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LETTERS TO THE EDITOR

SUPPRESSION OF VIBRATION IN THE AXIALLY MOVING KIRCHHOFF STRING BY BOUNDARY CONTROL

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1. introduction

Axially moving string-like continua such as threads, wires, magnetic tapes, belts, band-saws, chains, and cables have been subjects of the study of researchers in recent years; see survey papers [1–3] for extensive lists of references. Researchers have derived and studied different linear and non-linear mathematical models which describe the dynamics of such systems; see, e.g., references [4–28]. Recently, the important problem of designing stabilizing controllers to suppress the vibration of axially moving string-like continua has received attention by researchers; see, e.g., references [29–35]. Controllers in these references, except those in references [32] and [35], are designed for the linear models of axially moving strings. One way to describe the dynamics of an axially moving string is to model it as the moving Kirchhoff string; see, e.g., references [9] and [26]. The axially moving Kirchhoff string is represented by a non-linear partial differential equation. Our goal in this note is to show that the linear boundary control is a stabilizing controller for the axially moving Kirchhoff string. We achieve this goal by using an approach analogous to that in reference [35]. To the best of our knowledge, this note is the first to present the application of the boundary control to the moving Kirchhoff string.

We consider the axially moving string in Figure 1. The string is pulled at a constant speed through two eyelets which are distanced from each other by 1. One of the eyelets is fixed and the other one can move freely in the direction of the Y-axis. A control input force, denoted by u in Figure 1, can be applied to the free-to-move eyelet transversally. By transversal we mean in the direction of Y.

The dynamics of the string in Figure 1 can be represented by the following nonlinear partial differential equation (see, e.g., references [9] and [26]):

$$y_{tt}(x,t) + 2v y_{xt}(x,t) = \left(1 - v^2 + b\int_0^1 y_x^2(x,t)\,\mathrm{d}x\right) y_{xx}(x,t), \tag{1a}$$

for all $x \in (0, 1)$ and $t \geqslant 0$. In equation (1a), $y(\cdot,\cdot) \in \mathbb{R}$ denotes the transversal displacement of the string, $y_x := \partial y/\partial x$, $y_{xx} := \partial^2 y/\partial x^2$, $y_{tt} := \partial^2 y/\partial t^2$, $y_{xt} := \partial^2 y/\partial x\,\partial t$, and $b > 0$ is a constant real number, and $v \geqslant 0$ is proportional to the speed of the string trhough the eyelets. In realistic physical situations, $v < 1$.

The tension in the string represented by equation (1a) is *not* constant and is given by

$$T(t) = 1 + b\int_0^1 y_x^2(x,t)\,\mathrm{d}x,$$

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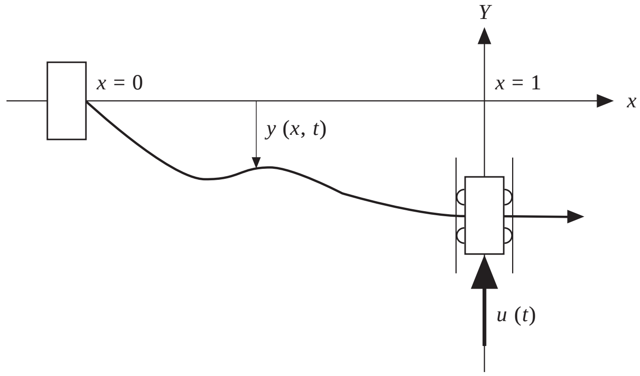

Figure 1. The string is pulled at a constant speed through two eyelets. The eyelet at $x = 0$ is fixed and the one at $x = 1$ can move freely in the direction of the axis Y. The control input force $u(t) = -ky_t(1, t)$ for all $t \geqslant 0$, where $k > 0$ is a constant real number, is applied to the free-to-move eyelet in the direction of Y.

for all $t \geqslant 0$ (see reference [36]). Having the tension T, we have the following boundary conditions:

$$y(0, t) = 0, \qquad \left(1 - v^2 + b\int_0^1 y_x^2(x, t)\,\mathrm{d}x\right) y_x(1, t) = u(t), \tag{1b, c}$$

for all $t \geqslant 0$. The boundary condition in equation (1b) states that the string is fixed at $x = 0$. The boundary condition in equation (1c) represents the balance of forces applied to the string at $x = 1$ in the direction of Y.

