Journal of Sound and Vibration (1998) **214**(4), 589–603
Article No. sv971468

RESPONSE OF NON-LINEAR DISSIPATIVE SHOCK ISOLATORS

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(*Received* 13 *September* 1996, *and in final form* 3 *December* 1997)

In this paper, a simple technique combining the straightforward perturbation method with Laplace transform has been developed to determine the transient response of a single degree-of-freedom system in the presence of non-linear, dissipative shock isolators. Analytical results are compared with those obtained by numerical integration using the classical Runge–Kutta method. Three types of input base excitations, namely, the rounded step, the rounded pulse and the oscillatory step are considered. The effects of nonlinear damping on the response are discussed in detail. Both the positive and negative coefficients of the nonlinear damping term have been considered. It has been shown that a critical value of the positive coefficient maximizes the peak values of relative and absolute displacements. This is true for any power-law damping force with an index greater than 1. On the other hand, the overall performance of a shock isolator improves if the nonlinear damping term is symmetric and quadratic with a negative coefficient.

1. INTRODUCTION

Non-linearity is ubiquitous in nature. Linearity is an approximation to reality. In shock and vibration systems, isolators such as air springs, elastomeric dampers, and wire-rope isolators are inherently nonlinear. Assumption of Hooke's law for springs and linear viscous damping for dampers is done just for mathematical simplicity. Sometimes the amplitudes of steady-state vibration are small enough to justify the assumption of linearity. However, the transient displacements may often be sufficiently large when the nonlinearity in springs and dampers cannot be ignored.

Ravindra and Mallik [1–3] have studied the effect of nonlinear damping on the performance of vibration isolators under harmonic loading. They observed bifurcations, chaos and strange attractors due to the presence of nonlinearity in springs and dampers in vibration isolators. They concluded that a strictly dissipative nonlinear damping may be used as a passive control strategy to suppress various instabilities occurring in nonlinear vibration isolation systems. The major objective of the present study is to ascertain the effect of nonlinear damping on the response of shock isolators.

Linear shock isolation problems are discussed in several books [4–7]. Snowdon [7] presented the response of nonlinear shock isolator modelled as a nonlinear elastic (tangent and inverse tangent elasticity) spring parallel to a viscous damper. He concluded that a soft spring (inverse tangent elasticity) performs better than a hard spring. In [8], Snowdon compared the performance of a dual-phase damper mounting system with that of a linear, simple mounting system. Guntur and Sankar [9] reported the performance of different kinds of dual-phase damping shock mounts. Hundal [10] reviewed the literature on pneumatic shock absorbers and isolators.

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Hundal [11] has also compared the performances of shock isolators with linear and quadratic damping with a base input in the form of an acceleration pulse of rectangular shape. By using the concept of a variable friction force, Mercer and Rees [12] proposed a new form of shock isolator which is adaptive in its action but is still composed of entirely passive elements.

Various analytical methods are available in the literature [13–20] for transient analysis of non-linear systems, namely, Linearization method [14], Ultraspherical polynomial method [15–17], and Lighthill's extension of Poincare's perturbation method [18]. In this paper, a simple straightforward perturbation method along with Laplace transform is used. This method is applicable for any order of non-linearity, both in the restoring and damping forces, expressed in the form of polynomials. A cubic non-linearity in the restoring force is assumed, whereas various forms of nonlinearities so far as the damping force is concerned have been included. For example, a cubic type non-linearity over and above the common linear viscous damping, as exhibited by fluid-dampers at high velocities, has been given special attention. Numerical results are included for a typical elastomeric damper which can be modelled by a combination of linear viscous damping and a dissipative quadratic damping with a negative coefficient.

Three types of base excitations, namely, the rounded step, the rounded pulse and the oscillatory step given in [8] are considered. Analytical results obtained by the present method are compared with those obtained by direct numerical integration. For the quadratic damping case, results are obtained by numerical integration alone since the symmetric dissipative quadratic damping cannot be expressed in polynomial form. The effects of nonlinear damping on the response of various models of shock isolators are discussed in detail.

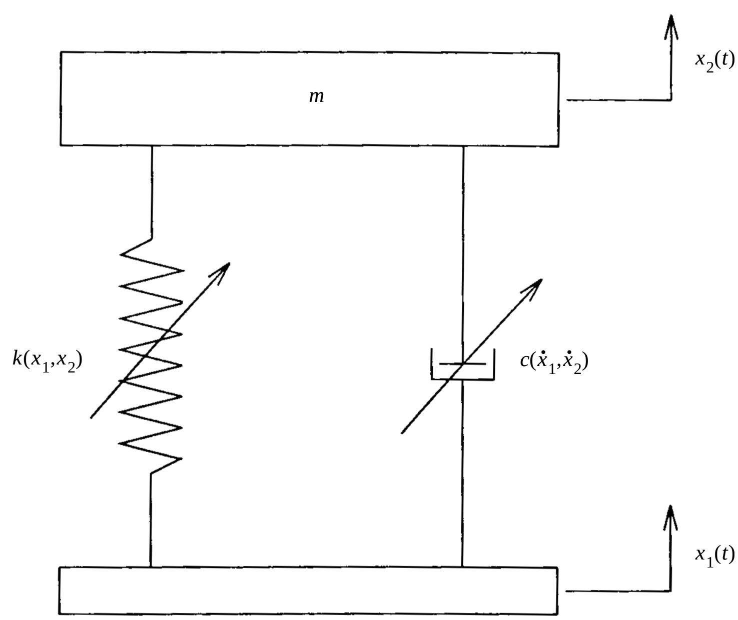

Figure 1. Non-linear base excited system.

