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NON-LINEAR VIBRATION OF A HINGED ORTHOTROPIC CIRCULAR PLATE WITH A CONCENTRIC RIGID MASS

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Non-linear free vibration of hinged orthotropic circular plates with a concentric rigid mass at the centre is studied by using the finite element method. Hamilton's principle is applied to derive the basis non-linear partial differential equations and associated boundary conditions for the problem of large amplitude of an orthotropic circular plate. The applications of the finite element method to the dynamic problem rely on the use of a variation principle to derive the necessary element property's equations. The assembled equations for the plate are formed by summing each of the element equations obtained in consideration of a single element. Then, the boundary conditions are imposed on the vector of nodal field variables, so that the appropriate boundary conditions are satisfied. The assembled equations form an eigenvalue problem and are solved for the unknown field variables. The relations between the fundamental frequencies and the amplitudes of non-linear vibrations of the circular orthotropic elastic plate with a rigid core are obtained. The results show that the frequency responses of the plate varies with changes of boundary conditions and the ratio between tangential and radial elastic constant.

1. INTRODUCTION

Anisotropic materials play an important role in modern technology. Many new anisotropic materials, such as reinforced plastics and composite materials, are being used in such fields as missiles, aircraft, space vehicles, pressure vessels and parts of structures to meet the special requirements. Since vibrations may be disastrous, reliable predications of their nature are of great importance. The vibration amplitude of sufficient magnitude may result in malfunction of delicate components.

According to von Karman's theorem, when the amplitude of vibration is the same order of magnitude as the thickness of the plate, its dynamic equations are non-linear and coupled. Due to the complexity of the governing equations, it is very difficult to obtain the exact analysis solutions. Thus approximate methods of analysis must be used [1].

Large amplitude vibrations of an isotropic circular plate, which has a concentric rigid mass at the centre and the same boundary conditions, have been studied by several authors [2, 3]. The solutions for the problem are obtained by using Kantorovich's averaging method. The authors also solved the problem by the finite element method and the results are similar to those by the Kantorovich averaging method [4]. However, the addition of a concentric rigid mass attached rigidly to the orthotropic circular plate has received limited attention [5].

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The present investigation is concerned with the axisymmetric non-linear vibrations of a hinged orthotropic circular plate with a concentric rigid mass, using the finite element method. By the variational principle the element equations are derived and the assembled equations for the plate are formed by summing each of the element equations. A non-linear eigenvalue problem is formed. Numerical solutions are obtained for the stated problem. Free vibrations of the hinged movable and immovable plate–mass are investigated for various orthotropic ratios, and corresponding fundamental frequencies are presented.

2. DERIVATION OF BASIC DIFFERENTIAL EQUATIONS

Consider a thin circular plate with a concentric rigid mass M_c as shown in Figure 1. The outer radius of the plate is a and constant thickness is h. The radius of the rigid mass is b and equals the inner radius of the plate. Let the origin of the polar co-ordinates (r, θ, z) be located at the centre of the middle plane. The plate material is assumed to be elastic, homogenous and cylindrical orthotropic.

The radial and circumferential strain–displacement relations derived from large deflection theory are as follows:

$$\varepsilon_r = u_r + \tfrac{1}{2}w_r^2 - z \cdot w_{rr}, \qquad \varepsilon_\theta = \frac{u}{r} - \frac{z}{r}\,w_r, \tag{1}$$

where $u(r, t)$ and $w(r, t)$ denote the radial and transverse components of the mid-plane displacement, respectively, while subscripts denote partial derivatives. The strain–stress relations in the polar co-ordinate system are:

$$\varepsilon_r = a_r \cdot \sigma_r + a_{r\theta} \cdot \sigma_\theta, \qquad \varepsilon_\theta = a_{r\theta} \cdot \sigma_r + a_\theta \cdot \sigma_\theta, \tag{2}$$

where σ_θ and σ_r are normal stresses in the tangential and raidal directions, respectively, the coefficients a_θ, $a_{r\theta}$ and a_r are the elastic constants, and the variables ε_θ and ε_r are normal strains in the tangential and radial directions, respectively. The stress–strain in equations (2) can be represented in the form:

$$\begin{bmatrix} \sigma_r \\ \sigma_\theta \end{bmatrix} = \frac{12 \cdot D}{c \cdot h^3} \cdot \begin{bmatrix} c & v \\ v & 1 \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta \end{bmatrix}. \tag{3}$$

Here $c = a_\theta / a_r$, $v = -a_{r\theta}/a_r$ and $D = ((a_\theta \cdot h^3)/12 \cdot (a_\theta \cdot a_r - a_{r\theta}^2))$ are defined, where c is elastic constant ratio, v is the Poisson ratio and D is the flexural rigidity of the plate.

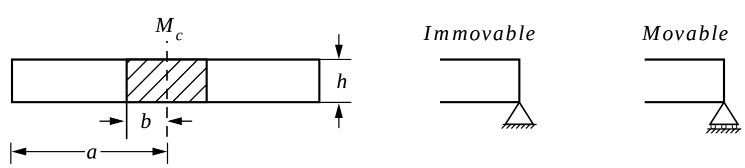

Figure 1. A circular plate with a concentric rigid mass and hinged boundary conditions.

