_h + p_0. \tag{64}$$

Inertia quantities are given by

$$[I^{(1)^{\alpha\beta}}, I^{(2)^{\alpha\beta}}, I^{(3)^{\alpha\beta}}, I^{(4)^{\alpha\beta}}, I^{(5)^{\alpha\beta}}, I^{(6)^{\alpha\beta}}] = \int_0^h \rho a^{\lambda v}[\mu^{\beta}_{v}\mu^{\alpha}_{\lambda}x_3\delta^{\beta}_{\lambda}\mu^{\alpha}_{v}, \mu^{\beta}_{v}h^{\alpha}_{\lambda}, x_3^2\delta^{\beta}_{\lambda}\delta^{\alpha}_{v}, h^{\beta}_{v}h^{\alpha}_{\lambda}, \delta^{\beta}_{\lambda}h^{\alpha}_{v}]\mu \, \mathrm{d}x_3$$

$$I^{(1)^{33}} = \int_0^h \rho\mu \, \mathrm{d}x_3. \tag{65}$$

The boundary conditions are

$$M^{(1)\alpha\beta}n_\beta = S^{(1)\alpha}, \quad \text{or } \delta u_\alpha = 0,$$
$$[\tfrac{1}{2}(M^{(2)\alpha\beta} + M^{(2)\beta\alpha})_{|\alpha} + F^{(2)\beta}]n_\beta = S^{(1)^3}, \quad \text{or } \delta w = 0,$$
$$M^{(3)\alpha\beta}n_\beta = S^{(2)\alpha}, \quad \text{or } \delta\gamma^0_\alpha = 0,$$
$$\tfrac{1}{2}[M^{(2)\alpha\beta} + M^{(2)\beta\alpha}]n_\beta = S^{3\alpha}, \quad \text{or } \delta w_{|\alpha} = 0. \tag{66}$$

The displacement equations of motion are deduced from equations (58)–(65) including the constitutive law given by equation (22).

3. APPPLICATIONS IN WAVE PROPAGATION

3.1. DISPERSION EQUATION

The solution, in terms of generalized displacements $u_1, u_2, w, \gamma^0_1, \gamma^0_2$, is assumed in the following form

$$u_1 = A_1\cos(nx_2)\exp(\mathrm{i}(\omega t - \lambda_1 x_1)), \qquad u_2 = A_2\sin(nx_2)\exp(\mathrm{i}(\omega t - \lambda_1 x_1)),$$
$$w = B\cos(nx_2)\exp(\mathrm{i}(\omega t - \lambda_1 x_1)),$$
$$\gamma^0_1 = C_1\cos(nx_2)\exp(\mathrm{i}(\omega t - \lambda_1 x_1)), \qquad \gamma^0_2 = C_2\sin(nx_2)\exp(\mathrm{i}(\omega t - \lambda_1 x_1)), \tag{68}$$

which characterizes the propagation of harmonic plane waves of wavenumber λ_1 and frequency ω; $\mathrm{i}^2 = -1$.

By substituting these expressions into the equations of motion given by equations (58) with equations (59)–(61) and (22), for free motions, five linear equations in terms of A_1, A_2, B, C_1, C_2 are obtained. For a non-trivial solution, the determinant of the coefficient matrix must vanish, resulting in the frequency equation

$$\det[\mathbf{K} - \omega^2\mathbf{M}] = 0, \tag{69}$$

where $\mathbf{K}$ represents a stiffness matrix, and $\mathbf{M}$ a mass matrix.

A sixth-order dispersion equation in λ_1^2, with the circumferential mode number n as parameter (n is an integer), is therefore obtained. For the sake of completeness, the elements of the determinantal frequency equation coefficients (denoted c_{ij}, $1 \leqslant i \leqslant 5$, $1 \leqslant j \leqslant 5$) are given in the Appendix. Five roots of the dispersion equation represent the axial wave number in the positive x_1 direction.

When the wavenumber λ_1 is set equal to zero, a number of cut-off frequencies, depending on the value of n are found, as may be observed in the following spectra.

Simplified expressions (asymptotic ones) for phase velocities can also be obtained when the wavelength is short ($\lambda_1 \to \infty$). Among those values, one will retain the shear-wave velocity of the medium, that will be denoted by c_T.

Geometric and physical parameters being given, the frequency equation (69) constitutes a transcendental relationship between the nondimensional wavenumber $\overline{\lambda}_1 = \lambda_1/\pi R$ (70), the number of circumferential waves n, and the non-dimensionalized frequency $\bar{\omega}$, defined by

$$\bar{\omega} = \omega R/\pi c_T. \tag{71}$$

When the constitutive law is viscoelastic, this latter equation admits complex wavenumbers. One recalls that the real part of the wavenumber represents harmonic

TABLE 1

Free vibration analysis of simply supported cylindrical isotropic shells; comparison of lowest natural frequency parameters $\tilde{\omega} = (\omega h/\pi)\sqrt{\rho/G}$, *where G is the shear modulus, n* = 0·3 *the Poisson's ratio, R the mean radius of the cylinder, a its length, for* $\lambda_1 = m\pi R/a = 4\pi$, *and m an integer* (SDT: *shear deformation theory*)