The initial displacement and velocity of the string are, respectively,

$$y(x, 0) = f(x), \qquad y_t(x, 0) = g(x), \tag{1d}$$

for all $x \in (0, 1)$, where $y_t := \partial y/\partial t$. We assume that $f \in C^1[0, 1]$, and that at least one of the functions f or g is not identically zero over [0, 1].

When the string does not move ($v = 0$), the system (1) represents the dynamics of a string known as the Kirchhoff string, which was originally studied by Kirchhoff in reference [37]. The Kirchhoff string has been studied by many researchers from the physical and mathematical points of view; see, e.g., references [38, pp. 220–254], [39], and [40] for extensive lists of references.

The control input u in equation (1c) is commonly known as the *boundary control*. In this note, we study the stabilization of the string in equation (1a) by u. More precisely, we study a u that results in $y(x, t) \to 0$ as $t \to \infty$ for all $x \in [0, 1]$. As a stabilizing control input, the following is proposed:

$$u(t) = -ky_t(1, t), \tag{2}$$

for all $t \geqslant 0$, where $k > 0$ is a constant real number. With this choice of u, the boundary control is the negative feedback of the transversal velocity of the string at $x = 1$, with the gain k. It is known that fixed *linear* strings represented by equation (1), in whcih $v = 0$ and $b = 0$, can be stabilized by the control law in equation (2); see, e.g., references [41–47]. Also, it is known that axially moving *linear* strings represented by equation (1), in which $v > 0$ and $b = 0$, can be stabilized by the control law in equation (2); see references [31] and [33]. Roughly speaking, the boundary control in equation (2) provides a dissipative effect in linear strings, because it is of the form of negative velocity feedback. This is in accordance with the well known fact that the negative velocity feedback increases damping in most finite dimensional inertial systems, such as, large flexible systems and robotic manipulators.

Our goal in this note is to show that the boundary control u in equation (2) stabilizes the non-linear axially moving non-linear string in equation (1), i.e., u results in $y(x, t)\to 0$ as $t\to\infty$ for $x\in[0, 1]$.

2. STABILIZATION BY BOUNDARY CONTROL

Our plan to establish the stability of the non-linear string represented by equations (1) and (2) is as follows. We define an energy like (Lyapunov) function of time for the string and denote it by $t\mapsto V(t)$. We show that V tends to zero exponentially.

The scalar-valued function V is defined as

$$V(t):=E(t)+\gamma\int_0^1[xy_t(x,t)y_x(x,t)+vxy_x^2(x,t)]\,\mathrm{d}x, \tag{3}$$

for all $t\geqslant 0$, where γ is a constant real number satisfying

$$0<\gamma<\min\left\{\frac{1-v^2}{1+2v},\frac{2(1-v^2)(k+v)}{1-v^2+k^2}\right\}, \tag{4}$$

and

$$E(t):=\frac{1}{2}\int_0^1[y_t^2(x,t)+(1-v^2)y_x^2(x,t)]\,\mathrm{d}x+\frac{b}{4}\left(\int_0^1 y_x^2(x,t)\,\mathrm{d}x\right)^2, \tag{5}$$

and $y(\cdot,\cdot)$ satisfies equations (1) and (2). From equations (3), (5), and (1d), we obtain

$$E(0)=\frac{1}{2}\int_0^1[g^2(x)+(1-v^2)f_x^2(x)]\,\mathrm{d}x+\frac{b}{4}\left(\int_0^1 f_x^2(x)\,\mathrm{d}x\right)^2, \tag{6a}$$

$$V(0)=E(0)+\gamma\int_0^1[xg(x)f_x(x)+vxf_x^2(x)]\,\mathrm{d}x, \tag{6b}$$

where $f_x(x):=\mathrm{d}f(x)/\mathrm{d}x$. Recall that at least one of the functions f or g is not identically equal to zero over [0, 1]. Furthermore, the function f, for which $f(0)=0$ by equation (1b), cannot assume a non-zero constant value over [0, 1]. Thus, $E(0)>0$.