By Hamilton's principle, it can be shown that, when an orthotropic circular plate with a concentric rigid mass at the centre undergoes finite amplitude axisymmetric vibration, the non-dimensional motion equations are as follows:

$$X_{\tau\tau} + c \cdot X_{\xi\xi\xi\xi} + \frac{2 \cdot c}{\xi} X_{\xi\xi\xi} - \frac{1}{\xi^2} X_{\xi\xi} + \frac{1}{\xi^3} X_\xi$$

$$= \frac{12 \cdot a}{h}\left(\frac{c}{\xi} \cdot U_\xi \cdot X_\xi + c \cdot U_{\xi\xi} \cdot X_\xi + c \cdot U_\xi \cdot X_{\xi\xi}\right.$$

$$\left. + \frac{c}{2} \cdot \frac{a}{h} \cdot \frac{1}{\xi} \cdot X_\xi^3 + \frac{3 \cdot c}{2} \cdot \frac{a}{h} \cdot X_\xi^2 \cdot X_{\xi\xi} + \frac{v}{\xi} \cdot U \cdot X_{\xi\xi} + \frac{v}{\xi} \cdot U_\xi \cdot X_\xi\right), \tag{4}$$

$$c \cdot U_{\xi\xi} + \frac{c}{\xi} \cdot U_\xi + c \cdot \frac{a}{h} \cdot X_\xi \cdot X_{\xi\xi} + \frac{c}{2} \cdot \frac{a}{h} \cdot \frac{1}{\xi} \cdot X_\xi^2 - \frac{1}{\xi^2} \cdot U - \frac{v}{2} \cdot \frac{a}{h} \cdot \frac{1}{\xi} \cdot X_\xi^2 = 0. \tag{5}$$

The following non-dimensional variables

$$X = \frac{w}{a}, \qquad \xi = \frac{r}{a}, \qquad R = \frac{b}{a}, \qquad \gamma = \frac{M_c}{\pi \cdot b^2 \cdot \rho \cdot h} \qquad U = \frac{u}{h} \qquad \tau = t \cdot \left[\frac{D}{\rho \cdot h \cdot a^4}\right]^{1/2},$$

have been used to transform the governing equations of motion and boundary conditions into the non-dimensional form. The derivation of the equations is shown in Appendix B. Let $c = 1{\cdot}0$, or $a_\theta = a_r = 1/E$ and $a_{r\theta} = -v/E$, where E is the material Young's modulus, equations (4) and (5) degenerate into the basic equations for the oscillation of an isotropic circular plate [4]. The associated boundary conditions in non-dimensional form are as follows.

In the case of hinged movable at the end $\xi = 1$

$$X = 0, \qquad c \cdot X_{\xi\xi} + \frac{v}{\xi} \cdot X_\xi = 0, \qquad c \cdot U_\xi + \frac{v}{\xi} \cdot U + \frac{c}{2} \cdot \frac{a}{h} \cdot X_\xi^2 = 0. \tag{6a}$$

In the case of hinged immovable at the end $\xi = 1$

$$X = 0, \qquad U = 0, \qquad c \cdot X_{\xi\xi} + \frac{v}{\xi} \cdot X_\xi = 0. \tag{6b}$$

In both cases at the end connected with rigid mass $\xi = R$

$$X_\xi = 0, \qquad U = 0, \qquad \xi \cdot \left(c \cdot X_{\xi\xi\xi} + \frac{c}{\xi} \cdot X_{\xi\xi} - \frac{1}{\xi^2} \cdot X_\xi\right)$$

$$- \frac{12 \cdot a}{h} \cdot \xi \cdot X_\xi \cdot \left(c \cdot U_\xi + \frac{c}{2} \cdot \frac{a}{h} \cdot X_\xi^2 + \frac{v}{\xi} \cdot U\right) + \frac{1}{2} \cdot \gamma \cdot R^2 \cdot \frac{\partial^2 X}{\partial \tau^2} = 0. \tag{6c}$$

3. FINITE ELEMENT ANALYSIS

An exact solution to the problem defined by equations (4)–(6) is at present unknown. Thus, the analysis of the problem is accomplished by numerical approaches. To solve problems of vibration with large amplitude there are several approximate methods such as the finite element method [4, 5] and Kantorovich's averaging method [2]. In this investigation the problem is studied by using the finite element method.