2. EQUATIONS OF MOTION

A single degree-of-freedom shock isolator system is shown in Figure 1, where the base is subjected to a shock displacement. The spring and the damper elements of the isolator are taken to be nonlinear with a cubic nonlinearity superimposed on a linear term. The equation of motion for the mass m, to be isolated, is

$$m\ddot{x}_2 + c_0(\dot{x}_2 - \dot{x}_1) + c_1(\dot{x}_2 - \dot{x}_1)^3 + k_0(x_2 - x_1) + k_1(x_2 - x_1)^3 = 0 \tag{1}$$

where x_1 and x_2 are the absolute displacements of the base and the mass m, respectively, and the dots denote derivatives with respect to time t. The initial conditions on x_2 are taken to be zero, i.e., $x_2 = \dot{x}_2 = 0$ at $t = 0$. This equation is written in the non-dimensional form as

$$\Delta'' + 2\zeta\Delta' + \delta^*(\Delta')^3 + \Delta + \varepsilon^*\Delta^3 = f(T) \tag{2}$$

where $\Delta = (x_2 - x_1)/x_{1max}$ is the relative displacement, between the mass m and the base, non-dimensionalized with respect to the maximum base displacement x_{1max}, the non-dimensionalized time $T = \omega_0 t$ with $\omega_0 = \sqrt{k_0/m}$, the primes denote derivatives with respect to T, and the other non-dimensional parameters are

$$\zeta = \frac{c_0}{2m\omega_0}, \qquad \delta^* = \frac{c_1\,\omega_0\,(x_{1max})^2}{m}, \qquad \varepsilon^* = \frac{k_1\,(x_{1max})^2}{k_0}, \qquad \text{and} \quad f(T) = -\frac{x_1''}{x_{1max}}.$$

The three commonly considered shock displacements of the base [7, 8], taken as input $x_1(t)$ in equation (1), are shown in Figure 2 and are expressed in functional forms as follows:

Case (*a*) *Rounded displacement step*:

$$x_1(t) = x_{1max}\,[1 - (1 + \gamma\omega_0 t)\,\mathrm{e}^{(-\gamma\omega_0 t)}] \tag{3}$$

for $t \geqslant 0$
with γ as the severity parameter. The nonhomogeneous term in equation (2) is then

$$f(T) = -\gamma^2(1 - \gamma T)\,\mathrm{e}^{(-\gamma T)} \tag{4}$$

for $T \geqslant 0$.

Case (*b*) *Unidirectional rounded displacement pulse*:

$$x_1(t) = x_{1max}\,(\mathrm{e}^2/4)\,(\gamma\omega_0 t)^2\,\mathrm{e}^{(-\gamma\omega_0 t)} \tag{5}$$

for $t \geqslant 0$.

Therefore $f(T)$ in equation (2) is

$$f(T) = -\frac{\mathrm{e}^2\gamma^2}{4}(2 - 4\gamma T + \gamma^2T^2)\,\mathrm{e}^{(-\gamma T)} \tag{6}$$

for $T \geqslant 0$.

The general form of forcing functions in equations (4) and (6) may be written as

$$f(T) = \left(\sum_{j=0}^{N} A_j\,T^j\right)\mathrm{e}^{aT}. \tag{7}$$

Case (*c*) *Oscillatory displacement step*:

$$x_1(t) = x_{1max}\,(0{\cdot}68684)\,[1 - \{\cos(\gamma\omega_0 t) + 0{\cdot}25\sin(\gamma\omega_0 t)\}\,\mathrm{e}^{(-0{\cdot}25\gamma\omega_0 t)}] \tag{8}$$

for $t \geqslant 0$.