	n				n			
	1	**2**	**3**	**4**	**1**	**2**	**3**	**4**
	$h/r = 0{\cdot}06$				$h/r = 0{\cdot}1$			
Exact-3D	0·08639	0·08748	0·08933	0·09199	0·20529	0·20802	0·21261	0·21906
Present	**0·08636**	**0·08746**	**0·08932**	**0·092001**	**0·20480**	**0·20800**	**0·21201**	**0·21890**
Touratier	0·08635	0·08745	0·08931	0·09200	0·20458	0·20733	0·21192	0·21839
Bhimaraddi	0·08639	0·08728	0·08911	0·09175	0·20478	0·20678	0·21132	0·21771
SDT	0·08611	0·08718	0·08902	0·09165	0·20360	0·20628	0·21077	0·21710
Flügge	0·09161	0·09290	0·09510	0·09824	0·23623	0·23995	0·24620	0·25502
	$h/r = 0{\cdot}12$				$h/r = 0{\cdot}18$			
Exact-3D	0·27491	0·27849	0·28447	0·29287	0·50338	0·50937	0·51934	0·53325
Present	**0·27421**	**0·27828**	**0·28390**	**0·29171**	**0·50048**	**0·50933**	**0·51890**	**0·53319**
Touratier	0·27361	0·27721	0·28321	0·29161	0·50002	0·50606	0·51610	0·53008
Bhimaraddi	0·27286	0·27641	0·28233	0·29064	0·49818	0·50418	0·51416	0·52808
SDT	0·27197	0·27547	0·28131	0·28951	0·49479	0·50058	0·51021	0·52366
Flügge	0·32960	0·33479	0·34349	0·35571	0·67100	0·68056	0·69634	0·71803

variations, and that the imaginary part represents spatial attenuation. A direct method has been used to determine the roots of the frequency equation: i.e., fixing the values of frequency, restricted to their real part, and searching for the values of the corresponding wavenumber. For a given value of the frequency, several values of the wavenumber are obtained, each one corresponding to a peculiar propagating mode.

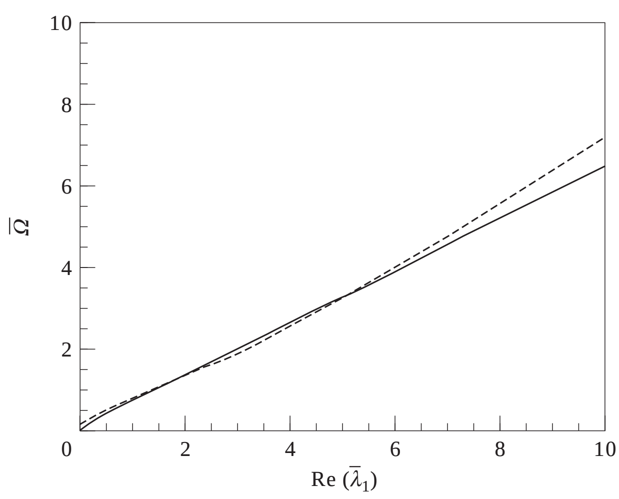

Figure 2. Non-dimensionalized frequency $\bar{\Omega} = (\omega h_2/\pi)\sqrt{\rho_2/\mu_2}$ versus real part of the non-dimensionalized wave number for axisymmetric mode spectrum ($n = 0$) in a three-layered elastic cylinder for $h/R = 0{\cdot}1$, where h_2 is the thickness of the second layer, ρ_2 its mass density, μ_2 the corresponding Lamé constant, R the mean radius of the cylinder, and h its thickness. ——, Model, ---; three-dimensional solution.

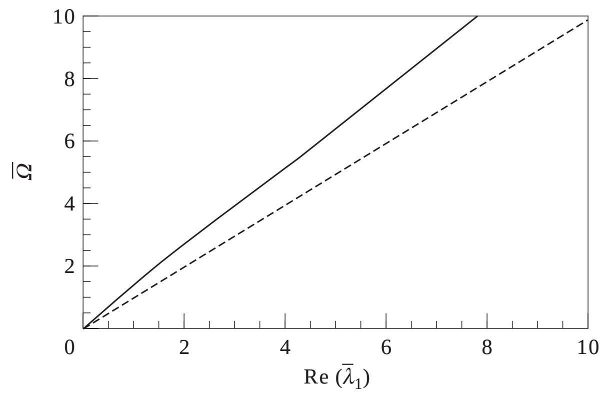

Figure 3. Non-dimensionalized frequency $\bar{\Omega} = (\omega h_2/\pi)\sqrt{\rho_2/\mu_2}$ versus real part of the non-dimensionalized wave number for axisymmetric mode spectrum ($n = 0$) in a three-layered elastic cylinder for $h/R = 1$, where h_2 is the thickness of the second layer, ρ_2 its mass density, μ_2 the corresponding Lamé, constant, R the mean radius of the cylinder, and h its thickness. Key as Figure 2.

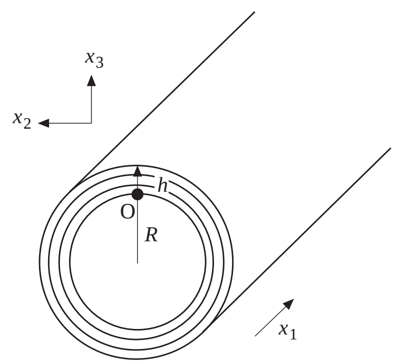

Figure 4. The viscoelastic cylinder.

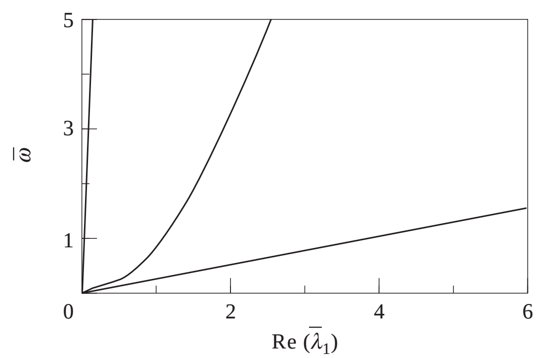

Figure 5. Non-dimensionalized frequency versus real parts of the non-dimensionalized wave numbers for axisymmetric mode spectrum ($n = 0$) in a three-layered cylinder with a viscoelastic internal layer.