Since the plate problems considered herein, as shown in Figure 1, are axisymmetric, the ring elements are chosen. The normalized region of the plate $[R, 1]$ is divided into n elements of width $2s$, where $s = (1 - R)/2n$. Each element has three nodes, which are two exterior nodes and one interior node. Each nodal point has three degrees of freedom (X, X_ξ, U), representing the transverse defletion X, slope X_ξ, and the in-plane displacement U at the jth nodal point of the ith element. The values of displacements and slopes within ith element are expressed as

$$\{X^{(e)}(\xi, \tau)\} = [\mathbf{N}^{(e)}] \cdot \{\mathbf{W}^{(e)}\}, \qquad \{U^{(e)}(\xi, \tau)\} = [\mathbf{L}^{(e)}] \cdot \{\mathbf{V}^{(e)}\} \tag{7a, b}$$

where $\{\mathbf{W}^{(e)}\}$ and $\{\mathbf{V}^{(e)}\}$ are nodal variable vectors, defined as

$$\{\mathbf{W}^{(e)}\}^T = (X_{(j)}, X_{(j)\xi}, X_{(j+1)}, X_{(j+1)\xi}, X_{(j+2)}, X_{(j+2)\xi}),$$

$$\{\mathbf{V}^{(e)}\}^T = (U_{(j)}, U_{(j+1)}, U_{(j+2)}).$$

Also $[\mathbf{N}^{(e)}]$ and $[\mathbf{L}^{(e)}]$ are row vectors of shape functions that play a more important role in all finite element analysis. Hermitian functions are used for interpolation, and then shape functions are

$$[\mathbf{N}^{(e)}] = \begin{bmatrix} 3/4 & -1/2 & -5/4 & 1 & 0 & 0 \\ 1/4 & -1/4 & -1/4 & 1/4 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \\ 1 & 0 & -2 & 0 & 1 & 0 \\ -3/4 & -1/2 & 5/4 & 1 & 0 & 0 \\ 1/4 & 1/4 & -1/4 & -1/4 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} X_5 \\ X_4 \\ X_3 \\ X_2 \\ X_1 \\ X_0 \end{bmatrix} \tag{8a}$$

$$[\mathbf{L}^{(e)}] = \begin{bmatrix} 1/2 & -1/2 & 0 \\ -1 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \cdot \begin{bmatrix} X_2 \\ X_1 \\ X_0 \end{bmatrix}.$$

Their derivations are shown in the appendices of reference [4].

If the whole domain of the plate contains n elements, the representation of the field variable over the whole plate is given by

$$X(\xi, \tau) = \sum [\mathbf{N}_j^{(e)}] \cdot \{\mathbf{W}_j^{(e)}\} = \mathbf{N}^T\mathbf{W}, \qquad U(\xi, \tau) = \sum [\mathbf{L}_j^{(e)}] \cdot \{\mathbf{V}_j^{(e)}\} = \mathbf{L}^T\mathbf{V} \tag{9a, b}$$

where vector

$$\mathbf{W}^T = (X_{(1)}, X_{(1)\xi}, X_{(2)}, X_{(2)\xi}, \ldots, X_{(2n+1)}, X_{(2n+1)\xi}),$$

represents the unknown nodal values of the transverse deflections and slope, and where vector

$$\mathbf{V}^T = (U_{(1)}, U_{(2)}, \ldots, U_{(2n+1)}),$$

represents the unknown nodal value of the in-plane displacements.

Consider the strain–displacement relations for an elastic media with large deformations (von Karma's plate theory). The strain vector $\{\boldsymbol{\varepsilon}\}$ is expressed as

$$\{\boldsymbol{\varepsilon}\} = \begin{bmatrix} u_r \\ u/r \\ \cdots \\ -w_{rr} \\ -w_r/r \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\cdot w_r^2 \\ 0 \\ \cdots \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \varepsilon_p' \\ \cdots \\ \varepsilon_b \end{bmatrix} + \begin{bmatrix} \varepsilon_p'' \\ \cdots \\ 0 \end{bmatrix}$$

$$= \{\boldsymbol{\varepsilon}'\} + \{\boldsymbol{\varepsilon}''\}, \tag{10}$$

where vectors $\{\boldsymbol{\varepsilon}'\}$ and $\{\boldsymbol{\varepsilon}''\}$ are linear and non-linear strain vectors, respectively. The non-dimensional forms of the strain parts ε_p', ε_p'' and ε_b are

$$\varepsilon_p' = \frac{h_0}{a}\begin{bmatrix} U_\xi \\ U/\xi \end{bmatrix}, \qquad \varepsilon_b = \frac{-1}{a}\begin{bmatrix} X_{\xi\xi} \\ X_\xi/\xi \end{bmatrix}, \qquad \varepsilon_p'' = \begin{bmatrix} X_\xi^2/2 \\ 0 \end{bmatrix}. \tag{11}$$

For an orthotropic elastic material, the elastic constant matrix is

$$C = \begin{bmatrix} C_p & 0 \\ 0 & C_b \end{bmatrix}, \tag{12}$$

where

$$C_p = \frac{12\cdot D}{c\cdot h^3}C_1, \qquad C_b = \frac{12\cdot D\cdot z^2}{c\cdot h^3}C_1, \qquad C_1 = \begin{bmatrix} c & \upsilon \\ \upsilon & 1 \end{bmatrix},$$

and z is the distance from the middle plane along the thickness direction of the plate. To establish the global stiffness and mass matrices, the potential energy and kinetic energy are determined. By using equations (9)–(12), the total strain energy for the plate is obtained as

$$\Pi = \frac{1}{2}\cdot\int_\forall \varepsilon^T\cdot C\cdot\varepsilon\cdot \mathrm{d}v$$

$$= \frac{1}{2}\cdot\int_\forall \varepsilon'^T\cdot C\cdot\varepsilon'\cdot\mathrm{d}v + \frac{1}{2}\cdot\int_\forall(\varepsilon'^T\cdot C\cdot\varepsilon'' + \varepsilon''^T\cdot C\cdot\varepsilon' + \varepsilon''^T\cdot C\cdot\varepsilon'')\cdot\mathrm{d}v$$

$$= \Pi' + \Pi'', \tag{13}$$

in which the first term is a quadratic function representing the linear part of the strain energy, the second term is the non-linear part of the strain energy and dv is the volume unit.

