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CHANGES IN THE NATURAL FREQUENCIES OF REPEATED MODE PAIRS INDUCED BY CRACKS IN A VIBRATING RING

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The presence of cracks in solid structures can be detected by measuring changes in the natural frequencies of appropriate vibration modes. In the present work a simple energy-based model of a vibrating ring is presented to estimate the effect of a crack on the frequencies of repeated mode pairs. Introduction of a crack breaks the symmetry of the structure, causing the frequencies of the degenerate modes to split to a measurable extent. A simple expression is derived that relates the split in frequency to the frequency of the modes in the un-cracked ring, the dimensions of the ring and the crack length. The treatment is extended to the case of a ring that is not perfectly symmetrical prior to the introduction of a crack. In the latter case, when the crack is small, the mode pairs have a preferred orientation determined by the initial asymmetry. As the crack grows the mode pairs rotate to become aligned with the axis defining the location of the crack.

1. INTRODUCTION

The presence of defects in solid structures can be detected by measuring the variations in the natural frequencies of vibration modes (e.g., [1–3]). However, application of this method in practice is problematical because modal frequencies are sensitive to other types of defect such as variations in dimension, porosity, etc. In order to detect growing cracks it is desirable to be able to measure relative changes in frequency such as those that occur in the degenerate modes in symmetrical bodies [4].

In axi-symmetrical bodies, the radial-axial modes occur in pairs with identical frequencies [5] [the so-called sin $(n\theta)$ and cos $(n\theta)$ modes, e.g. Figure 1(a)]. However, if a crack nucleates and grows in an appropriate orientation, the axial symmetry is broken and the modal frequencies split by a small amount $\Delta\omega$, depending on the size of the crack. Srivasan and Kot [6] investigated the effect of a slot in the wall of a cylinder on the frequencies of radial-axial modes. They found that the slot had little influence on modal frequencies, although the mode shapes were found to be distorted. Srivasan and Kot investigated circumferential slots (plane of the cut normal to the axis). Such cracks are not in an orientation that will break axial symmetry. Wake *et al*. [7] investigated the effect of longitudinal slots (plane of the cut parallel to the axis) in tubes and found significant splits in the frequencies of many of the radial-axial modes.

The object of this paper is to present a simplified analytical model of frequency splits in the degenerate radial-axial vibration modes and is relevant to a tube containing a longitudinal crack. To avoid the complications of three-dimensional treatment, the vibration of a ring is modelled so that the deformation is confined to a plane. Previous

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work on the vibration of imperfect rings [8, 9] has shown that deviations from perfect axi-symmetry, modelled using added point masses and radial and torsional springs produces frequency splitting of the repeated modes. In the present work, a simple expression is derived relating the split in frequency to the frequency of the un-cracked ring, the ring dimensions and the crack length. The treatment is also extended to the case of a ring that is not perfectly symmetrical prior to the introduction of a crack. In this case, when the crack is small, the mode pairs initially have a preferred orientation determined by the initial asymmetry. As the crack grows the mode pairs rotate to an orientation determined by the location of the crack.

2. EFFECT OF A CRACK ON THE NATURAL FREQUENCIES OF REPEATED MODES

Consider the vibration of a ring of radius R, width h, density ρ and Young's modulus E [Figure 1(b)]. Apart from a modification to the elastic constants used, this is formally the same problem as that involving the Rayleigh vibration modes in an infinitely long tube.

The problem is simplified if we assume that $h \ll R$ and that strains are infinitesimal, so that the results of Love [10] can be used. In addition, only bending deformation is considered, where bending is characterised by the local deviation in curvature κ from the mean curvature $1/R$. For zero strain at the mid section of the ring, the relation between the components of the displacement u_r and u_θ, defined with respect to the cylindrical co-ordinates r, θ, is

$$\frac{\partial u_\theta}{\partial \theta} + u_r = 0. \tag{1}$$

Since the displacements are sinusoidal functions of θ, it follows that

$$\begin{bmatrix} u_r \\ u_\theta \end{bmatrix} = \begin{bmatrix} \cos(n\theta) & \sin(n\theta) \\ -\sin(n\theta)/n & \cos(n\theta)/n \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \tag{2}$$