Now, we prove a property of V.

Lemma 2.1. The function V satisfies

$$0\leqslant K_1E(t)\leqslant V(t)\leqslant K_2E(t), \tag{7}$$

for all $t\geqslant 0$, where $K_1>0$ and $K_2>0$ are constant real numbers given by

$$K_1=1-\gamma(1+2v)/(1-v^2),\qquad K_2=1+\gamma(1+2v)/(1-v^2). \tag{8a, b}$$

Proof. For the integral terms in equation (3), whose coefficient is γ, we have (the argument (x, t) of the functions is deleted)

$$\int_0^1 xy_t y_x \,\mathrm{d}x \leqslant \int_0^1 x|y_t|\,|y_x|\,\mathrm{d}x \leqslant \frac{1}{2}\int_0^1 y_t^2\,\mathrm{d}x + \frac{1}{2}\int_0^1 y_x^2\,\mathrm{d}x, \tag{9a}$$

$$\int_0^1 vxy_x^2\,\mathrm{d}x \leqslant v\int_0^1 y_x^2\,\mathrm{d}x, \tag{9b}$$

for all $t \geqslant 0$. Adding equations (9a) and (9b), we obtain

$$\int_0^1 (xy_t y_x + vxy_x^2)\,\mathrm{d}x \leqslant \frac{1}{2}\int_0^1 y_t^2\,\mathrm{d}x + \frac{1+2v}{2(1-v^2)}\int_0^1 (1-v^2)y_x^2\,\mathrm{d}x, \tag{10}$$

for all $t \geqslant 0$. Since

$$(1+2v)/(1-v^2) \geqslant 1, \tag{11}$$

for all $0 \leqslant v < 1$, we conclude that

$$\int_0^1 (xy_t y_x + vxy_x^2)\,\mathrm{d}x \leqslant \frac{1+2v}{1-v^2}\left(\frac{1}{2}\int_0^1 [y_t^2 + (1-v^2)y_x^2]\,\mathrm{d}x\right) \leqslant \frac{1+2v}{1-v^2}E(t), \tag{12a}$$

for all $t \geqslant 0$. Similarly, we obtain

$$\int_0^1 (xy_t y_x + vxy_x^2)\,\mathrm{d}x \geqslant -\frac{1+2v}{1-v^2}E(t), \tag{12b}$$

for all $t \geqslant 0$. Using inequalities (12) in equation (3), we obtain inequality (7). Note that $\gamma < (1-v^2)/(1+2v)$ by inequality (4). Therefore, K_1 and K_2 in equation (8) are positive real numbers. □

Remarks. (1) Since $(1+2v)/(1-v^2) \geqslant 1$ for all $0 \leqslant v < 1$, then γ in inequality (4) is at least less than 1.

(2) By inequality (7) and the fact that $E(0) > 0$, it is concluded that $V(0) > 0$. □

Next, we use equation (2) in equation (1c) and rewrites the boundary conditions as

$$y(0, t) = 0, \qquad y_x(1, t) = -ky_t(1, t)\Big/\left(1 - v^2 + b\int_0^1 y_x^2(x, t)\,\mathrm{d}x\right), \tag{13a, b}$$

for all $t \geqslant 0$. We now prove some identities for the functions satisfying equation (13).

Lemma 2.2. Let $y(\cdot, \cdot)$ satisfy the boundary conditions in equation (13). Then,

$$2\int_0^1 y_{xt}y_t\,\mathrm{d}x = y_t^2(1, t), \tag{14a}$$

$$\int_0^1 (y_{xx}y_t + y_{xt}y_x)\,\mathrm{d}x = -ky_t^2(1, t)\Big/\left(1 - v^2 + b\int_0^1 y_x^2\,\mathrm{d}x\right), \tag{14b}$$

$$\int_0^1 xy_{xt}y_t\,\mathrm{d}x = \frac{1}{2}y_t^2(1,t) - \frac{1}{2}\int_0^1 y_t^2\,\mathrm{d}x, \tag{14c}$$

$$\int_0^1 xy_{xx}y_x\,\mathrm{d}x = k^2y_t^2(1,t)\Big/2\left(1-v^2+b\int_0^1 y_x^2\,\mathrm{d}x\right)^2 - \frac{1}{2}\int_0^1 y_x^2\,\mathrm{d}x, \tag{14d}$$

for all $t \geqslant 0$.