TABLE 2

Non-dimensional frequency for axisymmetric mode spectrum ($n = 0$) *in a three-layered cylinder with a viscoelastic internal layer*

$\bar{\omega}$	1st mode [Re ($\bar{\lambda}_1^{(1)}$)]	2nd mode [Re ($\bar{\lambda}_1^{(2)}$)]	3rd mode [Re ($\bar{\lambda}_1^{(3)}$)]
0	0·000	0·000	—
0·5	1·059	0·559	0·011
1	2·122	0·796	0·023
2	4·264	1·135	0·045
3	6·416	1·400	0·064
5	10·725	1·826	0·094
5·5	11·802	1·919	0·100
7	15·032	2·177	0·115
7·5	16·107	2·258	0·120
8	17·182	2·335	0·124
9	19·753	2·534	0·133
9·5	20·410	2·904	0·136
10	21·487	3·197	0·139
11	23·637	3·817	0·146

3.2. VALIDATION OF THE THEORY IN THE CASE OF WAVE PROPAGATION

In order to assess the accuracy of the proposed laminated shell theory in dynamics, comparison with previous theories has been made for the case of a homogeneous elastic cylinder; comparison with the exact three-dimensional solution has also been made for a three-layered elastic cylinder for axisymmetric motion, which is the only case where an exact solution has been investigated [15, 17].

3.2.1. *Case of an isotropic elastic cylinder*

Table 1 contains non-dimensionalized natural frequencies for isotropic short cylindrical shells obtained by using various theories: three-dimensional elasticity (Armenakas *et al*. [16]); present theory; Touratier theory [5]; Bhimaraddi theory [9]; shear-deformation theory with a shear correction factor equal to $\pi^2/12$, Mirsky and Hermann [10]; Flügge theory [11]. Only the most significant problem from the Bhimaraddi paper [9] has been retained: i.e., $\lambda_1 = m\pi R/a = 4\pi$. Comparisons of the above theories show that the maximum error in the present analysis is about $-0{\cdot}57\%$, whereas in the Touratier theory results are about $-0{\cdot}6\%$, Bhimaraddi results about -1%, the shear-deformation theory is about $-1{\cdot}8\%$, and the Flügge theory is about $+35\%$. The improvements due to the simultaneous refinements of the shear and membrane terms are apparent.

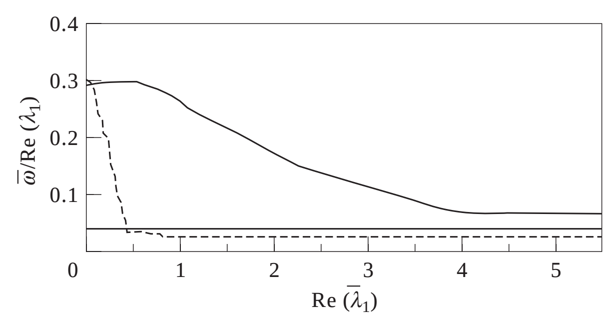

Figure 6. Non-dimensionalized phase velocity versus real parts of the non-dimensionalized wave numbers for axisymmetric mode spectra ($n = 0$) for three-layered cylinders. ——, Viscoelastic case; —–, elastic case.

TABLE 3

Non-dimensional phase velocity versus frequency axisymmetric mode spectra ($n = 0$) *in a three-layered cylinder with* (*a*) *a viscoelastic internal layer*, *and* (*b*) *an elastic internal layer*

	1st mode		2nd mode		3rd mode	
$\bar{\omega}$	Re ($\bar{\lambda}_1^{(1)}$)	$\bar{\omega}$/Re ($\bar{\lambda}_1^{(1)}$)	Re ($\bar{\lambda}_1^{(2)}$)	$\bar{\omega}$/Re ($\bar{\lambda}_1^{(2)}$)	Re ($\bar{\lambda}_1^{(3)}$)	$\bar{\omega}$/Re ($\bar{\lambda}_1^{(3)}$)
(a)						
0·000	0·000	0·048	0·000	0·300	—	—
0·051	1·059	0·048	0·559	0·091	0·011	4·636
0·101	2·122	0·048	0·796	0·127	0·023	4·391
0·203	4·264	0·048	1·135	0·179	0·045	4·511
0·304	6·416	0·047	1·400	0·217	0·064	4·750
0·506	10·725	0·047	1·826	0·277	0·094	5·383
0·557	11·802	0·047	1·919	0·290	0·100	5·570
0·709	15·032	0·047	2·177	0·326	0·115	6·165
0·760	16·107	0·047	2·258	0·337	0·120	6·333
0·810	17·182	0·047	2·335	0·347	0·124	6·533
0·912	19·753	0·046	2·534	0·360	0·133	6·857
0·962	20·410	0·047	2·904	0·331	0·136	7·074
1·013	21·487	0·047	3·197	0·317	0·139	7·288
1·114	23·637	0·047	3·817	0·292	0·146	7·630
(b)						
0·000	0·000	0·048	—	—	—	—
0·101	2·116	0·048	0·790	0·128	0·029	3·483
0·152	3·174	0·048	0·970	0·157	0·043	3·535
0·203	4·232	0·048	1·122	0·181	0·057	3·561
0·253	5·290	0·048	1·258	0·201	0·072	3·514
0·304	6·347	0·048	1·381	0·220	0·086	3·535
0·355	7·405	0·048	1·500	0·237	0·100	3·550
0·405	8·463	0·048	1·603	0·253	0·115	3·522
0·456	9·521	0·048	1·704	0·278	0·129	3·535
0·506	10·579	0·048	1·801	0·281	0·143	3·538
0·608	12·695	0·048	1·982	0·307	0·172	3·535
0·709	14·811	0·048	2·151	0·330	0·201	3·527
0·810	16·927	0·048	2·310	0·351	0·229	3·537
0·912	19·043	0·048	2·461	0·371	0·258	3·535
1·013	20·567	0·048	2·746	0·369	0·315	3·216

3.2.2. *Case of a three-layered elastic cylinder*

An infinitely long traction-free circular cylindrical shell, of thickness h and inner radius R, composed of three orthotropic elastic layers, perfectly bonded at their interfaces is considered. The purpose of the study being the propagation of plane waves in the infinite length cylinder, no boundary conditions are requested.

