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APPLICATION OF THE ABSOLUTE NODAL CO-ORDINATE FORMULATION TO MULTIBODY SYSTEM DYNAMICS

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The floating frame of reference formulation is currently the most widely used approach in flexible multibody simulations. The use of this approach, however, has been limited to small deformation problems. In this investigation, the computer implementation of the new *absolute nodal co-ordinate formulation* and its use in the small and large deformation analysis of flexible multibody systems that consist of interconnected bodies are discussed. While in the floating frame of reference formulation a mixed set of absolute reference and local elastic co-ordinates are used, in the absolute nodal co-ordinate formulation only absolute co-ordinates are used. In the absolute nodal co-ordinate formulation, new interpretation of the nodal co-ordinates of the finite elements is used. No infinitesimal or finite rotations are used as nodal co-ordinates for beams and plates, instead, global slopes are used to define the element nodal co-ordinates. Using this interpretation of the element co-ordinates, beams and plates can be considered as isoparametric elements, and as a result, exact modelling of the rigid body dynamics can be obtained using the element shape function and the absolute nodal co-ordinates. Unlike the floating frame of reference approach, no co-ordinate transformation is required in order to determine the element inertia. The mass matrix of the finite elements is a constant matrix, and therefore, the centrifugal and Coriolis forces are equal to zero when the absolute nodal co-ordinate formulation is used. Another advantage of using the absolute nodal co-ordinate formulation in the dynamic simulation of multibody systems is its simplicity in imposing some of the joint constraints and also its simplicity in formulating the generalized forces due to spring-damper elements. The results obtained in this investigation show an excellent agreement with the results obtained using the floating frame of reference formulation when large rotation–small deformation problems are considered.

1. INTRODUCTION

The formulation of the equations of motion of flexible multibody systems using the finite element method has been a challenging problem, particularly when conventional non-isoparametric elements such as beams and plates are used. The nodal co-ordinates of these widely used elements include infinitesimal rotations. As a result, exact modelling of the rigid body dynamics cannot be obtained when these non-isoparametric elements are used [1]. Such a limitation poses a serious problem when flexible multibody systems are considered. Generally these systems consist of interconnected rigid and deformable bodies, each of which may undergo large rotations. For this reason, several formulations that lead

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to exact modelling of the rigid body inertia were proposed for the non-linear dynamic analysis of flexible multibody systems. Among these formulations is the floating frame of reference approach [2–6] which can be used to obtain accurate modelling of the rigid body dynamics and it also leads to zero strain under an arbitrary rigid body motion of the non-isoparametric finite elements. The floating frame of reference approach uses two sets of co-ordinates to describe the dynamics of deformable bodies that undergo large reference displacements. The large reference translations and rotations are described by a mixed set of absolute Cartesian and orientation co-ordinates defined in a global inertial frame of reference. The elastic displacements of the bodies are defined with respect to its co-ordinate system using the nodal co-ordinates of the finite elements. The body frame of reference is defined using an appropriate set of reference conditions that define a unique displacement field [6]. The equations of motion obtained using the floating frame of reference formulation exhibit a strong non-linear inertia coupling between the reference and elastic co-ordinates. The mass matrix is highly non-linear and the inertia forces include Coriolis and centrifugal forces which are quadratic in the velocities. The stiffness matrix, on the other hand, takes a simple form and it is the same as the stiffness matrix that appears in structural mechanics.

The use of two different types of frames of reference; global and local (inertial and non-inertial), to describe two different sets of co-ordinates (reference co-ordinates and elastic co-ordinates), leads to the complexity of the resulting inertia forces. If isoparametric finite elements, which have absolute nodal co-ordinates defined in the inertial frame of reference, are used to model the flexible bodies, much simpler expressions for the inertia forces can be obtained. Furthermore, the shape function and the nodal co-ordinates of the element can be used to obtain exact modelling of the rigid body dynamics provided that the finite element shape functions have a complete set of rigid body modes. In the absolute nodal co-ordinate formulation [7, 8], a new interpretation of the nodal co-ordinates is used in order to develop new isoparametric beam and plate elements. Unlike the work of Simo and Vu-Quoc [9], no finite rotations are used as nodal co-ordinates, and instead, global slopes are used as nodal co-ordinates. The use of finite rotations as nodal co-ordinates can lead to redundancy in representing the large rotation of the cross-section of the finite element [6, 7].

In addition to the fact that the absolute nodal co-ordinate formulation automatically captures the non-linear effects arising from the coupling between different modes of displacements, the formulation of the joint constraints and forces becomes simpler when this new approach is used in flexible multibody dynamics. It is the objective of this investigation to examine and demonstrate the use of this new finite element procedure in the small and large deformation analysis of flexible multibody systems that consist of interconnected bodies. Comparison will be made with the floating frame of reference formulation which is currently the most widely used computer procedure for the analysis of flexible multibody systems. Throughout the analysis presented in this paper, a two-dimensional beam element is used for demonstration purposes.

This paper is organized as follows. In section 2, the absolute nodal co-ordinate formulation is reviewed, and the constant element mass matrix and non-linear stiffness matrix are identified. In section 3, the formulation of the generalized forces, when the absolute nodal co-ordinate formulation is used, is presented. In this section, some of the fundamental differences between the absolute nodal co-ordinate formulation and other existing finite element procedures are shown. Because of the use of global slopes as element nodal co-ordinates, a new set of generalized moments must be used. In section 4, the formulation of the joint constraints in the absolute nodal co-ordinate formulation is discussed. Section 5, the relationship between co-ordinates used in the absolute nodal co-ordinate formulation and the floating frame of reference approach is presented. Examples are presented in section 6 and the numerical results obtained using the absolute nodal co-ordinate formulation are compared with the results obtained using the floating frame of reference approach. Summary and conclusions drawn from the analysis developed in this paper are presented in section 7.

2. ABSOLUTE NODAL CO-ORDINATE FORMULATION

In the mixed finite element formulations, displacements and displacement gradients are used as nodal co-ordinates. These conventional finite element mixed formulations, however, have serious limitations when flexible multibody applications are considered. For instance, most of the mixed formulations were used in the framework of incremental procedure and the shape functions employed often do not have a complete set of rigid body modes. Furthermore, in structural dynamics applications mixed formulations are often used with lumped masses. When a lumped mass formulation is used with conventional beam elements, exact modelling of rigid body dynamics cannot be obtained [6]. In the absolute nodal co-ordinate formulation used in this investigation, it is required that the element shape function has a complete set of rigid body modes that can describe arbitrary rigid body translational and rotational displacements. Global displacements and slopes are used as nodal co-ordinates. By so doing, exact modelling of the rigid body dynamics can be obtained provided that a consistent mass formulation is used.