Proof. From equation (13a), we have $y_t(0,t)=0$ for all $t \geqslant 0$. Thus, we obtain

$$2\int_0^1 y_{xt}y_t\,\mathrm{d}x = \int_0^1 (y_t^2)_x\,\mathrm{d}x = y_t^2(1,t), \tag{15}$$

for all $t \geqslant 0$. That is, equation (14a) holds.

Having $y_t(0,t)=0$ for all $t \geqslant 0$, we next obtain

$$\int_0^1 (y_{xx}y_t + y_{xt}y_x)\,\mathrm{d}x = \int_0^1 (y_xy_t)_x\,\mathrm{d}x = y_x(1,t)y_t(1,t), \tag{16}$$

for all $t \geqslant 0$. Using equation (13b) in equation (16), we obtain equation (14b).

Next, we write

$$\int_0^1 xy_{xt}y_t\,\mathrm{d}x = \frac{1}{2}\int_0^1 (xy_t^2)_x\,\mathrm{d}x - \frac{1}{2}\int_0^1 y_t^2\,\mathrm{d}x, \tag{17}$$

for all $t \geqslant 0$. Thus, equation (14c) follows.

Finally, we write

$$\int_0^1 xy_{xx}y_x\,\mathrm{d}x = \frac{1}{2}\int_0^1 (xy_x^2)_x\,\mathrm{d}x - \frac{1}{2}\int_0^1 y_x^2\,\mathrm{d}x = \frac{1}{2}y_x^2(1,t) - \frac{1}{2}\int_0^1 y_x^2\,\mathrm{d}x, \tag{18}$$

for all $t \geqslant 0$. Using equation (13b) in equation (18), we obtain equation (14d). □

Next, we compute the time-derivative of the function E.

Lemma 2.3. The time-derivative of the function E in equation (5), along the solution of the system (1a), (1d), and (13) (equivalently, the system (1) and (2)) satisfies

$$\dot{E}(t) = -(k+v)y_t^2(1,t) \leqslant 0, \tag{19}$$

for all $t \geqslant 0$.

Proof. From equation (5), we obtain

$$\dot{E}(t) = \int_0^1 [y_{tt}y_t + (1-v^2)y_{xt}y_x]\,\mathrm{d}x + b\int_0^1 y_x^2\,\mathrm{d}x\int_0^1 y_{xt}y_x\,\mathrm{d}x, \tag{20}$$

for all $t \geqslant 0$. Substituting y_{tt} from equation (1a) into equation (20), we obtain

$$\dot{E}(t) = -2v\int_0^1 y_{xt}y_t\,\mathrm{d}x + \left(1-v^2+b\int_0^1 y_x^2\,\mathrm{d}x\right)\int_0^1 (y_{xx}y_t + y_{xt}y_x)\,\mathrm{d}x, \tag{21}$$

for all $t \geqslant 0$. Using equations (14a) and (14b) in equation (21), we obtain inequality (19). □

Using the preliminary results obtained thus far, we next prove that the functions V and E tend to zero exponentially.

Theorem 2.4. The functions V and E, along the solution of the system (1a), (1d), and (13) (equivalently, the system (1) and (2)) satisfy

$$0 \leqslant V(t) \leqslant V(0)\,\mathrm{e}^{-\gamma t/K_2}, \qquad 0 \leqslant E(t) \leqslant (V(0)/K_1)\,\mathrm{e}^{-\gamma t/K_2}, \tag{22a, b}$$

for all $t \geqslant 0$, where K_1 and K_2 are given in equation (8).