Figures 2 and 3 show the frequency versus real part of the wave number for the first axisymmetric mode (fundamental torsional one) for $h/R = 0{\cdot}1$ and $h/R = 1$ respectively. The results obtained with the present theory are compared to those of the exact three-dimensional solution of Armenakas [17]. It can be observed that, for $h/R = 0{\cdot}1$, the present theory gives results close to the exact three-dimensional solution. The case $h/R = 1$, where the cylinder can be considered nearly as a solid one, has been examined in order to show the limitations of the model, as can be observed in Figure 4.

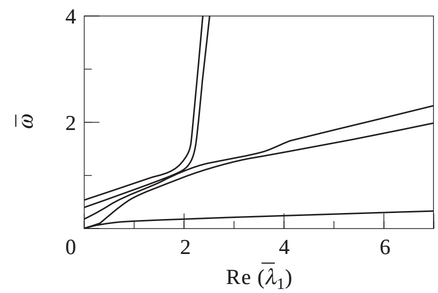

Figure 7. Non-dimensionalized frequency versus real parts of the non-dimensionalized wave numbers for asymmetric mode spectra ($n = 1$) for three-layered cylinders having viscoelastic internal layer.

3.3. APPLICATION TO A MULTILAYERED VISCOELASTIC CYLINDER

An infinitely long traction-free circular cylindrical shell, of thickness h and inner radius $R (R = 9{\cdot}5h)$, composed of three orthotropic layers of equal thickness, perfectly bonded at their interfaces is considered (see Figure 4). The internal layer of the cylinder is made of a viscoelastic polymer and the external skins are purely elastic, all the corresponding constitutive law being isotropic (Young's moduli E_1, E_3, Poisson's ratios ν_1, ν_3, mass densities ρ_1, ρ_3 respectively). A viscoelastic Kelvin–Voigt constitutive law, using complex Young moduli, has been retained for the viscoelastic layer, in the form

$$E_2 = E_2' + \mathrm{j}\omega E_2'', \tag{67}$$

where E_2' and E_2'' are constants. The Poisson's ratio and the mass density of the viscoelastic core will respectively be denoted ν_2, ρ_2.

TABLE 4

Non-dimensional frequency for asymmetric mode spectra ($n = 1$) in a three-layered cylinder with a viscoelastic internal layer

$\bar{\omega}$	1st mode Re $(\bar{\lambda}_1^{(1)})$	2nd mode Re $(\bar{\lambda}_1^{(2)})$	3rd mode Re $(\bar{\lambda}_1^{(3)})$	4th mode Re $(\bar{\lambda}_1^{(4)})$	5th mode Re $(\bar{\lambda}_1^{(5)})$
0·000	0·000	—	—	—	—
0·159	4·660	0·325	0·300	—	—
0·955	8·073	0·780	0·644	0·611	—
1·591	10·791	1·243	1·233	0·907	—
1·750	11·440	1·752	1·566	1·253	0·703
2·068	12·752	2·603	1·925	1·500	1·104
2·227	13·419	2·978	2·026	1·663	1·336
2·387	14·092	3·298	2·094	1·844	1·591
2·546	14·775	3·578	2·141	2·026	1·846
2·705	15·449	3·827	2·308	2·162	1·952
2·864	16·126	4·027	2·478	2·274	2·052
3·023	16·819	4·186	2·811	2·280	2·147
3·182	17·504	4·299	3·162	2·310	2·238
3·500	18·865	4·364	3·531	2·371	2·268
3·819	20·215	4·74	3·916	2·500	2·294
4·773	24·227	7·574	4·257	2·718	2·364
6·364	30·127	12·307	26·378	2·510	2·400

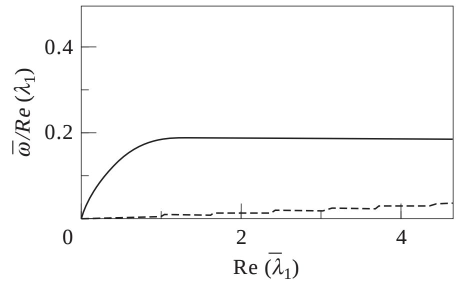

Figure 8. Non-dimensionalized phase velocity versus real parts of the non-dimensionalized wave numbers for first asymmetric mode spectra ($n = 1$) for three-layered cylinders. Key as Figure 6.

The mechanical properties of the different layers are as follows; steel skins, $\rho_1 = \rho_3 = 7800\ \mathrm{kg/m^3}$, $\nu_1 = \nu_3 = 0{\cdot}3$, $E_1 = E_3 = 210$ GPa, viscoelastic core, $\rho_2 = 980\ \mathrm{kg/m^3}$, $\nu_2 = 0{\cdot}453$, $E_2' = 100$ GPa, $E_2'' = 20\ \mathrm{GPa \cdot s^{-1}}$. In this paper, the frequency-versus-real part of the wave number for axisymmetric and asymmetric modes are presented.