The absolute nodal co-ordinate formulation is presented in several previous publications [1, 6, 7, 8, 10, 11]. In this section, for the sake of completeness of the presentation, this formulation is briefly reviewed. In the absolute nodal co-ordinate formulation, the co-ordinates of the material points are defined in the global system. These absolute co-ordinates, as shown in Figure 1, are defined in terms of the element shape function and the vector of nodal co-ordinates as

$$\mathbf{r} = \begin{bmatrix} r_x \\ r_y \end{bmatrix} = \mathbf{S}\mathbf{e}, \tag{1}$$

where $\mathbf{r}$ is the global position vector of an arbitrary point on the element, $\mathbf{S}$ is a global shape function that includes a complete set of rigid body modes, and $\mathbf{e}$ is the vector of

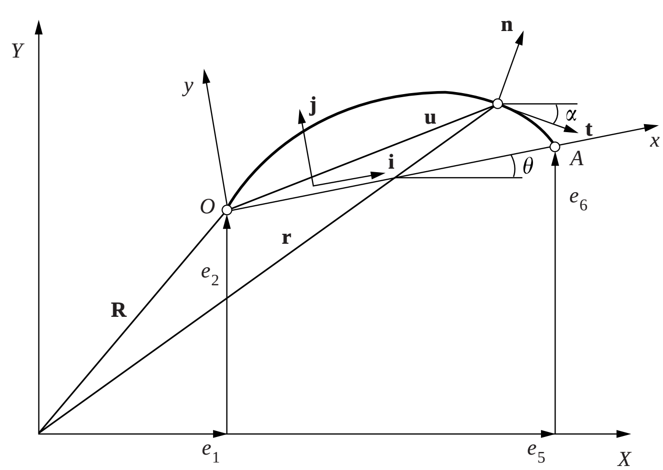

Figure 1. Planar beam element.

nodal co-ordinates that includes global displacements and slopes defined at the nodal points of the element.

2.1. DISPLACEMENT FIELD AND RIGID BODY KINEMATICS

In this paper, a planar beam element is used as an example to demonstrate the use of the finite element absolute nodal co-ordinate formulation in flexible multibody applications. Since the co-ordinates of the material points in this formulation are defined in a global frame of reference, there is no reason to use different polynomials to interpolate the displacement components. In this investigation, a cubic polynomial is used for both components of the displacement. In this case, the element shape function and the vector of nodal co-ordinates are defined as [6, 8, 11]

$$\mathbf{S} = \begin{bmatrix} 1-3\xi^2+2\xi^3 & 0 & l(\xi-2\xi^2+\xi^3) & 0 & 3\xi^2-2\xi^3 & 0 & l(\xi^3-\xi^2) & 0 \\ 0 & 1-3\xi^2+2\xi^3 & 0 & l(\xi-2\xi^2+\xi^3) & 0 & 3\xi^2-2\xi^3 & 0 & l(\xi^3-\xi^2) \end{bmatrix}, \tag{2}$$

$$\mathbf{e} = [e_1 \quad e_2 \quad e_3 \quad e_4 \quad e_5 \quad e_6 \quad e_7 \quad e_8]^T, \tag{3}$$

where the elements of the vector of nodal co-ordinates are defined as

$$\begin{aligned} &e_1 = r_x(x=0), \qquad e_2 = r_y(x=0), \qquad e_3 = \frac{\partial r_x(x=0)}{\partial x}, \qquad e_4 = \frac{\partial r_y(x=0)}{\partial x}, \\ &e_5 = r_x(x=l), \qquad e_6 = r_y(x=l), \qquad e_7 = \frac{\partial r_x(x=l)}{\partial x}, \qquad e_8 = \frac{\partial r_y(x=l)}{\partial x}, \end{aligned} \tag{4}$$

where x is the spatial co-ordinate along the element axis. Note that in the absolute nodal co-ordinate formulation no infinitesimal rotations are used as nodal co-ordinates, instead, global slopes are used. The initial values of the global slopes in the undeformed reference configuration can be determined using simple rigid body kinematics by utilizing the fact that equation (1) can be used to obtain exact modelling of the kinematics of rigid bodies. For instance, in an arbitrary undeformed reference configuration defined by the translations $r_x(x=0)$ and $r_y(x=0)$ and the rigid body rotation θ, the global position of an arbitrary point on the beam can be written as

$$\mathbf{r}(x) = \begin{bmatrix} r_x(x) \\ r_y(x) \end{bmatrix} = \mathbf{Se} = \begin{bmatrix} r_x(x=0) + x\cos\theta \\ r_y(x=0) + x\sin\theta \end{bmatrix}. \tag{5}$$

It follows that the global slopes in the undeformed reference configuration are defined as [6, 10, 11]

$$e_3 = e_7 = \cos\theta, \qquad e_4 = e_8 = \sin\theta. \tag{6}$$

A similar procedure can be used to determine the global slopes in the case of three-dimensional elements.

2.2. KINETIC ENERGY

The kinetic energy of the beam element is defined as

$$T = \frac{1}{2}\int_V \rho \dot{\mathbf{r}}^T \dot{\mathbf{r}}\,\mathrm{d}V = \tfrac{1}{2}\dot{\mathbf{e}}^T\left(\int_V \rho \mathbf{S}^T\mathbf{S}\,\mathrm{d}V\right)\dot{\mathbf{e}} = \tfrac{1}{2}\dot{\mathbf{e}}^T\mathbf{M}_a\dot{\mathbf{e}}, \tag{7}$$

where V is the volume, ρ is the mass density of the beam material, and $\mathbf{M}_a$ is the mass matrix of the element. Note that the mass matrix in equation (7) is symmetric and constant, and it is the same matrix that appears in linear structural dynamics. Using the shape function of equation (2), the mass matrix of the element can be evaluated as [6, 8, 11]

$$\mathbf{M}_a = \int_V \rho \mathbf{S}^T\mathbf{S}\,\mathrm{d}V = m\begin{bmatrix} \frac{13}{35} & 0 & \frac{11l}{210} & 0 & \frac{9}{70} & 0 & -\frac{13l}{420} & 0 \\ & \frac{13}{35} & 0 & \frac{11l}{210} & 0 & \frac{9}{70} & 0 & -\frac{13l}{420} \\ & & \frac{l^2}{105} & 0 & \frac{13l}{420} & 0 & -\frac{l^2}{140} & 0 \\ & & & \frac{l^2}{105} & 0 & \frac{13l}{420} & 0 & -\frac{l^2}{140} \\ & & & & \frac{13}{35} & 0 & -\frac{11l}{210} & 0 \\ & \text{symmetric} & & & & \frac{13}{35} & 0 & -\frac{11l}{210} \\ & & & & & & \frac{l^2}{105} & 0 \\ & & & & & & & \frac{l^2}{105} \end{bmatrix}, \tag{8}$$

where m is the mass of the beam and l is its length. It can be demonstrated that the use of this mass matrix leads to exact modelling of the rigid body inertia [8].