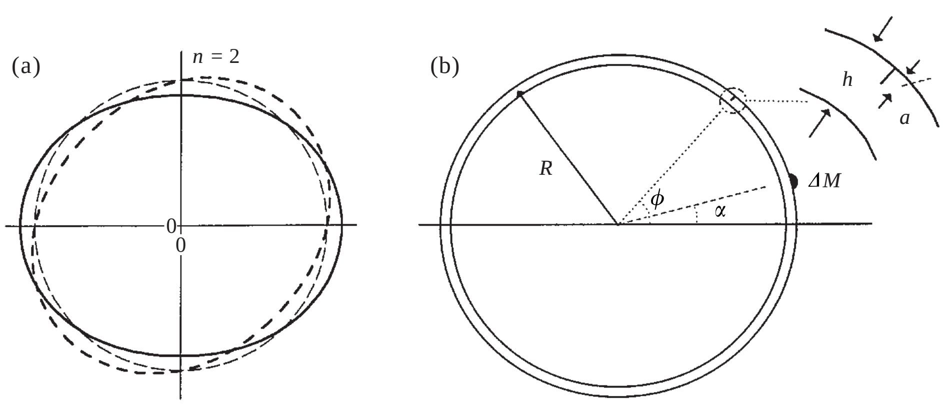

Figure 1. (a) Illustration of displacement produced by the cos $(n\theta)$ and sin $(n\theta)$ modes for $n = 2$. (b) Model representing cracked ring.

where n is an integer ($n > 1$) and q_1 and q_2 are generalised co-ordinates which are periodic functions of time. The local deviation of curvature from the mean value $1/R$ is given by

$$\kappa = \frac{1}{R^2}\frac{\partial}{\partial\theta}\left[u_\theta + \frac{\partial u_r}{\partial\theta}\right] = \frac{n^2-1}{R^2}[q_1\cos(n\theta) + q_2\sin(n\theta)]. \tag{3}$$

For displacements given by (2), the elastic strain energy V_0 and kinetic energy T can be written in the form:

$$V_0 = \tfrac{1}{2}m\omega_0^2[q_1^2 + q_2^2], \tag{4}$$

and

$$T = \tfrac{1}{2}m[\dot{q}_1^2 + \dot{q}_2^2]. \tag{5}$$

Here, the superior dots denote differentiation with respect to time t and m is the modal mass [11] defined by

$$m = \frac{n^2+1}{n^2}\pi\rho R h. \tag{6}$$

The natural frequency ω_0 of the nth mode is given by

$$\omega_0^2 = \frac{1}{12}\frac{n^2(n^2-1)^2}{(n^2+1)}\frac{E}{\rho}\frac{h^2}{R^4}. \tag{7}$$

The introduction of a thin crack affects the vibration of the ring mainly through its effect on the elastic strain energy. Under conditions of fixed boundary displacements the strain energy of the ring falls by an amount ΔV so that the potential energy can be written

$$V = V_0 - \Delta V.$$

Morassi [12] used a perturbation method to show that the sensitivity of the frequencies of a cracked beam is proportional to the modal strain energy density at the corresponding section of an un-cracked beam. In the present work, a fracture mechanics approach is used to obtain an estimate for ΔV in terms of crack length.

The connection between energy changes and fracture mechanics parameters is well established, (e.g. Paris and Sih [13]). The original hypothesis of Griffith [14] was that energy released by a reduction in elastic strain energy under fixed displacement conditions would be available to create new surfaces and to drive fracture. This is encapsulated in the definition of the free energy release rate G. For a crack of length a in a plate of unit thickness with constant displacements applied at the boundary, G is defined by

$$G = -\frac{\partial V}{\partial a}. \tag{8}$$

It is assumed that spontaneous crack extension occurs only when G reaches a critical value. For the present application G is assumed to be below the critical value. Thus, if G is known as a function of crack length, the reduction in energy produced by the introduction of a crack can be obtained from (8) by integration:

$$\Delta V = \int_0^a G\,\mathrm{d}a'. \tag{9}$$

In addition, G is also directly related to the stress intensity factor K that characterises the stress field of the crack at points close to the tip, i.e.,

$$G = (1 - \nu^2)K^2/E. \tag{10}$$

Expressions for K have been tabulated in the literature for a range of loading and geometrical cases so that ΔV can be estimated using equations (9) and (10).