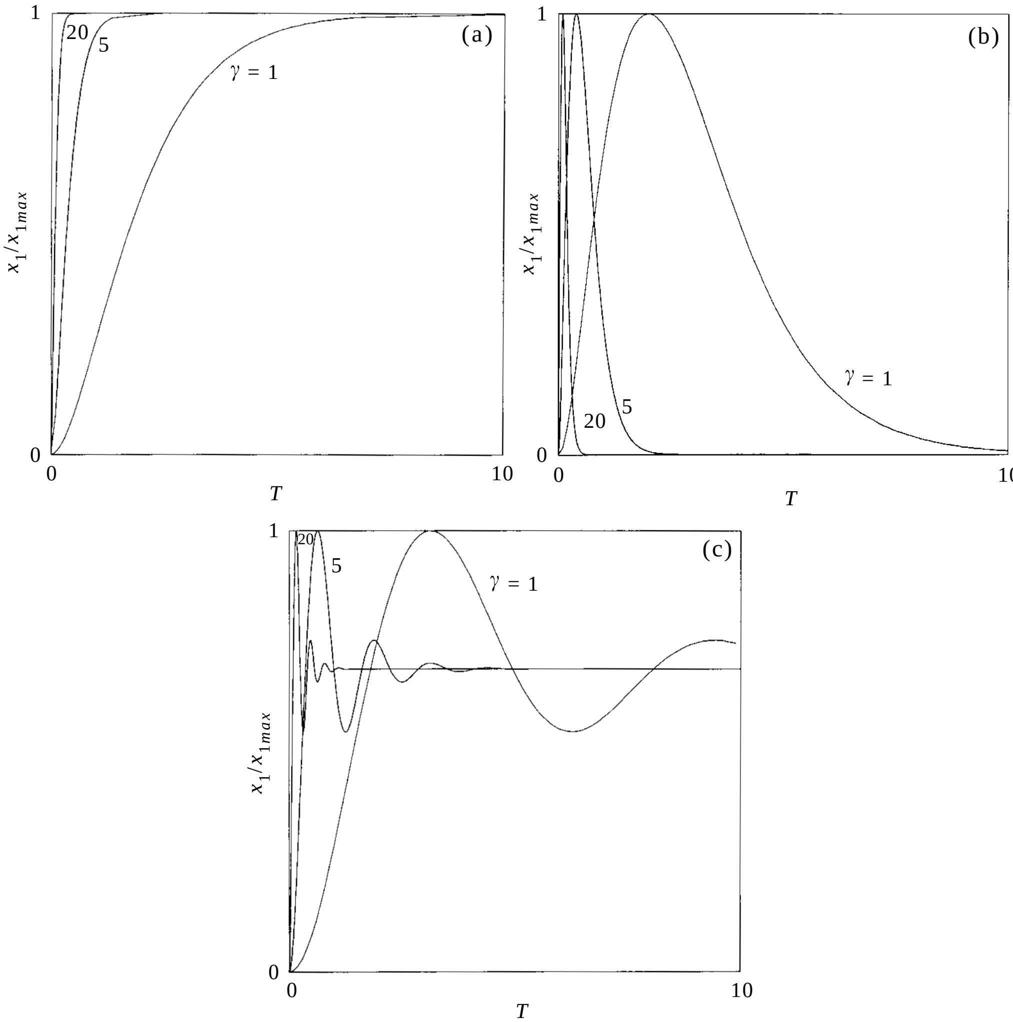

Figure 2. Different forms of base excited functions. (a) The rounded step displacement; (b) the rounded pulse displacement; and (c) the oscillatory displacement step.

The corresponding input function in equation (2) is

$$f(T) = -(17/16)\,(0{\cdot}68684)\gamma^2[\cos(\gamma T) - 0{\cdot}25\sin(\gamma T)]\,\mathrm{e}^{(-0{\cdot}25\gamma T)} \tag{9}$$

for $T \geqslant 0$.

In a general form, equation (9) may be rewritten as

$$f(T) = \left[\sum_{j=0}^{N}\{C_j\cos(\omega_j T) + D_j\sin(\omega_j T)\}\right]\mathrm{e}^{aT} \tag{10}$$

3. CLOSED FORM SOLUTIONS

It is desirable to have an analytical solution of equation (2) with $f(T)$ given by either equation (7) or (10). This would facilitate calculating transients at any given instant directly from the closed form solution rather than integrating numerically equation (2) from $T = 0$. To this end, the perturbation method [18–26] has been used to separate the terms of the same order and then the solutions of differential equations are obtained through Laplace transformation.

The parameters δ^* and ϵ^* in equation (2) are taken to be small and of the same order, when equation (2) can be rewritten as

$$\Delta'' + 2\zeta\Delta' + \Delta + \epsilon(\delta_1 (\Delta')^3 + \epsilon_1 \Delta^3) = f(T) \tag{11}$$

where, by putting ϵ_1 and δ_1 equal to zero or one, the effect of non-linearity, in only stiffness or in only damping or in both stiffness and damping, can be studied. The solution of equation (11) is perturbed in small parameter $\epsilon(\ll 1)$ as

$$\Delta(T) = \sum_{i=0}^{\infty} \epsilon^i \Delta_i (T). \tag{12}$$

Substitution of equation (12) in equation (11) and collection of like power terms of ϵ yields

$$\Delta_0'' + 2\zeta\Delta_0' + \Delta_0 = f(T) \tag{13}$$

and

$$\Delta_i'' + 2\zeta\Delta_i' + \Delta_i = -\epsilon_1 \left[\sum_{j=0}^{i-1} \sum_{n=0}^{j} \Delta_{i-j-1} \Delta_{j-n} \Delta_n \right] - \delta_1 \left[\sum_{j=0}^{i-1} \sum_{n=0}^{j} \Delta'_{i-j-1} \Delta'_{j-n} \Delta'_n \right] \tag{14}$$

for $i \geqslant 1$, with $\Delta_i = \Delta_i' = 0$ at $T = 0 \forall i > 0$.

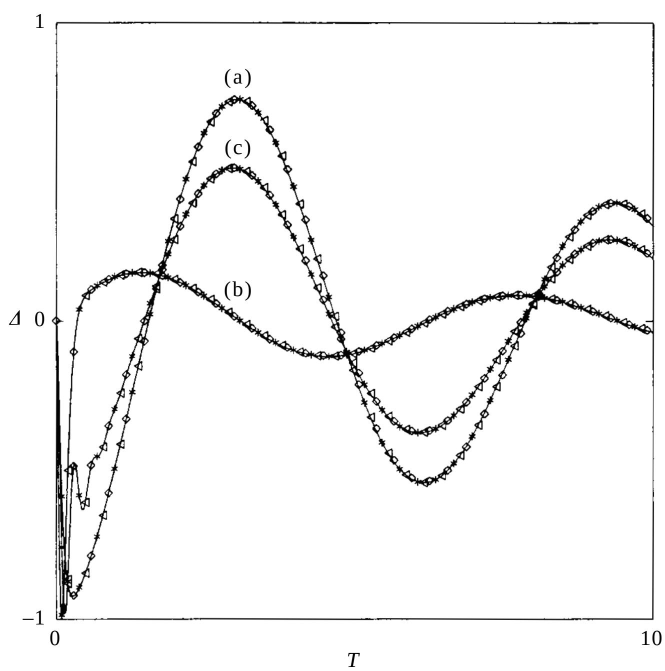

Figure 3. Comparison of analytical and numerical results for the cubic non-linearity in stiffness only. $\zeta = 0.1$, $\gamma = 20$, $\varepsilon = 0.01$, $\varepsilon_1 = 1$, $\delta_1 = 0$. $\Diamond$, for linear; *, for I order; $\triangleright$, for II order; —— for numerical. (a) The rounded step displacement; (b) the rounded pulse displacement; and (c) the oscillatory displacement step.