2.3. STRAIN ENERGY

While the absolute nodal co-ordinate formulation leads to a simple expression for the inertia forces, the use of this formulation results in a relatively complex expression for the elastic forces. In order to demonstrate this fact, a simple linear elastic model based on the classical beam theory is used in this section. If point O shown in Figure 1 is used as the reference point, the displacements of an arbitrary point on the beam relative to point O may be written as

$$\mathbf{u} = \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} (\mathbf{S}_1 - \mathbf{S}_{1O})\mathbf{e} \\ (\mathbf{S}_2 - \mathbf{S}_{2O})\mathbf{e} \end{bmatrix}, \tag{9}$$

where $\mathbf{S}_1$ and $\mathbf{S}_2$ are the rows of the element shape function matrix, and $\mathbf{S}_{1O}$ and $\mathbf{S}_{2O}$ are the rows of the element shape function matrix defined at point O. In order to define these relative displacements in the element co-ordinate system, two unit vectors $\mathbf{i}$ and $\mathbf{j}$ along the element axes are defined as

$$\mathbf{i} = \begin{bmatrix} i_x \\ i_y \end{bmatrix} = \frac{\mathbf{r}_A - \mathbf{r}_O}{|\mathbf{r}_\mathrm{A} - \mathbf{r}_O|}, \qquad \mathbf{j} = \begin{bmatrix} j_x \\ j_y \end{bmatrix} = \mathbf{k} \times \mathbf{i}, \tag{10}$$

where $\mathbf{k}$ is a unit vector along the Z-axis. The longitudinal and transverse deformations of the beam can then be defined as [6, 8, 11]

$$\mathbf{u}_d = \begin{bmatrix} u_l \\ u_t \end{bmatrix} = \begin{bmatrix} \mathbf{u}^T\mathbf{i} - x \\ \mathbf{u}^T\mathbf{j} \end{bmatrix} = \begin{bmatrix} u_x i_x + u_y i_y - x \\ u_x j_x + u_y j_y \end{bmatrix}. \tag{11}$$

The strain energy of the beam element due to the longitudinal and transverse displacements is given by

$$U = \frac{1}{2}\int_0^l \left(Ea\left(\frac{\partial u_l}{\partial x}\right)^2 + EI\left(\frac{\partial^2 u_t}{\partial x^2}\right)^2\right) \mathrm{d}x = \tfrac{1}{2}\mathbf{e}^T\mathbf{K}_a\mathbf{e}, \tag{12}$$

where E is the modulus of elasticity, a is the cross-sectional area, I is the second moment of area of the beam element, and $\mathbf{K}_a$ is the element stiffness matrix. This matrix is a non-linear function of the nodal co-ordinates. It can be shown that the strain energy can be expressed in terms of the following *stiffness shape integrals* [6, 8, 11]:

$$\begin{gathered}
\mathbf{A}_{11} = \frac{Ea}{l}\int_0^1 \left[\left(\frac{\partial \mathbf{S}_1}{\partial \xi}\right)^T\left(\frac{\partial \mathbf{S}_1}{\partial \xi}\right)\right] \mathrm{d}\xi, \qquad \mathbf{A}_{12} = \frac{Ea}{l}\int_0^1 \left[\left(\frac{\partial \mathbf{S}_1}{\partial \xi}\right)^T\left(\frac{\partial \mathbf{S}_2}{\partial \xi}\right)\right] \mathrm{d}\xi, \\
\mathbf{A}_{21} = \frac{Ea}{l}\int_0^1 \left[\left(\frac{\partial \mathbf{S}_2}{\partial \xi}\right)^T\left(\frac{\partial \mathbf{S}_1}{\partial \xi}\right)\right] \mathrm{d}\xi, \qquad \mathbf{A}_{22} = \frac{Ea}{l}\int_0^1 \left[\left(\frac{\partial \mathbf{S}_2}{\partial \xi}\right)^T\left(\frac{\partial \mathbf{S}_2}{\partial \xi}\right)\right] \mathrm{d}\xi, \\
\mathbf{B}_{11} = \frac{EI}{l^3}\int_0^1 \left[\left(\frac{\partial^2 \mathbf{S}_1}{\partial \xi^2}\right)^T\left(\frac{\partial^2 \mathbf{S}_1}{\partial \xi^2}\right)\right] \mathrm{d}\xi, \qquad \mathbf{B}_{12} = \frac{EI}{l^3}\int_0^1 \left[\left(\frac{\partial^2 \mathbf{S}_1}{\partial \xi^2}\right)^T\left(\frac{\partial^2 \mathbf{S}_2}{\partial \xi^2}\right)\right] \mathrm{d}\xi, \qquad (13) \\
\mathbf{B}_{21} = \frac{EI}{l^3}\int_0^1 \left[\left(\frac{\partial^2 \mathbf{S}_2}{\partial \xi^2}\right)^T\left(\frac{\partial^2 \mathbf{S}_1}{\partial \xi^2}\right)\right] \mathrm{d}\xi, \qquad \mathbf{B}_{22} = \frac{EI}{l^3}\int_0^1 \left[\left(\frac{\partial^2 \mathbf{S}_2}{\partial \xi^2}\right)^T\left(\frac{\partial^2 \mathbf{S}_2}{\partial \xi^2}\right)\right] \mathrm{d}\xi, \\
\mathbf{A}_1 = Ea\int_0^1 \left(\frac{\partial \mathbf{S}_1}{\partial \xi}\right)^T \mathrm{d}\xi, \qquad \mathbf{A}_2 = Ea\int_0^1 \left(\frac{\partial \mathbf{S}_2}{\partial \xi}\right)^T \mathrm{d}\xi,
\end{gathered}$$

where the explicit forms of these matrices obtained using the shape function of equation (2) are given in the Appendix. Using these stiffness shape integrals, the generalized elastic forces of the element can be calculated from [6, 8, 11]

$$\begin{aligned}\left(\frac{\partial U}{\partial \mathbf{e}}\right)^T &= \mathbf{A}_{11}\mathbf{e}i_x^2 + \mathbf{A}_{22}\mathbf{e}i_y^2 + (\mathbf{A}_{12} + \mathbf{A}_{21})\mathbf{e}i_x i_y - \mathbf{A}_1 i_x - \mathbf{A}_2 i_y + \mathbf{B}_{11}\mathbf{e}j_x^2 + \mathbf{B}_{22}\mathbf{e}j_y^2 \\ &\quad + (\mathbf{B}_{12} + \mathbf{B}_{21})\mathbf{e}j_x j_y + (\mathbf{e}^T\mathbf{A}_{11}\mathbf{e}i_x + \tfrac{1}{2}\mathbf{e}^T(\mathbf{A}_{12} + \mathbf{A}_{21})\mathbf{e}i_y - \mathbf{A}_1^T\mathbf{e})\left(\frac{\partial i_x}{\partial \mathbf{e}}\right)^T \\ &\quad + (\mathbf{e}^T\mathbf{A}_{22}\mathbf{e}i_y + \tfrac{1}{2}\mathbf{e}^T(\mathbf{A}_{12} + \mathbf{A}_{21})\mathbf{e}i_x - \mathbf{A}_2^T\mathbf{e})\left(\frac{\partial i_y}{\partial \mathbf{e}}\right)^T \\ &\quad + (\mathbf{e}^T\mathbf{B}_{11}\mathbf{e}j_x + \tfrac{1}{2}\mathbf{e}^T(\mathbf{B}_{12} + \mathbf{B}_{21})\mathbf{e}j_y)\left(\frac{\partial j_x}{\partial \mathbf{e}}\right)^T \\ &\quad + (\mathbf{e}^T\mathbf{B}_{22}\mathbf{e}j_y + \tfrac{1}{2}\mathbf{e}^T(\mathbf{B}_{12} + \mathbf{B}_{21})\mathbf{e}j_x)\left(\frac{\partial j_y}{\partial \mathbf{e}}\right)^T, \end{aligned} \tag{14}$$