After substitution of equations (11) and (12) into equation (13), and integration through the thickness of the plate h, the linear and non-linear parts of strain energy can be written as

$$\Pi' = \frac{1}{2}\cdot\frac{12\cdot D}{c\cdot h^2}\iint {\varepsilon^T}_p\cdot C_1\cdot\varepsilon_p'\cdot\mathrm{d}s + \frac{1}{2}\cdot\frac{D}{c}\iint\varepsilon_b^T\cdot C_1\cdot\varepsilon_b\cdot\mathrm{d}s, \tag{14a}$$

$$\Pi'' = \frac{1}{2} \cdot \frac{12 \cdot D}{c \cdot h^2} \iint (\varepsilon^T{}_p \cdot C_1 \cdot \varepsilon''_p + \varepsilon''^T{}_p \cdot C_1 \cdot \varepsilon'_p + \varepsilon''^T_p \cdot C_1 \cdot \varepsilon''{}_p) \cdot \mathrm{d}s, \tag{14b}$$

After substitution of equations (9a) and (b) into equation (11), the strain parts appearing in equation (14) can be written as

$$\varepsilon'_p = \frac{h_0}{a} \begin{bmatrix} L_\xi^T \\ L^T/\xi \end{bmatrix} \cdot \mathbf{V}, \qquad \varepsilon_b = -\frac{1}{a} \begin{bmatrix} N_{\xi\xi}^T \\ N_\xi^T/\xi \end{bmatrix} \cdot \mathbf{W}, \qquad \varepsilon''_p = \begin{bmatrix} \frac{1}{2} X_\xi^2 \\ 0 \end{bmatrix}. \tag{15}$$

Thus, their first variations are

$$\delta\varepsilon'_p = \frac{h_0}{a} \begin{bmatrix} L_\xi^T \\ L^T/\xi \end{bmatrix} \cdot \delta\mathbf{V}, \qquad \delta\varepsilon_b = -\frac{1}{a} \begin{bmatrix} N_{\xi\xi}^T \\ N_\xi^T/\xi \end{bmatrix} \cdot \delta\mathbf{W},$$

$$\delta\varepsilon''_p = \begin{bmatrix} X_\xi \\ 0 \end{bmatrix} \cdot N_\xi^T \cdot \delta\mathbf{W}. \tag{16}$$

Now, the first variation of the total strain energy is

$$\delta\Pi = \begin{bmatrix} \delta\mathbf{V} \\ \delta\mathbf{W} \end{bmatrix}^T \cdot \begin{bmatrix} \dfrac{12 \cdot D}{c} K_p & 0 \\ 0 & \dfrac{D}{c} K_b \end{bmatrix} \cdot \begin{bmatrix} \mathbf{V} \\ \mathbf{W} \end{bmatrix}$$

$$+ \begin{bmatrix} \delta\mathbf{V} \\ \delta\mathbf{W} \end{bmatrix}^T \cdot \begin{bmatrix} 0 & \dfrac{12 \cdot D}{c \cdot h} K'_p \\ \dfrac{6 \cdot D \cdot a}{c \cdot h} K'^T_p & \dfrac{12 \cdot D \cdot a^2}{c \cdot h^2} K''_p \end{bmatrix} \cdot \begin{bmatrix} \mathbf{V} \\ \mathbf{W} \end{bmatrix}, \tag{17}$$

where

$$K_p = \iint \left(c \cdot L_\xi \cdot L_\xi^T + \frac{v}{\xi} L \cdot L_\xi^T + \frac{v}{\xi} L_\xi \cdot L^T + \frac{1}{\xi^2} L \cdot L^T \right) \cdot \mathrm{d}s,$$

$$K_b = \iint \left(c \cdot N_{\xi\xi} \cdot N_{\xi\xi}^T + \frac{v}{\xi} N_{\xi\xi} \cdot N_\xi^T + \frac{v}{\xi} N_\xi \cdot N_{\xi\xi}^T + \frac{1}{\xi^2} N_\xi \cdot N_\xi^T \right) \cdot \mathrm{d}s,$$

$$K'_p = \iint (X_\xi) \cdot \left(c \cdot L_\xi + \frac{v}{\xi} L \right) \cdot N_\xi^T \cdot \mathrm{d}s,$$

$$K''_p = \iint (\tfrac{1}{2} X_\xi^2) \cdot (c \cdot N_\xi \cdot N_\xi^T) \cdot \mathrm{d}s.$$