TABLE 5

Non-dimensional phase velocity versus frequency for asymmetric mode spectra ($n = 1$) *in a three-layered cylinder with* (a) *a viscoelastic internal layer and* (b) *an elastic internal layer*

	1st mode				
$\bar{\omega}$	Re ($\bar{\lambda}_1^{(1)}$)	$\bar{\omega}$/Re ($\bar{\lambda}_1^{(1)}$)	$\bar{\omega}$	Re ($\bar{\lambda}_1^{(1)}$)	$\bar{\omega}$/Re ($\bar{\lambda}_1^{(1)}$)
(a)			(b)		
0·000	0·000	—	0·000	0·000	—
0·955	8·073	—	0·477	3·204	—
1·591	10·791	—	0·509	3·417	—
1·750	11·440	2·489	0·573	3·843	—
2·068	12·752	1·873	0·636	4·268	—
2·227	13·419	1·667	0·700	4·694	—
2·387	14·092	1·500	0·796	5·331	—
2·546	14·775	1·379	0·859	5·755	—
2·705	15·449	1·467	0·955	6·390	—
2·864	16·126	1·473	1·018	6·815	—
3·023	16·819	1·482	1·114	7·450	—
3·182	17·504	1·564	1·177	7·873	—
3·500	18·865	1·684	1·273	8·508	—
3·819	20·215	2·081	1·432	9·565	—
4·773	24·227	2·692	1·591	10·622	—
			1·909	12·735	13·636
			2·068	13·791	13·605
			2·227	14·848	13·579
			2·387	15·904	13·563
			2·546	16·961	13·543
			2·705	18·018	13·525
			2·864	19·074	13·509
			3·023	20·131	13·496
			3·182	21·188	13·483
			3·500	23·301	13·462
			3·819	25·415	13·447
			4·455	29·644	13·459
			4·773	31·758	13·445
			6·364	42·333	13·426

3.3.1. *Axisymmetric modes*

For axisymmetric motion ($n = 0$), a number of terms in the frequency determinant (69) vanish, reducing the frequency equation to

$$c_{22}\left(c_{55} - \frac{c_{25}c_{52}}{c_{22}}\right)\begin{vmatrix} c_{11} & c_{13} & c_{14} \\ c_{31} & c_{33} & c_{34} \\ c_{41} & c_{43} & c_{44} \end{vmatrix} = 0, \tag{72}$$

so that only three modes, among which two fundamental ones (with a zero cut-off frequency) exist.

3.3.1.1. Frequency spectrum. Figure 5 and Table 2 show the frequency-versus real part of the wave number for axisymmetric mode spectrum, in the case of the viscoelastic cylinder described above. The three different modes, among which the fundamental torsional and longitudinal modes, clearly appear.

In the viscoelastic case studied, the fundamental torsional and longitudinal modes are coupled, as can be proved by developing equation (72); in an elastic case, they would on the contrary be uncoupled, as can also be proved by developing equation (72). The remaining mode (torsional upper mode) is an upper one, with a non-zero cut-off frequency.

3.3.1.2. Phase–velocity spectrum; comparison with the equivalent elastic case. Figure 6 shows the phase velocity versus real part of the wave number for the two first axisymmetric modes (torsional and longitudinal) spectra, in the case of the viscoelastic cylinder and for the elastic case.

Viscoelasticity affects the first fundamental torsional mode, which becomes weakly dispersive, with an asymptotic velocity numerically found equal to the shear-wave velocity in the cylinder (c_T). The fundamental longitudinal mode is, on the contrary, much more sensitive to viscoelasticity. Frontwave phase-velocities are however the same in both cases; the mode tends to become non-dispersive as Re $(\bar{\lambda}_1)$ increases, with an asymptotic velocity that is numerically found to be equal to c_T, for the elastic and viscoelastic internal layer (see Tables 3(a, b)). This figure shows that even for a viscoelastic constitutive law for the core, torsional waves are weakly dispersive (not dispersive in the elastic case), while longitudinal waves are dispersive.

3.3.2. *Asymmetric modes*

3.3.2.1. Frequency spectrum. Figure 7 and Table 4 show the frequency-versus real part of the wave number for asymmetric mode spectra, in the case of the three-layered cylinder with a viscoelastic internal layer. The five different modes, among which is the fundamental flexural mode, clearly appear.

3.3.2.2. Phase-velocity spectrum; comparison with the equivalent elastic case. Figure 8 and Tables 5(a, b) show the phase velocity-versus real part of the wave number for the first asymmetric mode (flexural) spectra, in the case of the three-layered cylinder with a viscoelastic or an elastic core.

In the viscoelastic case, phase velocity is much more sensitive to the variations of the real part of the wave number. As for the longitudinal motion, it can also be observed that, at a fixed wavenumber, the flexural harmonic waves propagate with larger phase velocities in the viscoelastic cylinder than in the equivalent elastic one. In both cases, as usually, this flexural mode tends to become non-dispersive as Re $(\bar{\lambda}_1)$ increases.

4. CONCLUSIONS

A new refined two-dimensional laminated shell theory, which allows the continuity requirements for displacements and stresses at layer interfaces to be satisfied exactly, is proposed. The model, which keeps only five generalized displacements, also takes into account refinements of membrane and shear terms. The efficiency of this new kinematics for the modelling of shells in dynamics is proved through comparison with previous theories in a case for which an exact three-dimensional theory is known (Armenakas [15, 16]). The results presented here show the improvements due to the theory, for which no need for shear correction factors is requested The model is then applied to the determination of the dispersive behaviour of a circular cylindrical viscoelastic shell. The influence of viscoelasticity, as concerns dispersion, clearly appears.