where

$$\begin{aligned}\left(\frac{\partial i_x}{\partial \mathbf{e}}\right)^T &= \left(\frac{\partial i_y}{\partial \mathbf{e}}\right)^T \\ &= D[-(e_6 - e_2)^2 \quad (e_5 - e_1)(e_6 - e_2) \quad 0 \quad 0 \quad (e_6 - e_2)^2 \quad -(e_5 - e_1)(e_6 - e_2) \quad 0 \quad 0]^T, \\ \left(\frac{\partial i_y}{\partial \mathbf{e}}\right)^T &= -\left(\frac{\partial j_x}{\partial \mathbf{e}}\right)^T \\ &= D[(e_5 - e_1)(e_6 - e_2) \quad -(e_5 - e_1)^2 \quad 0 \quad 0 \quad -(e_5 - e_1)(e_6 - e_2) \quad (e_5 - e_1)^2 \quad 0 \quad 0]^T,\end{aligned} \tag{15}$$

$$D = \frac{1}{((e_5 - e_1)^2 + (e_6 - e_2)^2)^{3/2}}.$$

2.4. EQUATIONS OF MOTION

Using the principle of virtual work in dynamics and the expression of the kinetic and strain energies given by equations (7) and (12), the equation of motion of the finite element can be written as

$$\mathbf{M}_a\ddot{\mathbf{e}} = \mathbf{Q}, \tag{16}$$

where $\mathbf{Q}$ is the vector of generalized external nodal forces including the elastic forces. Note that centrifugal and Coriolis forces are equal to zero since the mass matrix is constant. The equations of motion of the deformable body can be obtained by assembling the equations of its elements using a standard finite element procedure.

3. FORMULATION OF THE GENERALIZED EXTERNAL FORCES

It is clear from the analysis presented in the preceding section that there are several fundamental differences between the absolute nodal co-ordinate formulation and some of the existing finite element procedures. One of these differences is the fact that there is no need to use co-ordinate transformation in order to determine the element mass matrix. Another difference is attributed to the formulation of the stiffness matrix which is highly non-linear in the case of the absolute nodal co-ordinate formulation even in the case of simple linear elastic model.

Another fundamental difference is due to the nature of the co-ordinates used in the absolute nodal co-ordinate formulation. These co-ordinates do not include infinitesimal or finite rotations. As such, attention must be paid to the definition of the generalized forces associated with the global slopes of the finite element. In this section, the definition of the generalized forces in the absolute nodal co-ordinate formulation is discussed.

3.1. FORCE VECTOR

The virtual work due to an externally applied force $\mathbf{F}$ acting on an arbitrary point on the element is given by $\mathbf{F}^T\delta\mathbf{r}$, where $\mathbf{r}$ is the position vector of the point of application of the force and $\delta\mathbf{r}$ is the virtual change in the vector $\mathbf{r}$. In order to obtain the generalized forces associated with the absolute nodal co-ordinates it is necessary to express $\delta\mathbf{r}$ in terms of the virtual displacements of these nodal co-ordinates. To this end, one can write

$$\mathbf{F}^T\delta\mathbf{r} = \mathbf{F}^T\mathbf{S}\delta\mathbf{e} = \mathbf{Q}_F^T\delta\mathbf{e}, \tag{17}$$

where $\mathbf{Q}_F = \mathbf{S}^T\mathbf{F}$ is the vector of generalized forces associated with the element nodal co-ordinates. For example, the virtual work due to the distributed gravity of the finite element can be obtained using the shape function of equation (2) as

$$\int_v [0 \quad -\rho g]\mathbf{S}\delta\mathbf{e}\,\mathrm{d}V = mg\begin{bmatrix} 0 & -\dfrac{1}{2} & 0 & -\dfrac{l}{12} & 0 & -\dfrac{1}{2} & 0 & \dfrac{l}{12}\end{bmatrix}\delta\mathbf{e}, \tag{18}$$

which defines the vector of generalized distributed gravity forces as

$$\mathbf{Q}_F = mg\begin{bmatrix} 0 & -\dfrac{1}{2} & 0 & -\dfrac{l}{12} & 0 & -\dfrac{1}{2} & 0 & \dfrac{l}{12}\end{bmatrix}^T. \tag{19}$$

3.2. MOMENT

When a moment M acts at a cross-section of the beam, the virtual work due to this moment is given by $M\delta\alpha$, where α is the angle of rotation of the cross-section. The orientation of a co-ordinate systen whose origin is rigidly attached to this cross-section (see Figure 1) can be defined using the following transformation matrix:

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \frac{1}{d^{1/2}}\begin{bmatrix} \dfrac{\partial r_x}{\partial x} & -\dfrac{\partial r_y}{\partial x} \\ \dfrac{\partial r_y}{\partial x} & \dfrac{\partial r_x}{\partial x} \end{bmatrix}, \qquad d = \left(\frac{\partial r_x}{\partial x}\right)^2 + \left(\frac{\partial r_y}{\partial x}\right)^2. \tag{20}$$

Using the elements of the planar transformation matrix given in the preceding equation, one has

$$\sin\alpha = d^{-1/2}\left(\frac{\partial r_y}{\partial x}\right), \qquad \cos\alpha = d^{-1/2}\left(\frac{\partial r_x}{\partial x}\right). \tag{21}$$

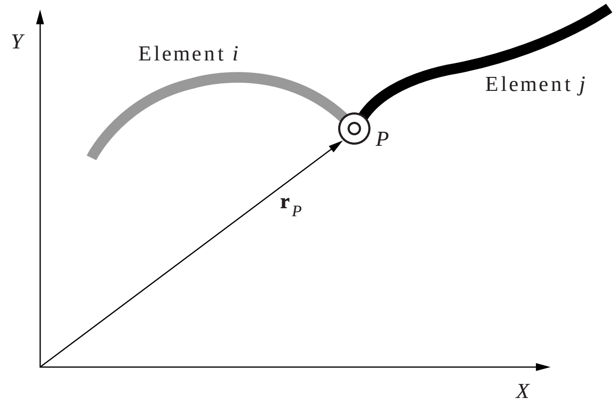

Figure 2. Revolute (pin) joint between two elements.