For a beam of unit thickness subject to bending with a curvature κ, an expression for K (e.g., Ewalds and Wanhill [15]) gives

$$\Delta V = \tfrac{1}{9} E h^4 P(a/h)\kappa^2, \tag{11}$$

where the dimensionless function $P(a/h)$ is given by

$$P(a/h) = \int_0^{a/h} f(x)^2 \, \mathrm{d}x, \tag{12}$$

and

$$f(x) = \frac{3}{2}\sqrt{x}\,\frac{[1{\cdot}99 - x(1-x)(2{\cdot}15 - 3{\cdot}93x + 2{\cdot}7x^2)]}{(1+2x)(1-x)^{3/2}}. \tag{13}$$

For small crack lengths ($a/h \ll 1$), equations (12) and (13) give the following approximation

$$P(a/h) \approx 4{\cdot}5(a/h)^2. \tag{14}$$

The strain energy of the cracked ring vibrating with a displacement given by equation (2) can now be written as

$$V = \frac{1}{2} m\omega_0^2(q_1^2 + q_2^2) - \frac{1}{9} E h^4 P(a, h)\left[\frac{n^2 - 1}{R^2}\right]^2 (q_1 \cos(n\varphi) + q_2 \sin(n\varphi))^2, \tag{15}$$

where φ is defines the angular location of the crack (Figure 1).

The equation of motion of the cracked ring can now be obtained from Lagrange's equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} + \frac{\partial V}{\partial q_1} = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_2}\right) - \frac{\partial T}{\partial q_2} + \frac{\partial V}{\partial q_2} = 0. \tag{16}$$

Using (5), (6) and (15) in (16) gives:

$$\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} \omega_0^2 - \frac{\delta}{2}(1 + \cos(2n\varphi)) & -\frac{\delta}{2}\sin(2n\varphi) \\ -\frac{\delta}{2}\sin(2n\varphi) & \omega_0^2 - \frac{\delta}{2}(1 - \cos(2n\varphi)) \end{bmatrix} \cdot \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = 0, \tag{17}$$

where

$$\delta = \frac{2}{9}\cdot\frac{n^2(n^2-1)^2}{(n^2+1)}\cdot\frac{E h^3 P(a/h)}{\pi\rho R^5}. \tag{18}$$

The parameter δ represents the effect of the crack and determines the coupling between the generalised co-ordinates q_1 and q_2.

If the natural frequency of modal vibration is p and the generalised co-ordinates are represented in the form $q_1 = q_1^0 \exp(\mathrm{i}pt)$ and $q_2 = q_2^0 \exp(\mathrm{i}pt)$, an expression for p can be obtained from the eigenvalues of (17). Similarly, the angular location of the vibration modes can be obtained from the ratio of the eigenvectors. Expressing p in the form $p = \omega_0 - \Delta\omega$ and considering the radial motion of the ring, we obtain the following description of the modes:

(i) A node of one of the modes occurs at $\theta = \varphi$. There is no change in modal strain energy and its frequency is identical to ω_0, i.e.,

$$\Delta\omega = 0. \tag{19}$$

(ii) The other mode has an anti-node located at the crack, i.e., it is rotated by an angle $\pi/2n$ relative to its partner. The crack is located at the point of maximum curvature for this mode. From (11), the change in ΔV in the modal strain energy is a maximum and the frequency of the mode is reduced by the amount

$$\Delta\omega = \frac{\delta}{2\omega_0} + O(\delta^2). \tag{20}$$

Using (7) and (18) in (20), the total split in the frequencies of the degenerate modes, expressed as a fraction of the initial frequency, is to a first approximation

$$\frac{|\Delta\omega|}{\omega_0} = \frac{4}{3\pi}\,\frac{h}{R}\,P(a/h). \tag{21}$$

Thus, the fractional split in frequency is independent of the mode number.