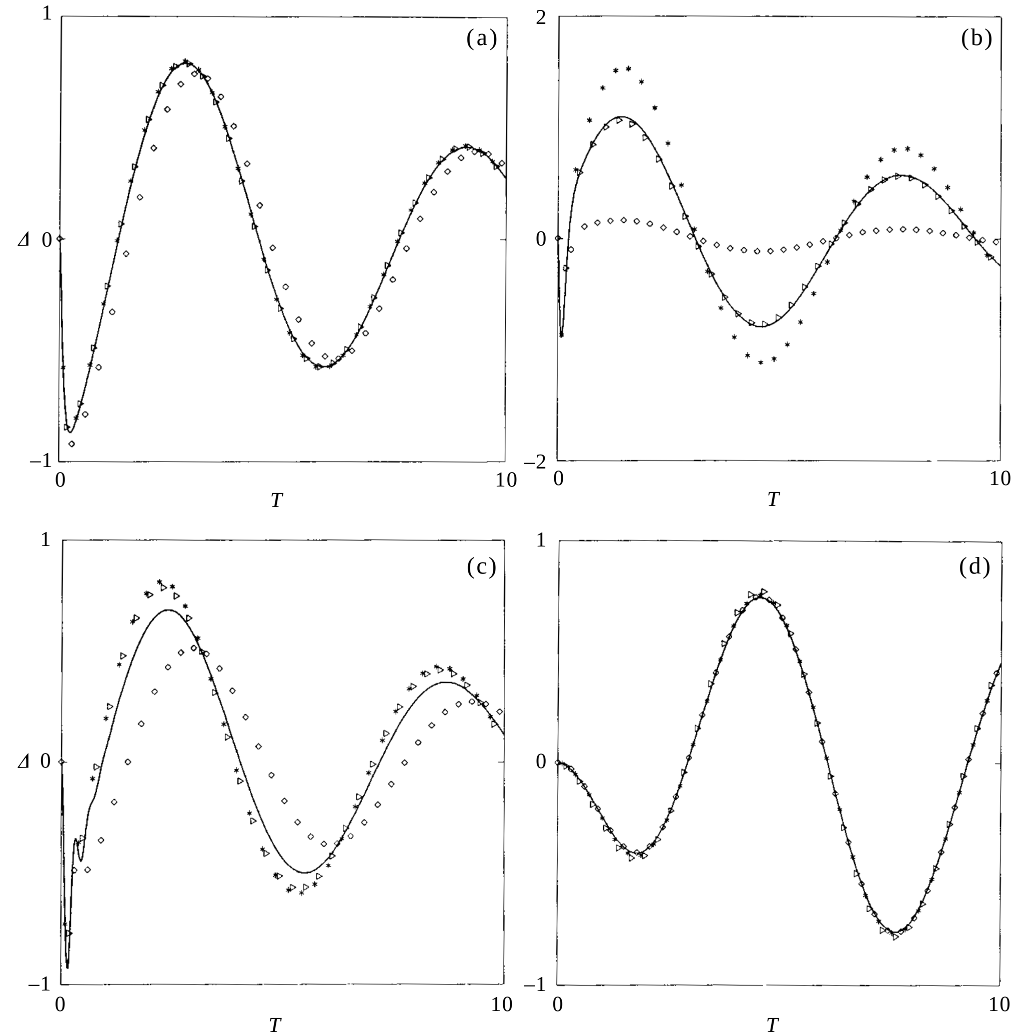

Figure 4. Comparison of analytical and numerical results for the cubic nonlinearity in damping only. $\zeta = 0.1$, $\varepsilon = 0.01$, $\varepsilon_1 = 0$, $\delta_1 = 1$. $\Diamond$, for linear; *, for I order; $\triangleright$, for II order; —— for numerical. (a) The rounded step displacement for $\gamma = 20$; (b) the rounded pulse displacement for $\gamma = 20$; (c) the oscillatory displacement step for $\gamma = 20$; and (d) the oscillatory displacement step for $\gamma = 1$.