Using these two equations, it can be shown that

$$\delta\alpha = \frac{\frac{\partial r_x}{\partial x}\delta\left(\frac{\partial r_y}{\partial x}\right) - \frac{\partial r_y}{\partial x}\delta\left(\frac{\partial r_x}{\partial x}\right)}{d}. \tag{22}$$

If the concentrated moment M is applied, for example, at node O of the element, the generalized forces due to this moment are defined as

$$\mathbf{Q}_M = \begin{bmatrix} 0 & 0 & \frac{-Me_4}{d} & \frac{Me_3}{d} & 0 & 0 & 0 & 0 \end{bmatrix}^T. \tag{23}$$

3.3. SPRING-DAMPER FORCES

The formulation of the generalized forces due to a springer-damper element connecting two finite elements is very simple as compared to the floating frame of reference formulation which leads to a highly non-linear complex expression for these forces [5]. In the absolute nodal co-ordinate formulation, the generalized forces due to a spring-damper element take a simple form due to the fact that absolute co-ordinates are used. If a and b are the nodes to which the ends of the spring-damper element are attached, the generalized forces acting at node b simply take the form

$$\mathbf{Q}_{SD} = k\begin{bmatrix} e_1^a - e_1^b \\ e_2^a - e_2^b \end{bmatrix} + c\begin{bmatrix} \dot{e}_1^a - \dot{e}_1^b \\ \dot{e}_2^a - \dot{e}_2^b \end{bmatrix}, \tag{24}$$

where k and c are the spring and damping coefficients, respectively.

4. FORMULATION OF CONSTRAINTS

The formulation of many of the constraint equations that describe mechanical joints in flexible multibody systems becomes relatively simple when the absolute nodal co-ordinate formulation is used. In many cases, these constraint equations take a complex non-linear form when the floating frame of reference approach is used. This is mainly due to the fact that in the floating frame of reference formulation, two sets of co-ordinates (reference and elastic) defined in two different frames of reference (global and body) are used. In the absolute nodal co-ordinate formulation, only one set of absolute co-ordinates defined in one global co-ordinate system is used. As a consequence, many of the constraint equations become simple and linear. For instance, the revolute joint constraints which are highly non-linear in the floating frame of reference formulation [5, 6] become simple and linear when the absolute nodal co-ordinate formulation is used. Figure 2 shows two elements i and j which are connected by a revolute joint at point P. The constraint equations for the revolute joint can be written as

$$\mathbf{r}^i_P = \mathbf{r}^j_P, \tag{25}$$

which can be written in terms of the element co-ordinates as

$$\mathbf{S}^i_P\mathbf{e}^i = \mathbf{S}^j_P\mathbf{e}^j, \tag{26}$$

where $\mathbf{S}^i_P$ and $\mathbf{S}^j_P$ are the shape functions of the elements i and j evaluated at point P, and $\mathbf{e}^i$ and $\mathbf{e}^j$ are the vectors of nodal co-ordinates of the two elements. If point P is selected as a nodal point on the two elements, the constraint equation of the revolute joint reduces to

$$\begin{bmatrix} e^i_5 - e^j_1 \\ e^i_6 - e^j_2 \end{bmatrix} = \mathbf{0}, \tag{27}$$

where e^i_5 and e^i_6 are the absolute translational nodal co-ordinates of element i at node P, and e^j_1 and e^j_2 are the absolute translational nodal co-ordinates of element j at node P.

5. COMPARISON WITH THE FLOATING FRAME OF REFERENCE FORMULATION

In the floating frame of reference formulation, not all co-ordinates represent absolute variables, since the configuration of the body is described using a mixed set of absolute reference and local deformation co-ordinates. The reference co-ordinates define the location and the orientation of a selected body co-ordinate system. The deformation of the body is described using a set of local shape functions and a set of deformation co-ordinates defined in the body co-ordinate system. In the floating frame of reference formulation, it is assumed that there is no rigid body motion between the body and its co-ordinate system. Using Figure 1 and the reference and deformation co-ordinates, the global position vector of an arbitrary point on the centreline of the beam element can be written as [6]

$$\mathbf{r} = \mathbf{R} + \mathbf{A}\mathbf{u}, \tag{28}$$

where $\mathbf{R} = \mathbf{R}(t)$ defines the global position vector of the origin of the selected beam co-ordinate system, $\mathbf{A} = \mathbf{A}(t)$ is the transformation matrix that defines the orientation of the selected beam co-ordinate system with respect to the inertial frame, and $\mathbf{u} = \mathbf{u}(x, t)$ is the local position vector of the arbitrary point defined with respect to the origin of the beam co-ordinate system. The local position vector $\mathbf{u}$ may be represented in terms of local shape functions $\mathbf{S}_l(x)$ as

$$\mathbf{u}(x, t) = \mathbf{S}_l(x)\mathbf{q}_f(t), \tag{29}$$

where $\mathbf{q}_f(t)$ is the vector of time dependent deformation co-ordinates which can also be used in the finite element formulation to interpolate the local position as well as the deformation. When the kinematic description of equation (28) is used, it is assumed that there is no rigid body motion between the beam and its co-ordinate system. As a consequence, it is required that the local shape function matrix $\mathbf{S}_l(x)$ contains no rigid body modes. Using equations (28) and (29), the motion of the flexible beam can be described using the floating frame of reference formulation as

$$\mathbf{r} = \mathbf{R} + \mathbf{A}\mathbf{S}_l\mathbf{q}_f, \tag{30}$$

where the vector $\mathbf{q}_f(t)$ describes the local position and the deformation of an arbitrary point [6, 10], and the vector

$$\mathbf{q}_r(t) = \begin{bmatrix} \mathbf{R}(t) \\ \theta(t) \end{bmatrix} \tag{31}$$

describes the reference motion. In equation (31), θ is the angle that defines the orientation of the beam co-ordinate system. Therefore, the vector of generalized co-ordinates of the beam used in the floating frame of reference formulation can be written in a partitioned form as

$$\mathbf{q} = [\mathbf{R}^T \quad \theta \quad \mathbf{q}_f^T]^T = [\mathbf{q}_r^T \quad \mathbf{q}_f^T]^T. \tag{32}$$

Using equation (30) and the co-ordinate partitioning of equation (32), it can be shown that the mass matrix of the deformable beam in the case of the floating frame of reference formulation can be written in a partitioned form as [6]

$$\mathbf{M}_f = \begin{bmatrix} \mathbf{m}_{rr} & \mathbf{m}_{rf} \\ \mathbf{m}_{fr} & \mathbf{m}_{ff} \end{bmatrix}. \tag{33}$$

Unlike the absolute nodal co-ordinate formulation which leads to a simple and constant mass matrix, the mass matrix in the preceding equation is highly non-linear in the co-ordinates $\mathbf{q} = [\mathbf{q}_r^T \ \mathbf{q}_f^T]^T$ as a result of the dynamic coupling between the reference co-ordinates $\mathbf{q}_r$ and the deformation co-ordinates $\mathbf{q}_f$. In the case of planar motion, one has

$$\mathbf{q}_r = [R_x \quad R_y \quad \theta]^T, \qquad \mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \tag{34}$$

In this case of planar motion, it can be shown that the non-linear mass matrix and the Coriolis and centrifugal forces of the finite element can be expressed in terms of the following constant inertia shape integrals [6]:

$$\bar{\mathbf{S}} = \int_V \rho\mathbf{S}_l \,\mathrm{d}V, \qquad \mathbf{m}_{ff} = \int_V \rho\mathbf{S}_l^T\mathbf{S}_l \,\mathrm{d}V, \qquad \tilde{\mathbf{S}} = \int_V \rho\mathbf{S}_l^T\tilde{\mathbf{I}}\mathbf{S}_l \,\mathrm{d}V, \tag{35}$$

where ρ and V are the mass density and volume of the element, and

$$\tilde{\mathbf{I}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{36}$$

By establishing the relationship between the co-ordinates used in the floating frame of reference formulation and the co-ordinates used in the absolute nodal co-ordinate formulation, the non-linear mass matrix of equation (33) can be obtained using the constant mass matrix of equation (8) [10].