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APPENDIX

The elements of the determinantal frequency equation are given below, where c_{ij} is an element in the ith row, jth column:

$$c_{21} = M_{A_1}^{(1)^{21}} + M_{A_1}^{(1)^{22}}, \qquad c_{22} = M_{A_2}^{(1)^{21}} + M_{A_2}^{(1)^{22}} - \omega^2 I^{(1)^{22}},$$

$$c_{23} = M_B^{(1)^{21}} + M_B^{(1)^{22}} - \omega^2(n/\alpha_2)I^{(2)^{11}}, \qquad c_{24} = M_{c_1}^{(1)^{21}} + M_{c_1}^{(1)^{22}},$$

$$c_{25} = M_{c_2}^{(1)^{21}} + M_{c_2}^{(1)^{22}} - \omega^2 I^{(3)^{22}}, \qquad c_{51} = M_{A_1}^{(3)^{21}} + M_{A_1}^{(3)^{22}},$$

$$c_{52} = M_{A_2}^{(3)^{21}} + M_{A_2}^{(3)^{22}} - \omega^2 I^{(3)^{22}}, \qquad c_{53} = M_B^{(3)^{21}} + M_B^{(3)^{22}} - \omega^2(n/\alpha_2)I^{(6)^{22}}$$

$$c_{54} = M_{c_1}^{(3)^{21}} + M_{c_1}^{(3)^{22}}, \qquad c_{55} = M_{c_2}^{(3)^{21}} + M_{c_2}^{(3)^{22}} + N_{c_2}^{(3)^{2}} - \omega^2 I^{(5)^{22}},$$

$$c_{31} = M_{A_1}^{(2)^{11}} + M_{A_1}^{(2)^{12}} + M_{A_1}^{(2)^{21}} + N_{A_1}^{(1)^{3}} - \mathrm{i}\omega^2\lambda_1 I^{(2)^{11}},$$

$$c_{32} = M_{A_2}^{(2)^{11}} + M_{A_2}^{(2)^{12}} + M_{A_2}^{(2)^{21}} + M_{A_2}^{(2)^{22}} + N_{A_2}^{(1)^{3}} - \mathrm{i}\omega^2(n/\alpha_2)I^{(2)^{22}},$$

$$c_{33} = M_B^{(2)^{11}} + M_B^{(2)^{12}} + M_B^{(2)^{21}} + M_B^{(2)^{22}} + N_B^{(1)^{3}} - \omega^2[I^{(1)^{33}} + \lambda_1^2 I^{(4)^{11}} + a^{22}n^2 I^{(4)^{22}}],$$

$$c_{34} = M_{c_1}^{(2)^{11}} + M_{c_1}^{(2)^{12}} + M_{c_1}^{(2)^{21}} + M_{c_1}^{(2)^{21}} + N_{c_1}^{(1)^{3}} - \mathrm{i}\omega^2\lambda_1 I^{(6)^{11}},$$

$$c_{35} = M^{(2)^{11}}c_2 + M^{(2)^{12}}c_2 + M_{c_2}^{(2)^{21}} + M_{c_2}^{(2)^{21}} + N_{c_2}^{(1)^{3}} - \mathrm{i}\omega^2(n/\alpha_2)I^{(6)^{22}}.$$

Here

$$M_{A_1}^{(1)^{11}} = -\mathrm{i}\lambda_1^2\int_0^h C_{1111}\mu\,\mathrm{d}x_3, \qquad M_A^{(1)^{11}} = \mathrm{i}\lambda_1 n\int_0^h \frac{C_{1122}}{\alpha_2}(\mu_2^2)^2\mu\,\mathrm{d}x_3,$$

$$M_B^{(1)^{11}} = -\mathrm{i}\lambda_1\int_0^h [C_{1111}(x_3\lambda_1^2 - b_{11}) + C_{1122}(a^{22}x_3n^2 - b_{22})\mu_2^2]\mu\,\mathrm{d}x_3,$$

$$M_{C_1}^{(1)^{11}} = -\lambda_1^2\int_0^h C_{1111}h_1^1\mu\,\mathrm{d}x_3, \qquad M_{C_2}^{(1)^{11}} = \lambda_1 n\int_0^b \frac{C_{1122}}{\alpha_2}h_2^2\mu\,\mathrm{d}x_3,$$

$$M_{A_1}^{(1)^{12}} = -n\int_0^h a^{22}C_{1212}\mu\,\mathrm{d}x_3, \qquad M_{A_2}^{(1)^{12}} = -\mathrm{i}\lambda_1\int_0^h \frac{C_{1212}}{\alpha_2}(\mu_2^2)^2\mu\,\mathrm{d}x_3,$$

$$M_B^{(1)^{12}} = -\mathrm{i}\lambda_1\int_0^h a^{22}C_{1212}(\mu_1^1 + \mu_2^2)x_3\mu\,\mathrm{d}x_3, \qquad M_{C_1}^{(1)^{12}} = -n\int_0^h C_{1212}(h_1^1)^2\mu\,\mathrm{d}x_3,$$

$$M_{C_2}^{(1)^{12}} = -\mathrm{i}\lambda_1\int_0^h \frac{C_{1212}}{\alpha_2}(h_2^2)^2\mu\,\mathrm{d}x_3, \qquad M_{A_1}^{(1)^{21}} = \mathrm{i}\lambda_1 n\int_0^h \frac{C_{1212}}{\alpha_2}(\mu_2^2)^2\mu\,\mathrm{d}x_3,$$

$$M_{A_2}^{(1)^{21}} = -\lambda_1^2\int_0^h C_{1212}(\mu_2^2)^4\mu\,\mathrm{d}x_3, \qquad M_B^{(1)^{21}} = -\lambda_1^2\int_0^h \frac{C_{1212}}{\alpha_2}(\mu_1^1 + \mu_2^2)x_3\mu\,\mathrm{d}x_3,$$