Finally, using the approximation (14) when $a \ll h$, we obtain

$$\frac{|\Delta\omega|}{\omega_0} \approx 1{\cdot}9\,\frac{h}{R}\left(\frac{a}{h}\right)^2. \tag{22}$$

This shows that the frequency split is approximately proportional to the square of the crack length.

In principle, equations (21) and (22) show that it is possible to detect the growth of radial cracks in a ring or a cylinder by measuring the split in frequency of the degenerate modes. In practice, however, the ring may not be perfectly symmetrical prior to the introduction of the crack. The effect of this is now investigated.

3. MODE PAIR FREQUENCY SPLITTING PRODUCED BY CRACKS IN A NON-SYMMETRICAL RING

The effect of asymmetry can be simulated [8, 9] by adding a small mass ΔM at the angular location $\theta = \alpha$ [Figure 1(b)]. Such additional mass may occur for a number of reasons. For instance, manufacturing imperfections could produce an equivalent effect. In vibration testing, the attachment of a load cell and vibrator will have the same effect as an additional mass. Similarly, deliberately machining slots of finite width to simulate cracks in an experiment will produce a local reduction in mass. If the total mass of the ring is M, the ratio $\varepsilon = \Delta M/M$ can be used to characterise this asymmetry. The resulting equations of motion are similar to those shown in (17), except that the folloiwng mass-related matrix [8, 9] appears in front of the acceleration terms:

$$\cdot \begin{bmatrix} 1+\epsilon+\epsilon\dfrac{n^2-1}{n^2+1}\cos 2n\alpha & \epsilon\dfrac{n^2-1}{n^2+1}\sin 2n\alpha \\ \epsilon\dfrac{n^2-1}{n^2+1}\sin 2n\alpha & 1+\epsilon-\epsilon\dfrac{n^2-1}{n^2+1}\cos 2n\alpha \end{bmatrix} \tag{23}$$

After some tedious manipulation the eigensolution yields the following results to a first order in ϵ.

When there is no crack, the frequency split for a mode pair is produced by the additional mass only, and is given by

$$\frac{|\Delta\omega|}{\omega_0}=\frac{n^2-1}{n^2+1}\epsilon=s_M\,. \tag{24}$$

This split is a function of mode number. When there is no additional mass, the frequency split is produced by the crack only and

$$\frac{|\Delta\omega|}{\omega_0}=\frac{\delta}{2\omega_0^2}=s_C\,. \tag{25}$$

To characterise the combined effect of additional mass and crack we introduce the following parameter

$$\lambda=\frac{s_C}{s_M}\,. \tag{26}$$

After some algebraic manipulation it is found that frequency split due to the combined action of additional mass and crack is

$$\frac{|\Delta\omega|}{\omega_0}=s_M[\lambda^2+2\lambda\cos(2n(\varphi-\alpha))+1]^{1/2}. \tag{27}$$

When the additional mass and the crack are at the same location, $\alpha=\varphi$ and (27) gives

$$\frac{|\Delta\omega|}{\omega_0}=s_C+s_M\,. \tag{28}$$

As before, the nodes in the mode pair are shifted relative to each other by an angle $\pi/2n$, but the orientation of the pair with respect to the position $\theta=0$ is determined by the value of λ. For instance, assuming that the mass is set arbitrarily at the position $\alpha=0$, the orientation of a node of one mode of the pair is at the angle ψ, given by

$$\tan(n\psi)=-\frac{\lambda\cos(2n\varphi)+1-\sqrt{\lambda^2+2\lambda\cos(2n\varphi)+1}}{\lambda\sin(2n\varphi)+1}\,. \tag{29}$$

Inspection of (27) and (29) shows that for the limit $\lambda\to 0$, $\psi\to 0$ and $|\Delta\omega|/\omega_0\to s_M$. When $\lambda\to\infty$, we have the limits: $\psi\to\varphi$ and $|\Delta\omega|/\omega_0\to s_C$, as required on physical grounds. Thus, initially when $\lambda=0$, the alignment of the mode pair is set up to be orientated with the additional mass. As the crack grows and the ratio λ increases, the modal alignment rotates in an attempt to become aligned with the crack. The frequency split is then dominated by the effect of the crack. Some numerical evaluations using (27) and (29) are shown in Figure 2. It can be seen that the change from mass-dominated behaviour to