5.1. SLOPE RELATIONSHIP

Using equation (28), the global position vector of an arbitrary point on the beam element can be written using the floating frame of reference formulation as

$$\mathbf{r}(x, t) = \begin{bmatrix} r_x \\ r_y \end{bmatrix} = \begin{bmatrix} R_x + u_x \cos\theta - u_y \sin\theta \\ R_y + u_x \sin\theta + u_y \cos\theta \end{bmatrix}, \tag{37}$$

where u_x and u_y are the position co-ordinates of the arbitrary point defined with respect to the beam co-ordinate system. It follows in the case of a slender beam element that

$$\frac{\partial r_x}{\partial x} = \frac{\partial u_x}{\partial x}\cos\theta - \frac{\partial u_y}{\partial x}\sin\theta, \qquad \frac{\partial r_y}{\partial x} = \frac{\partial u_x}{\partial x}\sin\theta + \frac{\partial u_y}{\partial x}\cos\theta. \tag{38}$$

The slope relationship plays a fundamental role in defining the relationship between the co-ordinates used in the absolute nodal co-ordinate formulation and the co-ordinates used in the floating frame of reference formulation.

5.2. CO-ORDINATE TRANSFORMATION

In the remainder of this section, the relationship between the co-ordinates used in the floating frame of reference formulation and the co-ordinates used in the absolute nodal co-ordinate formulation is presented [10]. In the case of the absolute nodal co-ordinate formulation, the global element shape function defined by equation (2) is used. In the floating frame of reference formulation, it is assumed that the origin of the beam co-ordinate system is located at point O and one of the axes connects points O and A. In this case, the local shape function can be obtained from the global shape function of equation (2) as

$$\mathbf{S}_l = \begin{bmatrix} l(\xi - 2\xi^2 + \xi^3) & 0 & 3\xi^2 - 2\xi^3 & l(\xi^3 - \xi^2) & 0 \\ 0 & l(\xi - 2\xi^2 + \xi^3) & 0 & 0 & l(\xi^3 - \xi^2) \end{bmatrix}. \tag{39}$$

Note that this local shape function does not include any rigid body modes. The vector $\mathbf{q}_f$ in this case can be defined as

$$\mathbf{q}_f = [q_1 \quad q_2 \quad q_3 \quad q_4 \quad q_5]^T, \tag{40}$$

where q_3 is the local x co-ordinate of the node at A defined in the beam co-ordinate system, and

$$q_1 = \frac{\partial u_x(x=0)}{\partial x}, \qquad q_2 = \frac{\partial u_y(x=0)}{\partial x}, \qquad q_4 = \frac{\partial u_x(x=l)}{\partial x}, \qquad q_5 = \frac{\partial u_y(x=l)}{\partial x}. \tag{41}$$

The vector $\mathbf{e}$ of equation (3) used in the absolute nodal co-ordinate formulation can be expressed in this case in terms of the components of the vector

$$\mathbf{q} = [R_x \quad R_y \quad \theta \quad q_1 \quad q_2 \quad q_3 \quad q_4 \quad q_5]^T \tag{42}$$

of the floating frame of reference formulation using equation (38) as

$$\mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \\ e_8 \end{bmatrix} = \begin{bmatrix} R_x \\ R_y \\ q_1 \cos\theta - q_2 \sin\theta \\ q_1 \sin\theta + q_2 \cos\theta \\ R_x + q_3 \cos\theta \\ R_y + q_3 \sin\theta \\ q_4 \cos\theta - q_5 \sin\theta \\ q_4 \sin\theta + q_5 \cos\theta \end{bmatrix}. \tag{43}$$

Using this vector, it can be shown that

$$\mathbf{Se} = \mathbf{R} + \mathbf{AS}_t\mathbf{q}_f = \mathbf{r}. \tag{44}$$

This equation demonstrates the equivalence of the kinematic descriptions used in the floating frame of reference formulation and the absolute nodal co-ordinate formulation. Therefore, the co-ordinate transformation of equation (43) can be used to obtain the non-linear mass matrix and the inertia shape integrals used in the floating frame of reference formulation from the constant mass matrix used in the absolute nodal co-ordinate formulation, as demonstrated in reference [10].

6. APPLICATIONS

In order to demonstrate the use of the absolute nodal co-ordinate formulation in the dynamic simulation of flexible multibody systems, two examples are considered in this section. The results obtained using the absolute nodal co-ordinate formulation are compared with the results obtained using the floating frame of reference formulation. The two examples considered are the free falling of a flexible pendulum under its own weight, and a flexible slider–crank mechanism driven by a moment applied to the crankshaft. Both the crankshaft and the connecting rod of the slider–crank mechanism are assumed to be flexible bodies. It is important, however, to point out that the floating frame of reference formulation can only be used in the case of small deformation because the deformation of the bodies is expressed in terms of infinitesimal rotations and linear mode shapes. The absolute nodal co-ordinate formulation, on the other hand, can be used in the small as well as in the large deformation analysis.

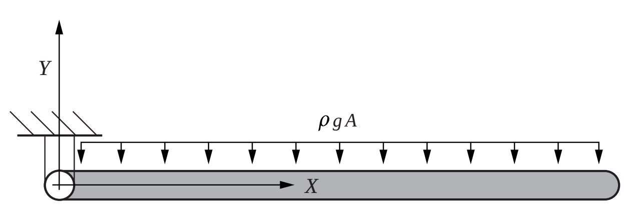

Figure 3. Free falling of a flexible pendulum.

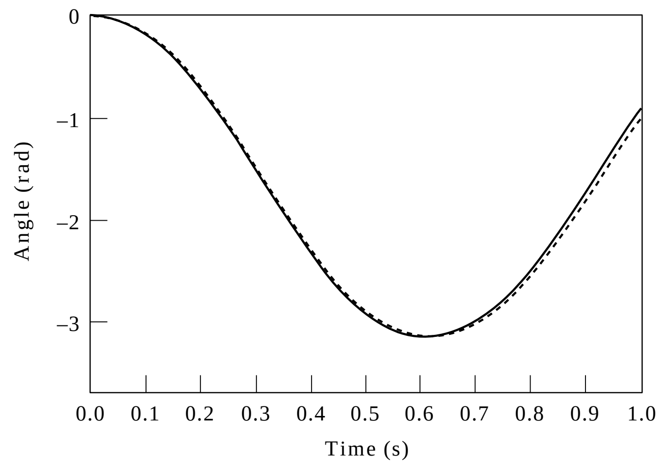

Figure 4. Angular orientation of the pendulum: ——, absolute nodal co-ordinate formulation; – – –, floating frame of reference formulation.