$$M_{C_1}^{(1)^{21}} = \mathrm{i}\lambda_1 n\int_0^h \frac{C_{1212}}{\alpha_2}(h_1^1)^2(\mu_2^2)^2\mu\,\mathrm{d}x_3, \qquad M_{C_2}^{(1)^{21}} = -\lambda_1^2\int_0^h C_{1212}(h_2^2)^2(\mu_2^2)^2\mu\,\mathrm{d}x_3,$$

$$M_{A_1}^{(1)^{22}} = \mathrm{i}\lambda_1 \int_0^h \frac{C_{1122}}{\alpha_2} (\mu_2^2)^2 \mu \, \mathrm{d}x_3, \qquad M_{A_2}^{(1)^{22}} = -n \int_0^h a^{22} C_{2222} (\mu_2^2)^4 \mu \, \mathrm{d}x_3,$$

$$M_B^{(1)^{22}} = -\lambda_1 \int_0^h \left[\frac{C_{1122}}{\alpha_2} (x_3 \lambda_1^2 - b_{11}) + C_{2222} (a^{22} x_3 n^2 - b_{22}) \mu_2^2 \right] (\mu_2^2)^2 \mu \, \mathrm{d}x_3,$$

$$M_{C_1}^{(1)^{22}} = \mathrm{i}\lambda_1 \int_0^h \frac{C_{1122}}{\alpha_2} h_1^1 (\mu_2^2)^2 \mu \, \mathrm{d}x_3, \qquad M_{C_2}^{(1)^{22}} = -n \int_0^h a^{22} C_{2222} h_2^2 (\mu_2^2)^2 \mu \, \mathrm{d}x_3,$$

$$M_{A_1}^{(2)^{11}} = \mathrm{i}\lambda_1^3 \int_0^h C_{1111} x_3 \mu \, \mathrm{d}x_3, \qquad M_{A_2}^{(2)^{11}} = -\lambda_1^2 n \int_0^h \frac{C_{1122}}{\alpha_2} (\mu_2^2)^2 x_3 \mu \, \mathrm{d}x_3,$$

$$M_B^{(2)^{11}} = -\lambda_{1_1}^2 \int_0^h [C_{1111} (x_3 \lambda_1^2 - b_{11}) + C_{1122} (a^{22} x_3 n^2 - b_{22}) \mu_2^2] \mu \, \mathrm{d}x_3,$$

$$M_{C_1}^{(2)^{11}} = \mathrm{i}\lambda_1^3 \int_0^h C_{1111} h_1^1 x_3 \mu \, \mathrm{d}x_3, \qquad M_{C_2}^{(2)^{11}} = -\lambda_1^2 n \int_0^h \frac{C_{1122}}{\alpha_2} h_2^2 x_3 \mu \, \mathrm{d}x_3,$$

$$M_{A1}^{(2)^{12}} = \mathrm{i}\lambda_1 n^2 \int_0^h a^{22} C_{1212} x_3 \mu \, \mathrm{d}x_3, \qquad M_{A_2}^{(2)^{12}} = -\lambda_1^2 n \int_0^h \frac{C_{1212}}{\alpha_2} (\mu_2^2)^2 x_3 \mu \, \mathrm{d}x_3,$$

$$M_B^{(2)^{12}} = -\lambda_1^2 n^2 \int_0^h a^{22} C_{1212} (\mu_1^1 + \mu_2^2) x_3^2 \mu \, \mathrm{d}x_3, \qquad M_{C_1}^{(2)^{12}} = \mathrm{i}\lambda_1 n^2 \int_0^h a^{22} C_{1212} (h_1^1)^2 x_3 \mu \, \mathrm{d}x_3,$$

$$M_{C_2}^{(2)^{12}} = -\lambda_1^2 n \int_0^b \frac{C_{1212}}{\alpha_2} (h_2^2)^2 x_3 \mu \, \mathrm{d}x_3, \qquad M_{A_1}^{(2)^{21}} = \mathrm{i}\lambda_1 n^2 \int_0^h a^{22} C_{1212} x_3 \mu_2^2 \mu \, \mathrm{d}x_3,$$

$$M_{A_2}^{(2)^{21}} = -\lambda_1^2 n \int_0^h \frac{C_{1212}}{\alpha_2} (\mu_2^2)^3 x_3 \mu \, \mathrm{d}x_3,$$

$$M_B^{(2)^{21}} = -\lambda_1^2 n^2 \int_0^h a^{22} C_{1212} (\mu_1^1 + \mu_2^2) x_3^2 (\mu_2^2)^2 \mu \, \mathrm{d}x_3,$$

$$M_{c_1}^{(2)^{21}} = \mathrm{i}\lambda_1 n^2 \int_0^h a^{22} C_{1212} (h_1^1)^2 x_3 \mu_2^2 \mu \, \mathrm{d}x_3, \qquad M_{C_2}^{(2)^{21}} = -\lambda_1^2 n \int_0^h \frac{C_{1212}}{\alpha_2} (h_2^2)^2 x_3 \mu_2^2 \mu \, \mathrm{d}x_3,$$

$$M_{A_1}^{(2)^{22}} = \mathrm{i}\lambda_1 n^2 \int_0^h a^{22} C_{1122} x_3 \mu_2^2 \mu \, \mathrm{d}x_3, \qquad M_{A_2}^{(2)^{22}} = -n^3 \int_0^h a^{22} \frac{C_{1122}}{\alpha_2} (\mu_2^2)^3 x_3 \mu \, \mathrm{d}x_3,$$