Proof. From equation (3) we obtain

$$\dot{V}(t) = \dot{E}(t) + \gamma \int_0^1 (xy_{tt}y_x + xy_t y_{xt} + 2vxy_{xt}y_x)\,\mathrm{d}x, \tag{23}$$

for all $t \geqslant 0$. Substituting y_{tt} from equation (1a) into equation (23), we obtain

$$\dot{V}(t) = \dot{E}(t) + \gamma \int_0^1 xy_{xt}y_t\,\mathrm{d}x + \gamma\left(1 - v^2 + b\int_0^1 y_x^2\,\mathrm{d}x\right)\int_0^1 xy_{xx}y_x\,\mathrm{d}x, \tag{24}$$

for all $t \geqslant 0$. Using equations (19), (14c), and (14d) in equation (24), we obtain

$$\dot{V}(t) = -\gamma E(t) - (k+v)y_t^2(1,t) - \frac{\gamma b}{4}\left(\int_0^1 y_x^2\,\mathrm{d}x\right)^2 + \frac{\gamma}{2}y_t^2(1,t) + \gamma k^2 y_t^2(1,t)\Big/2\left(1 - v^2 + b\int_0^1 y_x^2\,\mathrm{d}x\right), \tag{25}$$

for all $t \geqslant 0$. Neglecting the third term of equation (25) and $\int_0^1 y_x^2\,\mathrm{d}x$ in the last term of this equation, we obtain

$$\dot{V}(t) \leqslant -\gamma E(t) - (k+v)y_t^2(1,t) + \gamma y_t^2(1,t)/2 + \gamma k^2 y_t^2(1,t)/2(1-v^2), \tag{26}$$

for all $t \geqslant 0$. Therefore,

$$\dot{V}(t) \leqslant -\gamma E(t) - F(t), \tag{27}$$

for all $t \geqslant 0$, where

$$F(t) := [(k+v) - \gamma(1 - v^2 + k^2)/2(1-v^2)]y_t^2(1,t). \tag{28}$$

Having γ satisfying inequality (4), we conclude that the coefficient of $y_t^2(1,\cdot)$ in equation (28) is positive, and hence $F(t) \geqslant 0$ for all $t \geqslant 0$. Using the non-negativeness of F in inequality (27), we obtain

$$\dot{V}(t) \leqslant -\gamma E(t), \tag{29}$$

for all $t \geqslant 0$. Using inequality (7) in inequality (29), we obtain the following differential inequality:

$$\dot{V}(t) \leqslant -(\gamma/K_2)V(t), \tag{30}$$

for all $t \geqslant 0$, with the initial condition $V(0) > 0$ given in equation (6b). By a comparison theorem given in reference [48, p. 3] or reference [49, p. 2], we conclude that V in inequality (30) satisfies $V(t) \leqslant V(0)\,\mathrm{e}^{-\gamma t/K_2}$ for all $t \geqslant 0$. Note that by inequality (7), we have

$V(t) \geqslant 0$ for all $t \geqslant 0$. Thus, inequality (22a) holds. By inequalities (7) and (22a), we conclude that inequality (22b) holds. □

Finally, we show that the boundary control u in equation (2) stabilizes the non-linear string in equation (1).

Corollary 2.5. The solution of the system (1a), (1d), and (13) (equivalently, the system (1) and (2)), $y(x, t) \to 0$ as $t \to \infty$ for all $x \in [0, 1]$.

Proof. For the system (1a), (1d), and (13) we choose the Lyapunov function V in equation (3). Then, by Theorem 2.4, the function E tends to zero exponentially. From equation (5), we conclude that $y_x(x, t) \to 0$ as $t \to \infty$ for all $x \in [0, 1]$. Since, $y(0, t) = 0$ for all $t \geqslant 0$, we conclude that $y(x, t) \to 0$ as $t \to \infty$ for all $x \in [0, 1]$. □

3. CONCLUSION

In this note, the Lyapunov technique has been used to prove that the non-linear axially moving Kirchhoff string represented by equation (1) can be stabilized by the linear boundary control in equation (2). The boundary control is the negative feedback of the transversal velocity of the string at one end.

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