The series in equation (12) is a Cauchy product

$$\sum_{i=0}^{\infty} \epsilon^i \text{ and } \sum_{i=0}^{\infty} \Delta_i(T). \text{ For } \epsilon \ll 1, \sum_{i=0}^{\infty} \epsilon^i$$

is a convergent geometric series and

$$\sum_{i=0}^{\infty} \Delta_i(T)$$

is bounded for $\zeta > 0$, $\gamma > 0$ since the solution of equations (13) and (14) are bounded. Therefore, the series

$$\sum_{i=0}^{\infty} \epsilon^i \Delta_i(T)$$

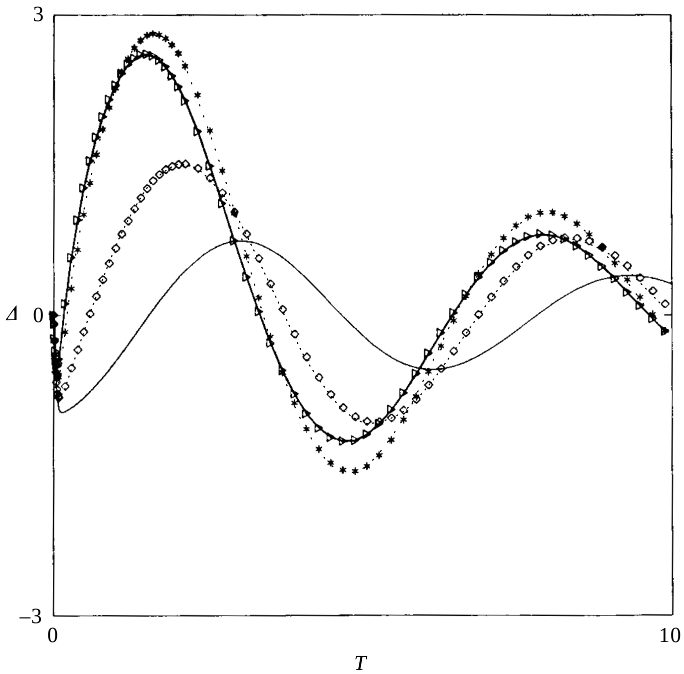

Figure 5. The effect of nonlinear cubic damping on the relative displacement for the rounded displacement step. $\gamma = 50$, $\zeta = 0.1$, $\varepsilon^* = 0$. ——, $\delta^* = 0$; $\diamondsuit$, $\delta^* = 0.01$; *, $\delta^* = 0.05$; $\triangleright$, $\delta^* = 0.1$.

is convergent [27]. It is to be noted here that the damping term is retained in equation (13) in order to avoid secular terms and thereby ensuring bounded solutions [20].

The focus here is on transients alone and over a short time duration when the solution of equation (11) were observed to give good convergence with only two terms [i.e., $i = 2$ in equation (12)].

3.1. ZEROTH ORDER SOLUTION

With the initial conditions $\Delta = \Delta' = 0$ at $T = 0$, the solutions of equation (13), for the three forcing functions in equations (4) (6) and (9), are obtained using Laplace transformation and are listed below:

Cases (a) and (b): corresponding to the generalized forcing function of equation (7),

$$\Delta_0(T) = \mathrm{e}^{(-\zeta T)}\,[P_1 \cos(\beta T) + Q_1 \sin(\beta T)]/\beta + \mathrm{e}^{(aT)}\left[\sum_{j=0}^{N} (B_{j+1}/j!)T^j\right] \quad (15)$$

where $\beta = \sqrt{(1-\zeta^2)}$. The other terms, namely P_1, Q_1 and B_{j+1}, are given in case (i) of the Appendix.

Case (c): the solution of equation (13) with $f(T)$ given by equation (10) is

$$\Delta_0(T) = \mathrm{e}^{(-\zeta T)}\,[P_2 \cos(\beta T) + Q_2 \sin(\beta T)]/\beta + \mathrm{e}^{aT}\left[\sum_{j=0}^{N} (R_j \cos(\omega_j T) + S_j \sin(\omega_j T))\right] \quad (16)$$

where P_2, Q_2, R_j and S_j are given in case (ii) of the Appendix.