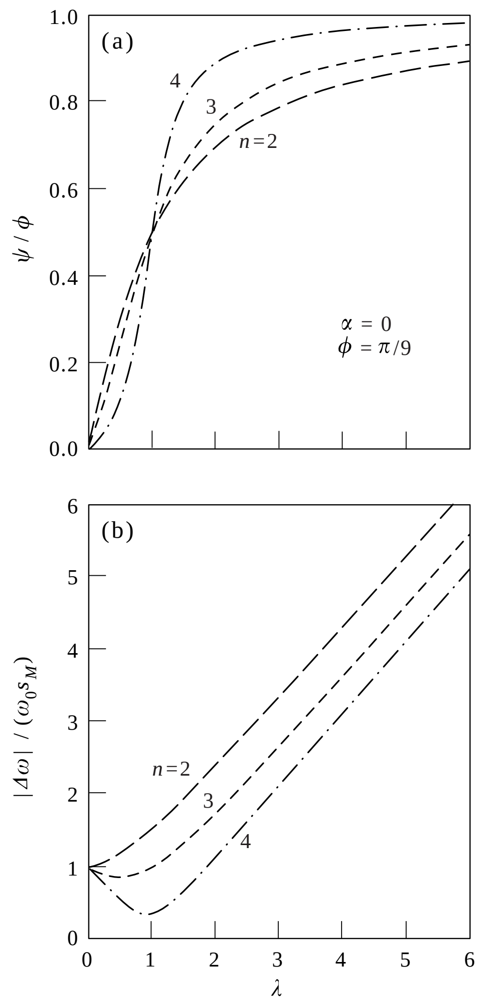

Figure 2. (a) Effect of the parameter λ (see text) on the mode pair orientation. When $\psi = \varphi$ the mode pair is aligned with the crack. (b) Effect of λ on the fractional split in the mode pair frequency.

crack-dominated behaviour becomes significant when λ is of the order unity, i.e., when $s_C \approx s_M$.

4. DISCUSSION

Equations (21) and (22) show that the growth of a radial crack in a ring can be detected by observing the split in frequency of the degenerate modes. As an example, take $h/R \approx 0{\cdot}1$ and $a/h \approx 0{\cdot}2$, giving $|\Delta\omega| \approx 8 \times 10^{-3}\omega_0$. Since ω_0 is likely to be of the order of 1 kHz or higher for a range of materials with $n \geqslant 2$, it is possible in principle to detect the growth of cracks in a symmetrical ring by measuring the split in frequency of the degenerate modes with a precision >1 Hz.

A split in frequency is produced by the crack causing a break in the symmetry of the structure. No real structure is perfectly symmetrical and the effect of a small amount of initial asymmetry is indicated in equation (27) and illustrated in Figure 2. It is clear that s_C has to be larger than s_M to detect crack growth. If it is smaller, crack growth could produce an ambiguous effect. On the other hand, if s_C is comparable with s_M the phenomenon of mode pair rotation indicated in (29) and illustrated in Figure 2(a) might also be used to detect growth of a crack.

For practical reasons, machined slots, rather than fatigue cracks, are often used to investigate the effect of defects in experiments. Equation (27) can be used to estimate the difference between slots and cracks. Since no material is removed in growth of a fatigue crack, the frequency split is given by the s_C parameter alone (ignoring the effect of any crack closure, e.g., Gudmudson [16] and Gounaris *et al*. [17]). In contrast when a slot is machined, a mass per unit length equal to ρab is removed, where b is the width of the cut. From (22), (24) and (27), the split in frequency of a mode pair produced by a slot is

$$\frac{|\Delta\omega|}{\omega_0} \approx s_C\left[1 - \frac{1}{4\pi}\,\frac{n^2-1}{n^2+1}\,\frac{b}{a}\right]. \tag{30}$$

The slot only behaves like a crack when the second term inside the brackets in (30) becomes small compared with unity. For instance, with $n = 2$, we require that the aspect ratio of the slot a/b to be significantly greater than 0·05 in order to be able to ignore the loss of mass. This result shows that careful consideration must be given to aspect ration of such slots in experiments designed to validate prediction techniques.