The total kinetic energy of the plate can be determined as

$$T = \frac{1}{2} \cdot \begin{bmatrix} \dfrac{\partial \mathbf{V}}{\partial \tau} \\ \dfrac{\partial \mathbf{W}}{\partial \tau} \end{bmatrix}^T \cdot \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{D}{c} M \end{bmatrix} \cdot \begin{bmatrix} \dfrac{\partial \mathbf{V}}{\partial \tau} \\ \dfrac{\partial \mathbf{W}}{\partial \tau} \end{bmatrix}, \tag{18}$$

where

$$M = \iint N \cdot N^T \cdot \mathrm{d}s + \gamma \cdot \pi \cdot R^2 \cdot (N \cdot N^T)|_{\xi = R}.$$

By applying Hamilton's principle, Euler–Lagrange equations for the plate can be obtained and written in the matrix form

$$\begin{bmatrix} 0 & 0 \\ \mathbf{0} & \frac{D}{c} M \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial^2 \mathbf{V}}{\partial \tau^2} \\ \frac{\partial^2 \mathbf{W}}{\partial \tau^2} \end{bmatrix} + \begin{bmatrix} \frac{12 \cdot D}{c} K_p & 0 \\ 0 & \frac{D}{c} K_b \end{bmatrix} \cdot \begin{bmatrix} \mathbf{V} \\ \mathbf{W} \end{bmatrix}$$

$$\begin{bmatrix} + & 0 & \frac{12 \cdot D}{c} \cdot \frac{a}{h} K'_p \\ & \frac{12 \cdot D}{c} \cdot \frac{a}{h} \cdot \frac{1}{2} K_p'^T & \frac{12 \cdot D}{c} \cdot \frac{a^2}{h^2} \cdot K''_p \end{bmatrix} \cdot \begin{bmatrix} \mathbf{V} \\ \mathbf{W} \end{bmatrix} = 0,$$

or in a reduced form as

$$K_p \cdot \mathbf{V} + \frac{a}{h} \cdot K'_p \cdot \mathbf{W} = 0$$

$$M \cdot \frac{\partial^2 \mathbf{W}}{\partial \tau^2} + K_b \cdot \mathbf{W} + 12 \cdot \frac{a}{h} \cdot \frac{1}{2} \cdot K_p'^T \cdot \mathbf{V} + 12 \cdot \frac{a^2}{h^2} \cdot K''_p \cdot \mathbf{W} = 0. \tag{19}$$

Equations (19) are the motion equations describing the non-linear free vibrations of an orthotropic circular plate with a concentric rigid mass. The following boundary conditions are imposed on equations (19) in the case of (a) hinged–movable,

$$\mathbf{V}^T = (0, U_{(2)}, U_{(3)}, \ldots, U_{(2n-1)}, U_{(2n)}),$$

$$\mathbf{W}^T = (X_{(1)}, 0, X_{(2)}, X_{(2)\xi}, \ldots, X_{(2n-1)}, X_{(2n-1)\xi}, 0, X_{(2n)\xi}), \tag{20a}$$

(b) hinged–immovable,

$$\mathbf{V}^T = (0, U_{(2)}, U_{(3)}, \ldots, U_{(2n-1)}, 0),$$

$$\mathbf{W}^T = (X_{(1)}, 0, X_{(2)}, X_{(2)\xi}, \ldots, X_{(2n-1)}, X_{(2n-1)\xi}, 0, X_{(2n)\xi}). \tag{20b}$$

Because of the boundary conditions the coefficient matrices of vector **W** and vector **V** must be properly modified so that the original matrices remain unchanged.

4. NUMERICAL COMPUTATION

To determine the responses of non-linear vibrations of the circular plate, or to obtain the solutions of equations (19) and (20), the computational iteration procedure must be used, which is shown in reference [4].

Figures 2 and 3 present the results determined for hinged movable and immovable non-linear free vibrations with various values of stiffness parameter ($c = 0{\cdot}5$, $1{\cdot}0$, and $2{\cdot}0$) at radius ($R = 0{\cdot}1$). At the same amplitude, the resonant oscillation frequency of the system material with a higher stiffness parameter is higher than those with a lower stiffness parameter. The phenomenon shows that the effects of the elastic constant in the tangential