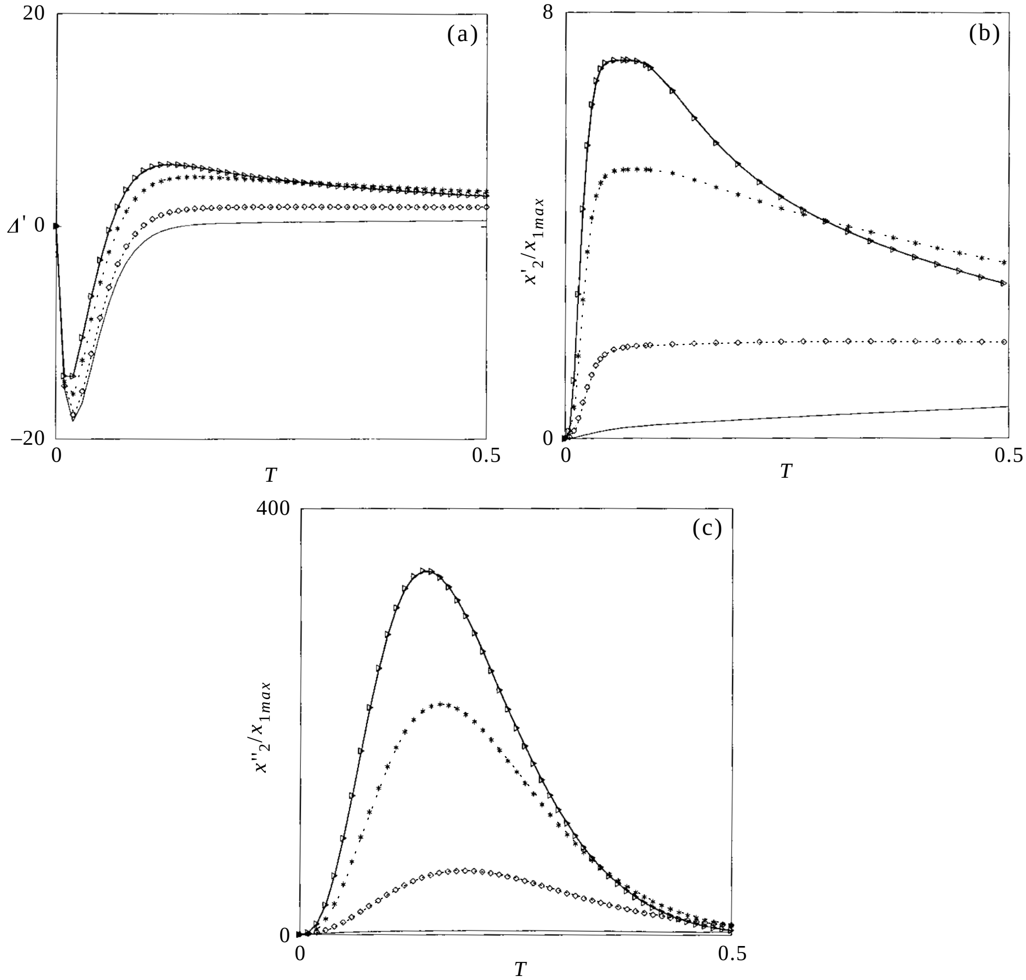

Figure 6. The effect of nonlinear cubic damping on the response for the rounded step. $\gamma = 50$, $\zeta = 0.1$, $\varepsilon^* = 0$. ——, $\delta^* = 0$; $\diamondsuit$, $\delta^* = 0.01$; $*$, $\delta^* = 0.05$; $\triangleright$, $\delta^* = 0.1$. (a) Relative velocity; (b) velocity; and (c) acceleration.

3.2. FIRST AND SECOND ORDER SOLUTIONS

For $i = 1$ and $i = 2$, equation (14) yields

$$\Delta_1'' + 2\zeta\Delta_1' + \Delta_1 = -[\epsilon_1(\Delta_0)^3 + \delta_1(\Delta_0')^3] \tag{17}$$

and

$$\Delta_2'' + 2\zeta\Delta_2' + \Delta_2 = -3[\epsilon_1(\Delta_0)^2\Delta_1 + \delta_1(\Delta_0')^2\Delta_1'], \tag{18}$$

respectively. Depending upon the type of shock loading, the solution given by equations (15) or (16) for Δ_0 can be used to obtain the forcing function in equation (17). The solution of equation (17) can then be obtained by using Laplace transformation with zero initial conditions. The same process is repeated for the solution of equation (18). When Δ_0 from equation (15) is substituted in equation (17) and the corresponding Δ_0 and Δ_1 in equation (18), the non-homogeneous terms can be expressed as a combination of three generalized

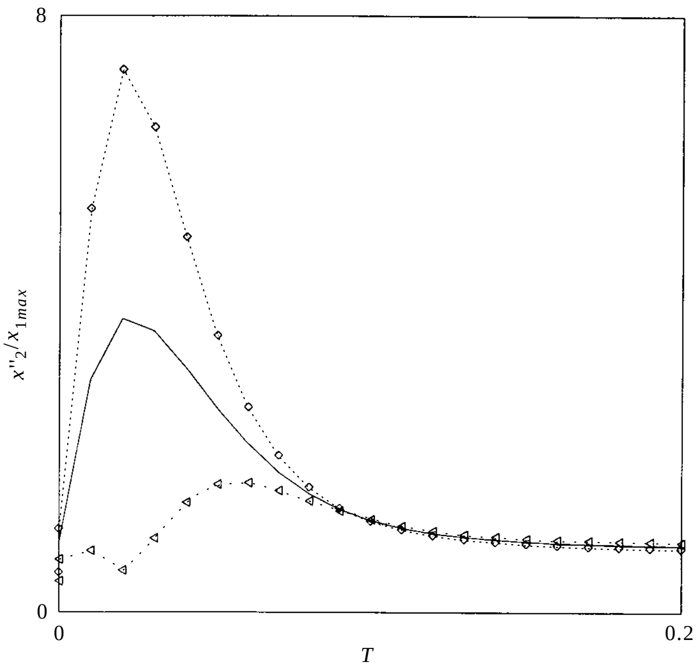

Figure 7. The effect of nonlinear quadratic damping on the acceleration for the rounded step. $\gamma = 50$, $\zeta = 0.1$, $\varepsilon^* = 0$. ——, $\delta^* = 0$; $\diamond$, $\delta^* = 0.01$; $\triangle$, $\delta^* = -0.01$.

functions: two of which are given by equations (7) and (10) and the third function is

$$e^{aT}\left[E\cos(\Omega T) + F\sin(\Omega T)\right]\left(\sum_{j=0}^{N} A_j T^j\right). \tag{19}$$

If Δ_0 from equation (16) is substituted in equation (17) and the corresponding Δ_0 and Δ_1 in equation (18), the homogeneous terms can be expressed in the form given by equation (10).

The Appendix lists the general forms of solutions of equations (17) and (18) corresponding to the non-homogeneous terms of the form of equations (7), (10) and (19).

4. RESULTS AND DISCUSSIONS

The results are presented in two parts: the first part to validate the accuracy of the closed form solutions and the second part to highlight the effects of non-linearity in isolator elements. The linear damping ratio ζ is taken to be 0·1 in all the results.

4.1. VALIDATION OF CLOSED FORM SOLUTIONS

The closed form solutions of equation (2), obtained through Laplace transformation of equations (13) and (15), were checked against the numerically integrated solutions of equation (11). Numerical integration was carried out by using the classical fourth order Runge–Kutta method.