6.1. FLEXIBLE PENDULUM

The first example considered in this section is the free falling of the flexible pendulum shown in Figure 3. The pendulum, which is horizontal in its initial position, falls under the effect of gravity. The beam has a length of 0·4 m, a cross-sectional area of 0·0018 m^2, a second

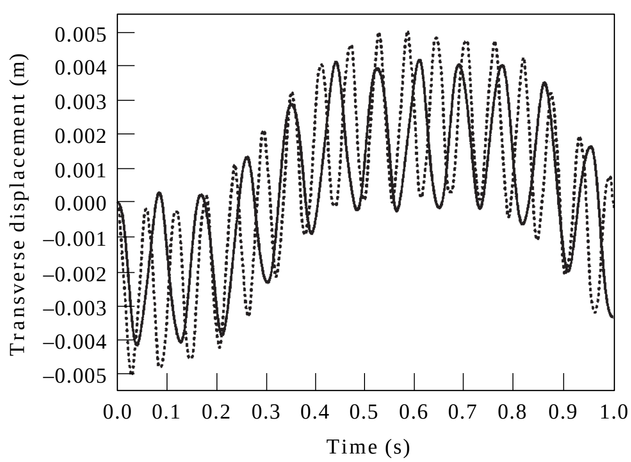

Figure 5. Transverse deformation of the tip point of the pendulum: key as in Figure 4.

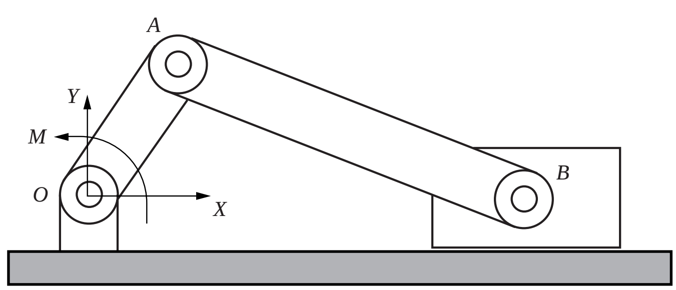

Figure 6. Slider–crank mechanism.

moment of area of 1·215 E-08 m^4, a mass density of 5540 kg/m^3 and a modulus of elasticity of 1·0 E 09 N/m^2. The beam is divided into 10 elements. In the floating frame of reference formulation, 10 elastic modes are used to describe flexibility of the pendulum rod. The body frame of reference of the flexible pendulum is assumed to be rigidly attached to its end at the pin joint. Note that in the absolute nodal co-ordinate formulation 42 degrees of freedom are used, as compared to 13 co-ordinates in the floating frame of reference formulation; 10 of them describe the elastic deformation.

Figure 4 shows the angular orientation of the flexible pendulum versus time obtained using the two formulations. A very good agreement can be observed between the two methods. Figure 5 shows the transverse displacement of the tip node of the pendulum versus time. The results presented in this figure show a good agreement between the absolute nodal co-ordinate formulation and the floating frame of reference formulation in the case of small deformation analysis.

6.2. FLEXIBLE SLIDER–CRANK MECHANISM

The second example used in this section to demonstrate the use of the absolute nodal co-ordinate formulation in the simulation of flexible multibody systems is the flexible slider–crank mechanism shown in Figure 6. The connecting rod is assumed to be much more flexible than the crankshaft and the slider block is assumed to be rigid and massless. In the initial position, both the connecting rod and crankshaft are assumed to be horizontal. The mechanism is assumed to be driven by a moment applied at the crankshaft. The crankshaft has a length of 0·152 m, a cross-sectional area of 7·854E-05 m^2, a second moment of area of 4·909E-10 m^4, a mass density of 2770 kg/m^3 and a modulus of elasticity of 1·0 E 09 N/m^2. The connecting rod is a beam of length 0·304 m, and has the same cross-sectional dimension and material properties as the crankshaft, with the exception of the modulus of elasticity which is assumed to be 0·5 E 08 N/m^2. In the dynamic model used in this study, the crankshaft is divided into three finite elements and the connecting rod is divided into eight elements. In the floating frame of reference formulation, three mode shapes are used to describe the flexibility of the crankshaft and five mode shapes are used for the connecting rod.

Two simulation cases were performed. In the first case, the moment applied at the crankshaft is given by

$$M(t) = [0{\cdot}01(1 - e^{-t/0{\cdot}167})]\ Nm. \tag{45}$$

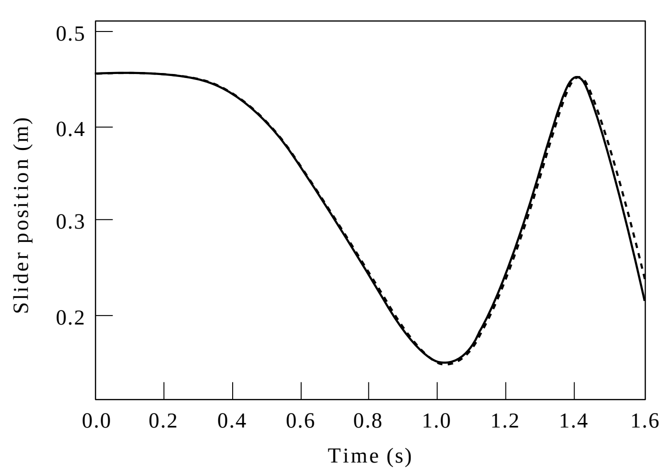

Figure 7. Co-ordinate of the slider block (moment defined by equation (45)): key as in Figure 4.

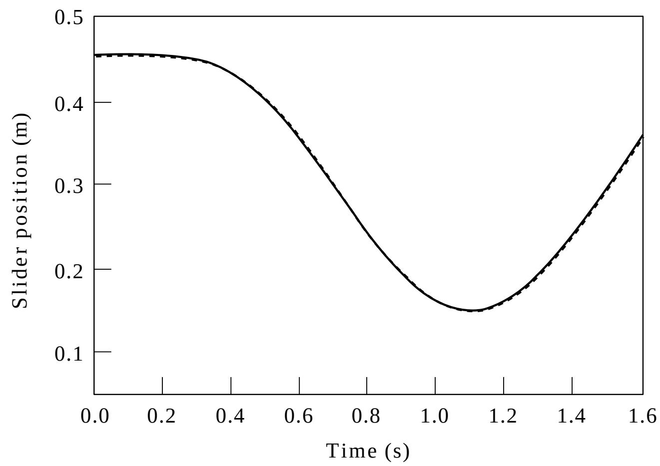

Figure 8. Co-ordinate of the slider block (moment defined by equation (46)): key as in Figure 4.