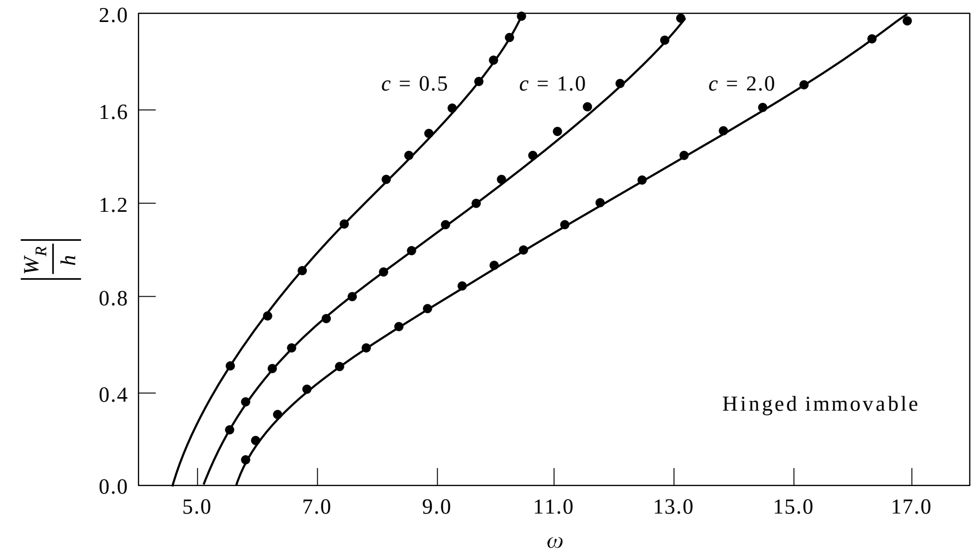

Figure 2. Non-linear free frequency responses of the annular plate with elastic constant ratio ($c = 0{\cdot}5$, $1{\cdot}0$ and $2{\cdot}0$), mass ratio ($\gamma = 1{\cdot}0$) and radius ratio ($R = 0{\cdot}1$) under hinged immovable boundary conditions.

direction upon oscillation frequency are higher than those in the radial direction. From shapes of the response curves in Figures 2 and 3, it seems that responses of the plate–mass system are similar to those of the hard-spring Duffing system. The results of Figure 4 show how the effects ofthe edge constraints influence the response of the plate–mass system. The resonant oscillation frequency of the hinged immovable plate is higher than that of the hinged movable plate on condition that the ratio of elastic constants is identical and the radius ratio held constant.

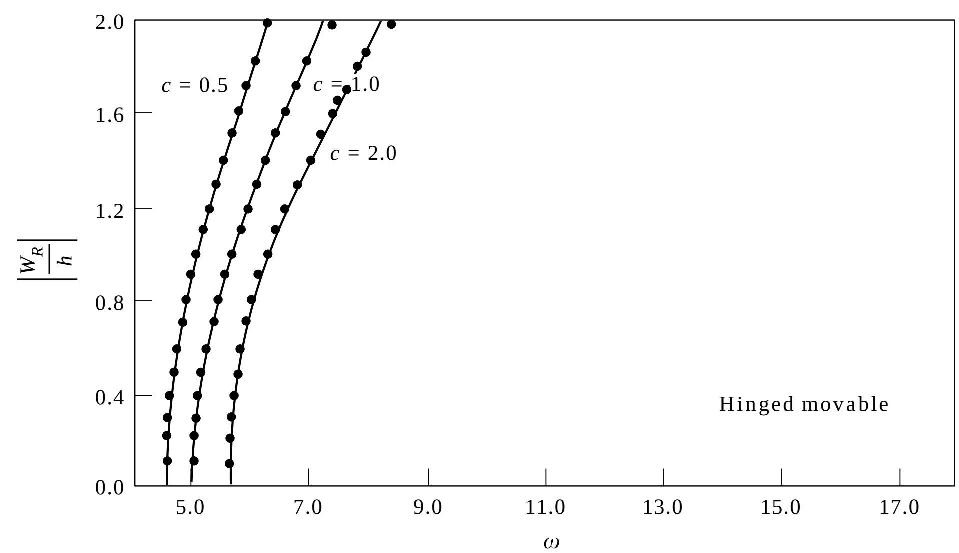

Figure 3. Non-linear free frequency responses of the annular plate with elastic constant ratio ($c = 0{\cdot}5$, $1{\cdot}0$ and $2{\cdot}0$), mass ratio ($\gamma = 1{\cdot}0$) and radius ratio ($R = 0{\cdot}1$) under hinged movable boundary conditions.