First, the non-linearity in stiffness alone is considered by taking $\epsilon = 0{\cdot}01$, $\epsilon_1 = 1$ and $\delta_1 = 0$ in equation (11). Figure 3(a), (b) and (c) show the comparative results for the three types of excitation, given by equations (4), (6) and (9), respectively. The severity parameter of the shock displacement, γ, is taken to be equal to 20 for all these results. It is clear from these figures that even the linear solution is very close to the exact numerical solution. This indicates that the effect of non-linearity in stiffness of the isolator is negligible. The same behaviour was observed for a large range of the severity parameter γ.

The closed form solutions for an isolator with non-linearity in damping alone are presented in Figure 4(a)–(c), with $\epsilon = 0{\cdot}01$, $\epsilon_1 = 0$, $\delta_1 = 1$ and $\gamma = 20$. It is evident from these figures that the non-linearity in damping affects the response, especially for rounded pulse and oscillatory step inputs [Figure 4(b) and (c)]. Further, as more terms are included in equation (12), the result progressively converges to the numerically integrated solution. It is evident from Figure 4(a) and (b), that closed form solutions match well with the numerical solution. Similar results were obtained over a large range of γ. On the other hand, for case (c) as seen from Figure 4(c) the two terms solution is still far away from the numerically integrated results. While varying γ for this case, it was observed that only when the value of γ is around 10 did the closed form solution match closely with the numerical solution. For a value of γ around 10, the effect of nonlinear damping is not very pronounced. The closed form solution again does not yield good results if the severity parameter γ is around 1; in fact, as shown in Figure 4(d), the solution diverges as more terms are taken in the closed form solution. This is expected near $\gamma = 1$ with the oscillatory input, because the oscillations in the input are at the linear natural frequency of the isolator which in turn gives rise to secular terms in the perturbed solutions.

4.2. EFFECT OF NONLINEAR DAMPING

The nonlinear damping term makes the isolator response differ appreciably from the response of a linearly damped isolator [e.g., see Figure 4(a)]. This difference increases as the severity parameter γ is increased. The effect of nonlinear damping on the response (relative displacement) to the rounded step input is highlighted in Figure 5. These results for various vlaues of δ^* have been obtained through numerical integration of equation (2) with $\epsilon^* = 0$. With increasing values of δ^*, the peak response values increase up to $\delta^* \approx 0{\cdot}05$, thereafter an increase in δ^* reduces the peak value. However, the peak values are always higher for $\delta^* > 0$ than that for $\delta^* = 0$. Similar trends were also observed for other types of inputs, namely, inputs of cases (b) and (c). In fact this trend was also observed for any power-law damping forces with the exponent greater than one.

The relative displacement of the mass is only one of the measures of the isolator performance. Other performance criteria of a shock isolator are indicated by the absolute displacement, the relative velocity, the absolute velocity and acceleration of the mass. The absolute displacement exhibits features similar to those shown in Figure 5 (for the relative displacement) and is not shown again. The effect of nonlinear damping on the other three criteria are depicted in Figure 6(a)–(c). Figure 6(a) shows that the peak value of the relative velocity is reduced by increasing the coefficient of the nonlinear, cubic damping term. However, Figure 6(b) and (c) clearly indicate that an increase in the same coefficient results in higher peaks in both the absolute velocity and the acceleration of the isolated mass.

With a shock displacement to the base, the initial velocity across the damping element is quite large. The cubic non-linearity in damping then causes a large excitation to the mass which in turn accounts for the high value of its response.

It may be mentioned here that a similar isolator system with various types of shock force excitation was also investigated. Especially the effect of non-linearities in damping was studied. It was found that the effect of nonlinear damping with positive coefficients on the response peaks was negligible, and an increase in the nonlinear damping coefficient always reduces the peak-response.

Fluid dampers, in certain range of velocity across them, can be better modelled by including a quadratic damping term. Some elastomeric isolators are also modelled by a combination of linear and quadratic damping forces. In the first case the coefficient of the quadratic term is positive, whereas in the latter, this coefficient is a small (compared to that of the linear term) negative number. Both these types of damping can be incorporated in equation (2) if the term $\delta^*(\Delta')^3$ is replaced by $\delta^* \mid \Delta' \mid \Delta'$ with δ^* positive or negative as the case may be. Figure 7 shows that the peak value of the absolute acceleration can be very effectively controlled with a negative δ^*. It was also seen that the effects of such quadratic damping terms are negligible so far as the relative and the absolute displacements are concerned.

5. CONCLUSIONS

A single degree-of-freedom, shock isolator system with nonlinear spring and damper has been considered with the base subjected to three types of shock displacements. A closed form solution has been approximated through Laplace transformation of the perturbed differential equations. It has been shown that for rounded step and pulse inputs, the two term closed form solution suffices to yield accurate response history. For the oscillatory displacement step input, however, the closed form solution is effective only for a limited range of the severity parameter γ. The advantage of having a closed form solution is in directly evaluating the transient response at any desired instant, rather than numerically integrating the equation of motion from $t = 0$.

The non-linearity in the restoring force has negligible effect on the response. On the other hand, the non-linearity in the damping term appreciably affects the response indices. In general, the performance of such an isolator with a positive nonlinear coefficient is worse than that of an isolator with linear damping. However, for elastomeric isolators having a small negative coefficient of the nonlinear damping term, there is a considerable reduction in the peak of the acceleration response.