In the second case, the moment is assumed to be

$$M(t) = \begin{cases} [0{\cdot}01(1 - e^{-t/0{\cdot}167})]\ Nm, & t \leqslant 0{\cdot}7, \\ 0, & t > 0{\cdot}7. \end{cases} \tag{46}$$

Two variables are used to compare the results obtained using the absolute nodal co-ordinate formulation and the floating frame of reference formulation. These are the X position of the slider block and the transverse deformation of the mid-point of the connecting rod. Figures 7 and 8 show the slider block position in the two cases of the applied moments. These two figures show good agreement between the results obtained using the absolute nodal co-ordinate formulation and the floating frame of reference approach. Figures 9 and 10 show the transverse deformation of the mid-point of the connecting rod. In the first case of the applied moment, when the velocity of the system increases as well as the inertia forces, the deformation becomes relatively large, and

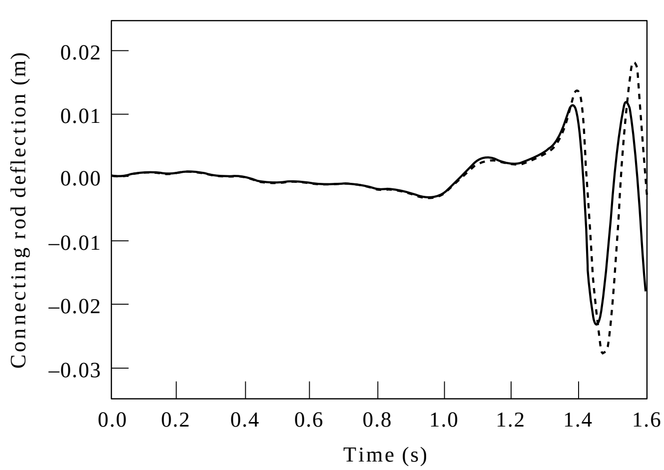

Figure 9. Deformation of the mid-point of the connecting rod (moment defined by equation (45)): key as in Figure 4.

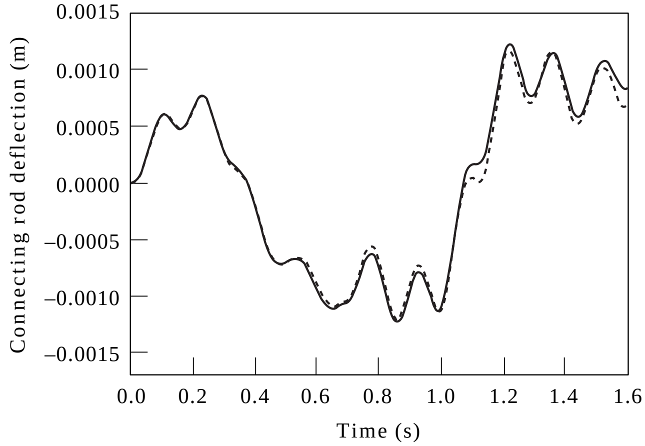

Figure 10. Deformation of the mid-point of the connecting rod (moment defined by equation (46)): key as in Figure 4.

differences between the solutions obtained using the two formulations can be observed. In the second case of the applied moment the transverse deformation remains relatively small. In this case, excellent agreement between the two formulations can be observed, as shown in Figure 10.

7. SUMMARY AND CONCLUSIONS

In the absolute nodal co-ordinate formulation, a new interpretation for the nodal co-ordinates is used. By using this new interpretation of the co-ordinates, a constant mass matrix can be obtained and as a result the Coriolis and centrifugal forces are equal to zero. The elastic forces, on the other hand, are highly non-linear functions of the element co-ordinates. The absolute nodal co-ordinate formulation can be effectively used in the large deformation problems [11] as well as flexible multibody applications as demonstrated in this paper. In addition to the constant simple mass matrix that appears in this formulation, the formulation of some of the joint constraints as well as forces can be very simple as compared to the floating frame of reference approach. Because of the nature of the co-ordinates used in the floating frame of reference formulation, such a method has only been used in the small deformation analysis of flexible multibody systems. The absolute nodal co-ordinate formulation does not suffer from this limitation and can be used in the small and large deformation analysis of flexible multibody systems. The applications used in this paper to compare the results obtained using the absolute nodal co-ordinate formulation and the results obtained using the floating frame of reference approach show excellent agreement between the two methods in the analysis of small deformations. Discrepancies can be observed between the results obtained using the two methods as the deformation increases.

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APPENDIX: STIFFNESS SHAPE INTEGRALS

The definition of the matrices that appear in the elastic forces can be made simpler if the nodal co-ordinates are rearranged as

$$\mathbf{e} = [e_1 \quad e_3 \quad e_5 \quad e_7 \quad e_2 \quad e_4 \quad e_6 \quad e_8]^T = [\mathbf{e}_x \quad \mathbf{e}_y]^T. \tag{A1}$$

Define the matrix **A** and **B** as

$$\mathbf{A} = Ea\begin{bmatrix} \frac{6}{5l} & \frac{1}{10} & -\frac{6}{5l} & \frac{1}{10} \\ \frac{1}{10} & \frac{2l}{15} & -\frac{1}{10} & -\frac{l}{30} \\ -\frac{6}{5l} & -\frac{1}{10} & \frac{6}{5l} & -\frac{1}{10} \\ \frac{1}{10} & -\frac{l}{30} & -\frac{1}{10} & \frac{2l}{15} \end{bmatrix}, \quad \mathbf{B} = EI\begin{bmatrix} \frac{12}{l^3} & \frac{6}{l^2} & -\frac{12}{l^3} & \frac{6}{l^2} \\ \frac{6}{l^2} & \frac{4}{l} & -\frac{6}{l^2} & \frac{2}{l} \\ -\frac{12}{l^3} & -\frac{6}{l^2} & \frac{12}{l^3} & -\frac{6}{l^2} \\ \frac{6}{l^2} & \frac{2}{l} & -\frac{6}{l^2} & \frac{4}{l} \end{bmatrix}. \tag{A2}$$

These matrices can be considered as the axial and bending stiffness matrices that appear in linear structural dynamics. By using the arrangement defined in equation (A1) and the matrices in equation (A2), the stiffness shape integrals that appear in the expression of the elastic forces are

$$\mathbf{A}_{11} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{A}_{22} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} \end{bmatrix}, \quad \mathbf{A}_{12} = \begin{bmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{A}_{21} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{A} & \mathbf{0} \end{bmatrix},$$

$$\mathbf{B}_{11} = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{B}_{22} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{bmatrix}, \quad \mathbf{B}_{12} = \begin{bmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{B}_{21} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B} & \mathbf{0} \end{bmatrix} \tag{A3}$$

and

$$\mathbf{A}_1 = [-Ea \quad Ea \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T, \qquad \mathbf{A}_2 = [0 \quad 0 \quad 0 \quad 0 \quad -Ea \quad Ea \quad 0 \quad 0]^T. \tag{A4}$$