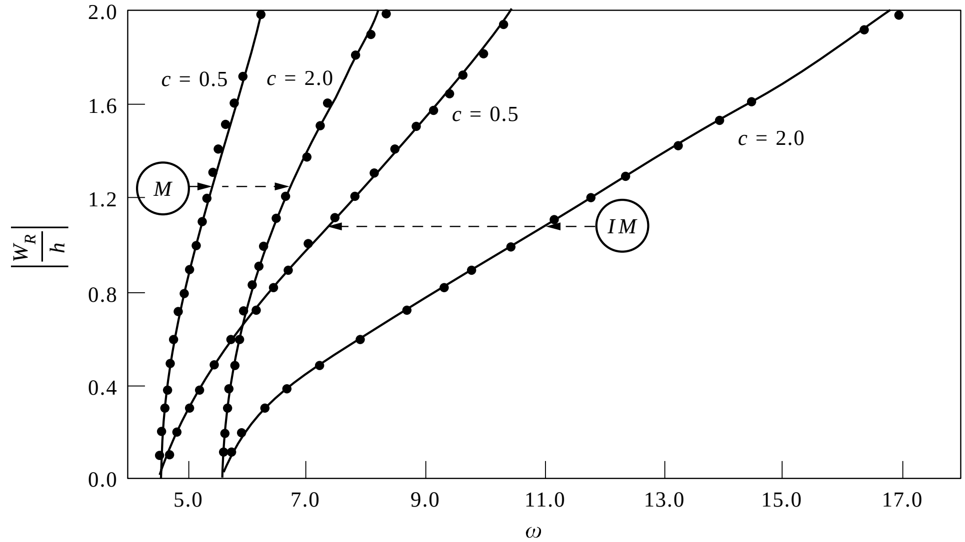

Figure 4. The comparisons of non-linear free frequency responses of the annular plate of radius ratio ($R = 0{\cdot}1$) and mass ratio ($\gamma = 1{\cdot}0$) between hinged movable—and immovable—boundary conditions.

5. CONCLUSIONS

The use of finite element method facilitated the solutions of the governing non-linear differential equations of motion. The method can easily be extended to investigate free or forced vibrations of a plate–mass under other boundary conditions. The non-linear behaviour of the plate–mass system under consideration is found to be dependent upon the specified elastic constant ratio. With the edge constraints considered, characteristics exhibited by the responses of the plate are similar to that of a hard-spring Duffing's system.

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APPENDIX A: NOMENCLATURE

r, θ, z	cylindrical co-ordinates used to describe the undeformed configuration of the plate
u, w	radial and transverse displacement of the middle plate, respectively
t	time variable
σ_θ, σ_r	stresses in the tangential and radial directions, respectively
$\varepsilon_\theta, \varepsilon_r$	strains in the tangential and radial directions, respectively
$a_\theta, a_{r\theta}, a_r$	the elastic constants
c	ratio of elastic constants $= a_\theta / a_r$
v	Poisson ratio $= -a_{r\theta}/a_r$

X, U	non-dimensional transverse and radial displacement, respectively
R	the ratio of the inner radius and the outer radius
D	flexural rigidity of the plate $= a_\theta \cdot h^3/12 \cdot (a_\theta \,.\, a_r - a_{r\theta}^2)$
M_c	concentric rigid mass
ρ	the material density of the plate
γ	the ratio of the concentric rigid mass and the mass of the plate with identical volume $= M_c/\pi \cdot b^2 \cdot \rho \cdot h$
τ	time variable

APPENDIX B: DERIVATION OF EQUATIONS

The derivation of equations (5) and (6) is as follows. By substituting equations (10) into equations (14), the strain energy can be written as

$$\Pi' = \frac{D}{c}\iint\left\{\frac{1}{2}\cdot w_{rr}^2 + \frac{1}{2}\cdot\frac{w_r^2}{r^2} + \frac{v}{r}w_r\cdot w_{rr}\right\}\cdot \mathrm{d}s + \frac{12\cdot D}{c\cdot h^2}\iint\left\{\frac{c}{2}u_r^2 + \frac{1}{2}\cdot\frac{u^2}{r^2} + \frac{v}{r}u\cdot u_r\right\}\cdot \mathrm{d}s, \tag{B1}$$

$$\Pi'' = \frac{12\cdot D}{c\cdot h^2}\iint\left\{\frac{c}{2}u_r\cdot w_r^2 + \frac{c}{8}\cdot w_r^4 + \frac{v}{2}\cdot\frac{u}{r}\cdot w_r^2\right\}\cdot \mathrm{d}s,$$

and total kinetic energy of the plate can be written as

$$T = \tfrac{1}{2}\rho\cdot h\cdot\iint\left(\frac{\partial w}{\partial t}\right)^2\cdot \mathrm{d}s + \tfrac{1}{2}M_c\cdot\left(\frac{\partial w}{\partial t}\right\}^2\Bigg|_{r=b}. \tag{B2}$$

From Hamilton's principle, one has

$$\delta\int_{t_2}^{t_1}(T - \Pi' - \Pi'')\cdot \mathrm{d}t = 0. \tag{B3}$$

After lengthy calculation and vanishing the first variation of the action integral, one has the following equation:

$$-\int_{t_2}^{t_1}\left\{2\pi\cdot\rho\cdot h\int_a^b\left(r\cdot\frac{\partial^2 w}{\partial t^2}\right)\cdot\delta w\cdot \mathrm{d}r\right\}\cdot \mathrm{d}t - \int_{t_2}^{t_1}\left\{M_c\cdot\left(\frac{\partial^2 w}{\partial t^2}\right)\cdot\partial w\Bigg|_b\right\}\cdot \mathrm{d}t$$

$$-\int_{t_2}^{t_1}\left\{2\pi\frac{D}{c}\int_a^b\left(c\cdot r\cdot w_{rrrr} + 2c\cdot w_{rrr} - \frac{1}{r}w_{rr} + \frac{1}{r^2}w_r\right)\cdot\delta w\cdot \mathrm{d}r\right\}\cdot \mathrm{d}t$$