Experimental work [7] has shown that commercially produced cylinders are sufficiently symmetrical to allow detection of machined slots by measurement of the frequency split of mode pairs with n in the range 2 to 4. The experimental technique used to measure frequency splits has been described in detail by Wake *et al*. The frequency response function (FRF) was obtained using a shaker and an accelerometer located at appropriate points on the surface of the tube. When a slot is introduced the FRF takes the form shown in Figure 3, allowing the frequency split to be measured.

It is important to note that the effect of slots on natural frequency can be quite different to that of fatigue cracks. Suppose a fatigue crack is formed in a tube by sustained pressure cycling between zero and a maximum internal pressure. If the tube is subsequently vibrated under zero pressure it is quite likely that the fatigue crack will remain closed during part or all of the vibration cycle, with a greatly reduced effect on natural frequency. Fatigue cracks in metals can remain closed even under applied tensile stress because of the residual

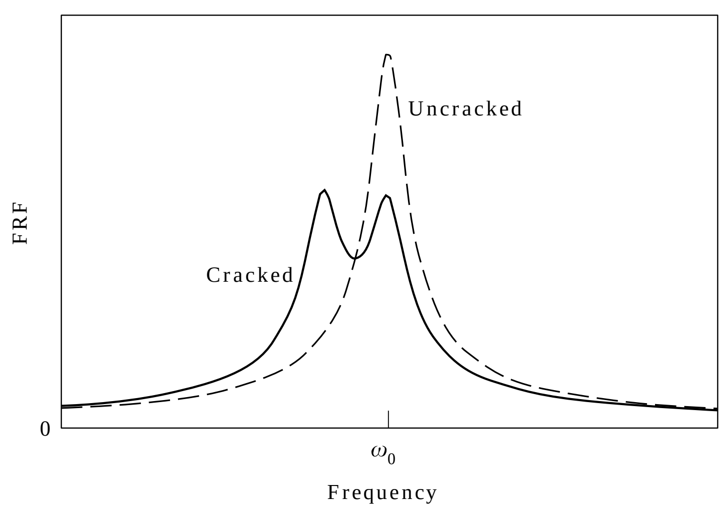

Figure 3. Effect of mode pair frequency splitting on the frequency response function.

compressive stress that develops during fatigue crack growth [18]. Thus, Gudmudson [16] showed that the vibration frequencies of a beam were much less affected by fatigue cracks than by slots of the same length. Gounaris *et al*. [17] also demonstrated that closed fatigue cracks did not have much influence on the natural frequencies of beams. In contrast to an unloaded body, a fatigue crack will be fully opened under a load equal to the maximum load in the fatigue cycle [19]. Thus in monitoring fatigue crack growth in gas cylinders by modal analysis, it would be advisable to perform the analysis with the cylinder filled to a significant fraction of the maximum working pressure. Finite element modelling [20] has shown that the crack sensitivity of modal frequencies is not much affected by internal pressure, although, of course, the frequencies of an un-cracked cylinder are increased by internal pressure.

The results on mode rotation [Figure 2(a)] suggest that, in principle, the onset of crack growth might be detected by this phenomenon, although it would probably be difficult to deduce crack length in terms of angle of orientation because of the non-linear nature of the relations. Mode orientation in a real ring with an initially high degree of symmetry should be sensitive to the introduction of a crack, provided that the crack was not introduced at an anti-node. By measuring acceleration at a number of points around the circumference of the ring the mode shape and thus the orientation can be determined. Mode rotations in cylinders following the introduction of a slot have been observed in this way in experimental work [20].

The present simplified model is limited because it is two-dimensional. Practical cases of interest, involving, say, crack growth in cylinders under the action of membrane stresses are three-dimensional [7]. However, it is believed, that a procedure similar to that used here, involving energy comparisons, could be extended to three-dimensional problems, as is being currently investigated by the authors.

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