$$M_B^{(2)^{22}} = -n^2 \int_0^h a^{22} [C_{1122} (x_3 \lambda_1^2 - b_{11}) + C_{2222} (a^{22} x_3 n^2 - b_{22}) \mu_2^2] \mu_2^2 \mu \, \mathrm{d}x_3$$

$$M_{C_1}^{(2)^{22}} = \mathrm{i}\lambda_1 n^2 \int_0^h a^{22} C_{1122} h_1^1 x_3 \mu_2^2 \mu \, \mathrm{d}x_3, \qquad M_{C_2}^{(2)^{22}} = -n^3 \int_0^h a^{22} \frac{C_{1122}}{\alpha_2} h_2^2 x_3 \mu_2^2 \mu \, \mathrm{d}x_3,$$

$$M_{A_1}^{(3)^{11}} = -\lambda_1^2 \int_0^h C_{1111} h_1^1 \mu \, \mathrm{d}x_3, \qquad M_{A_2}^{(3)^{11}} = -\mathrm{i}\lambda_1 n \int_0^h \frac{C_{1122}}{\alpha_2} (\mu_2^2)^2 h_1^1 \mu \, \mathrm{d}x_3,$$

$$M_B^{(3)^{11}} = -\mathrm{i}\lambda_1 \int_0^h [C_{1111}(x_3\lambda_1^2 - b_{11}) + C_{1122}(a^{22} x_3 n^2 - b_{22})\mu_2^2] h_1^1 \mu \, \mathrm{d}x_3,$$

$$M_{C_1}^{(3)^{11}} = -\lambda_1^2 \int_0^h C_{1111} (h_1^1)^2 \mu \, \mathrm{d}x_3, \qquad M_{C_2}^{(3)^{11}} = -\lambda_1 n \int_0^h \frac{C_{1122}}{\alpha_2} h_1^1 h_2^2 \mu \, \mathrm{d}x_3,$$

$$M_{A_1}^{(3)^{12}} = -n^2 \int_0^h a^{22} C_{1212} h_1^1 \mu \, \mathrm{d}x_3, \qquad M_{A_2}^{(3)^{12}} = -\mathrm{i}\lambda_1 n \int_0^h \frac{C_{1212}}{\alpha_2} (\mu_2^2)^2 h_1^1 \mu \, \mathrm{d}x_3,$$

$$M_B^{(3)^{12}} = -\mathrm{i}\lambda_1 \int_0^h a^{22} C_{1212} (\mu_1^1 + \mu_2^2) \mu \, \mathrm{d}x_3, \qquad M_{C_1}^{(3)^{12}} = -n^2 \int_0^h a^{22} C_{1212} (h_1^1)^3 \mu \, \mathrm{d}x_3,$$

$$M_{C_2}^{(3)^{12}} = -\mathrm{i}\lambda_1 n \int_0^h \frac{C_{1212}}{\alpha_2} (h_2^2)^2 h_1^1 \mu \, \mathrm{d}x_3, \qquad M_{A_1}^{(3)^{21}} = \mathrm{i}\lambda_1 n \int_0^h \frac{C_{1212}}{\alpha_2} \mu_2^2 h_2^2 \mu \, \mathrm{d}x_3,$$

$$M_{A_2}^{(3)^{21}} = -\lambda_1^2 \int_0^h C_{1212} (\mu_2^2)^3 h_2^2 \mu \, \mathrm{d}x_3, \qquad M_B^{(3)^{21}} = -\lambda_1^2 n \int_0^h \frac{C_{1212}}{\alpha_2} (\mu_1^1 + \mu_2^2) \mu_2^2 h_2^2 x_3 \mu \, \mathrm{d}x_3,$$

$$M_{C_1}^{(3)^{21}} = \mathrm{i}\lambda_1 n \int_0^h \frac{C_{1212}}{\alpha_2} (h_1^1)^2 \mu_2^2 \mu \, \mathrm{d}x_3, \qquad M_{C_2}^{(3)^{21}} = -\lambda_1^2 \int_0^h C_{1212} (h_2^2)^3 \mu_2^2 \mu \, \mathrm{d}x_3,$$

$$M_{A_1}^{(3)^{22}} = \mathrm{i}\lambda_1 n \int_0^h \frac{C_{1122}}{\alpha_2} (\mu_2^2)^2 h_2^2 \mu \, \mathrm{d}x_3, \qquad M_{A_2}^{(3)^{22}} = -n^2 \int_0^h a^{22} C_{2222} (\mu_2^2)^3 h_2^2 \mu \, \mathrm{d}x_3,$$

$$M_B^{(3)^{22}} = -n \int_0^h \left[\frac{C_{1122}}{\alpha_2} (x_3\lambda_1^2 - b_{11}) + C_{2222}(a^{22} x_3 n^2 - b_{22}) \mu_2^2 \right] \mu_2^2 h_2^2 \mu \, \mathrm{d}x_3,$$

$$M_{C_1}^{(3)^{22}} = \mathrm{i}\lambda_1 n \int_0^h \frac{C_{1122}}{\alpha_2} h_1^1 h_2^2 \mu_2^2 \mu \, \mathrm{d}x_3, \qquad M_{C_2}^{(3)^{22}} = -n^2 \int_0^h a^{22} C_{2222} (h_2^2)^2 \mu_2^2 \mu \, \mathrm{d}x_3,$$

$$N_{C_1}^{(3)^1} = \int_0^h C_{1313} (h_{1,3}^1 + b_1^1 h_1^1)^2 \mu \, \mathrm{d}x_3, \qquad N_{C_2}^{(3)^2} = \int_0^h C_{2323} (\mu_2^2 h_{2,3}^2 + b_2^2 h_2^2)^2 \mu \, \mathrm{d}x_3$$