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APPENDIX

In this Appendix the solution of linear second order differential equations with time-invariant coefficients for three types of generalized non-homogeneous functions with initial conditions $x = \dot{x} = 0$ at $t = 0$ are presented.

Consider the equation

$$\ddot{x} + 2\zeta\dot{x} + x = f(t) \tag{A1}$$

with initial conditions $x(0) = \dot{x}(0) = 0$.

Solution of equation (A1) is obtained by using Laplace transformations for three types of non-homogeneous functions $f(t)$.

Case (i)

$$f(t) = e^{at}\left[\sum_{m=0}^{N} A_m t^m\right]$$

This is the generalized form of both rounded step and pulse inputs. Solution of equation (A1) may be written as

$$x(t) = x_1(t) + x_2(t)$$

where

$$x_1(t) = e^{-\zeta t}[P_1 \cos(\beta t) + Q_1 \sin(\beta t)]/\beta$$

and

$$x_2(t) = e^{at}\left[\sum_{m=0}^{N} B_{m+1} t^m/m!\right]$$

where

$$\beta = \sqrt{1-\zeta^2};$$

$$\alpha = -\zeta + i\beta;$$

and

$$i = \sqrt{(-1)};$$

$$Q_1 + iP_1 = \left[\sum_{m=0}^{N} (N-m)!A_{N-m}(\alpha - a)^m\right]\bigg/(\alpha - a)^{N+1};$$

$$B_{N+1} = N!A_N/\Gamma;$$

$$B_N = [(N-1)!A_{N-1} - 2(a+\zeta)B_{N+1}]/\Gamma,$$

and

$$B_{N+1-m} = [(N-m)!A_{N-m} - 2(a+\zeta)B_{N+2-m} - B_{N+3-m}]/\Gamma;$$

for $2 \leqslant m \leqslant N$

where

$$\Gamma = a^2 + 2a\zeta + 1.$$

Case (ii)

$$f(t) = e^{at}\left[\sum_{m=0}^{N} \{C_m \cos(\omega_m t) + D_m \sin(\omega_m t)\}\right]$$

This function is a generalized form of the oscillatory step. Solution of equation (A1) may be written as

$$x(t) = x_1(t) + x_2(t)$$

where,

$$x_1(t) = e^{-\zeta t}[P_2 \cos(\beta t) + Q_2 \sin(\beta t)]/\beta$$

and

$$x_2(t) = e^{at}\left[\sum_{m=0}^{N} \{R_m \cos(\omega_m t) + S_m \sin(\omega_m t)\}\right]$$

where

$$\beta = \sqrt{1-\zeta^2};$$

$$\alpha = -\zeta + i\beta;$$

$$i = \sqrt{(-1)};$$

$$Q_2 + iP_2 = \sum_{m=0}^{N} \frac{C_m\,(\alpha - a) + D_m\,\omega_m}{(\alpha - a)^2 + \omega_m^2}$$

and

$$\alpha_m = a + i\omega_m\,;$$

$$S_m + iR_m = \frac{D_m + iC_m}{\alpha_m^2 + 2\zeta\alpha_m + 1}.$$

Case (*iii*)

$$f\,(t) = \mathrm{e}^{at}\,[E\cos\,(\Omega t) + F\sin\,(\Omega t)]\left(\sum_{m=0}^{N} A_m\,t^m\right)$$

This type of non-homogeneous terms occur in equations (17) and (18) and it is similar to equation (19). Solution of equation (A1) may be written as

$$x(t) = x_1\,(t) + x_2\,(t)$$

where

$$x_1\,(t) = \mathrm{e}^{-\zeta t}\,[P_3\cos\,(\beta t) + Q_3\sin\,(\beta t)]/\beta$$

and

$$x_2\,(t) = \mathrm{e}^{at}\left[\sum_{m=0}^{N}\{R_m\cos\,(\Omega t) + S_m\sin\,(\Omega t)\}t^m/m!\right]$$

where

$$\beta = \sqrt{(1-\zeta^2)};$$

$$\alpha_2 = -\zeta + i\beta;$$

and

$$i = \sqrt{(-1)};$$

$$U_m = \left[\sum_{k=0}^{\lceil m/2\rceil} (-1)^k \binom{m+1}{2k} (\alpha - a)^{m+1-2k}\Omega^{2k}\right]$$

$$V_m = \left[\sum_{k=0}^{\lfloor m/2\rfloor} (-1)^k \binom{m+1}{2k+1} (\alpha - a)^{m-2k}\Omega^{2k+1}\right]$$

$$G = (\alpha - a)^2 + \Omega^2;$$

$$Q_3 + iP_3 = \left[\sum_{m=0}^{N} m!A_m\,(EU_m + V_m\,F)G^{N-m}\right]\Big/ G^{N+1};$$

$$\alpha_2 = a + i\Omega;$$

$$\Gamma_2 = \alpha_2^2 + 2\zeta\alpha_2 + 1;$$

$$B_{N+1} = N!A_N/\Gamma_2;$$

$$B_N = B_{N+1}[(N-1)!A_{N-1} - 2(\alpha_2 + \zeta)B_{N+1}]/(N!A_N);$$

$$B_{N+1-m} = B_{N+1}[(N-m)!A_{N-m} - 2(\alpha_2 + \zeta)B_{N+2-m} - B_{N+3-m}]/(N!A_N);$$

for $2 \leqslant m \leqslant N$.

$$S_m + iR_m = (F + iE)B_{N+1-m} \qquad \text{for} \quad 0 \leqslant m \leqslant N.$$