$$+\int_{t_2}^{t_1}\left\{2\pi\frac{12\cdot D}{c\cdot h^2}\int_a^b(c\cdot u_r\cdot w_r + c\cdot r\cdot u_{rr}\cdot w_r + c\cdot r\cdot u_r\cdot w_{rr})\cdot\delta w\cdot \mathrm{d}r\right\}\cdot \mathrm{d}t$$

$$+\int_{t_2}^{t_1}\left\{2\pi\frac{12\cdot D}{c\cdot h^2}\int_a^b\left\{\frac{c}{2}w_r^3 + \frac{3}{2}c\cdot r\cdot w_r^2\cdot w_{rr} + v\cdot u\cdot w_{rr} + v\cdot u_r\cdot w_r\right)\cdot\delta w\cdot \mathrm{d}r\right\}\cdot \mathrm{d}t$$

$$+\int_{t_2}^{t_1}\left\{2\pi\frac{12\cdot D}{c\cdot h^2}\int_a^b\left\{c\cdot r\cdot u_{rr} + c\cdot u_r + \frac{c-v}{2}w_r^2 + c\cdot r\cdot w_r\cdot w_{rr} - \frac{u}{r}\right)\cdot\delta u\cdot \mathrm{d}r\right\}\cdot \mathrm{d}t$$

$$+\int_{t_2}^{t_1}\left\{2\pi\frac{D}{c}\left(c\cdot r\cdot w_{rrrr}+c\cdot w_{rr}-\frac{1}{r}w_r\right)\cdot\delta w\bigg|_a^b\right\}\cdot\mathrm{d}t$$

$$-\int_{t_2}^{t_1}\left\{2\pi\frac{12\cdot D}{c\cdot h^2}\left(c\cdot r\cdot u_r\cdot w_r+\frac{c}{2}r\cdot w_r^3+v\cdot u\cdot w_r\right)\cdot\delta w\bigg|_a^b\right\}\cdot\mathrm{d}t$$

$$-\int_{t_2}^{t_1}\left\{2\pi\frac{D}{c}(c\cdot r\cdot w_{rr}+v\cdot w_r)\cdot\delta w_r\bigg|_a^b\right\}\cdot\mathrm{d}t$$

$$-\int_{t_2}^{t_1}\left\{2\pi\frac{12\cdot D}{c\cdot h^2}\left(c\cdot r\cdot u_r+v\cdot u+\frac{c}{2}r\cdot w_r^2\right)\cdot\delta u\bigg|_a^b\right\}\cdot\mathrm{d}t=0. \tag{B4}$$

For equation (A4) to hold, the integrands in the integrals have to vanish separately. The governing equations are yielded as

$$\frac{\partial^2 w}{\partial t^2}+\frac{D}{c\cdot\rho\cdot h}\left(c\cdot w_{rrrr}+\frac{2\cdot c}{r}w_{rrr}-\frac{1}{r^2}w_{rr}+\frac{1}{r^3}w_r\right)$$

$$=\frac{12\cdot D}{c\cdot\rho\cdot h^3}\left(\frac{c}{r}u_r\cdot w_r+c\cdot u_{rr}\cdot w_r+c\cdot u_{rr}\cdot w_r+c\cdot u_r\cdot w_{rr}\right.$$

$$\left.+\frac{c}{2}\cdot\frac{1}{r}w_r^3+\frac{3}{2}c\cdot w_r^2\cdot w_{rr}+\frac{v}{r}uw_{rr}+\frac{v}{r}u_r\cdot w_r\right), \tag{B5}$$

$$c\cdot u_{rr}+\frac{c}{r}u_r+c\cdot w_r\cdot w_{rr}+\frac{c}{2}\cdot\frac{1}{r}w_r^2-\frac{1}{r^2}u-\frac{v}{2}\cdot\frac{1}{r}w_r^2=0, \tag{B6}$$

and the natural boundary conditions as

$$r\cdot\left(c\cdot w_{rr}+\frac{v}{r}\cdot w_r\right)\cdot\delta w_r\bigg|_a^b=0 \tag{B7a}$$

$$\left\{\frac{D}{c}\cdot r\cdot\left(c\cdot w_{rrr}+\frac{c}{r}\cdot w_{rr}-\frac{1}{r^2}w_r\right)-\frac{12\cdot D}{c\cdot h^2}\cdot r\cdot w_r\cdot\left(c\cdot u_r+\frac{c}{2}w_r^2+\frac{v}{r}\cdot u\right)\right\}\delta w\bigg|_a^b$$

$$+\frac{M_c}{2\cdot\pi}\left(\frac{\partial^2 w}{\partial t^2}\right)\cdot\delta w\bigg|_b=0, \tag{B7b}$$

$$r\cdot\left(c\cdot u_r+\frac{v}{r}\cdot u+\frac{c}{2}w_r^2\right)\cdot\delta u\bigg|_a^b=0. \tag{B7c}$$

Using the non-dimensional variables, the governing equations (B5) and (B6) are converted to the non-dimensional forms (4) and (5). Also by considering the physical conditions at each end of the plate, from the natural boundary conditions (B7a), (B7b) and (B7c) , one can yield the non-dimensional boundary conditions (6a), (6b) and (6c).