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ON THE POLYGONAL MEMBRANE WITH A CIRCULAR CORE

C. Y. Wang
Departments of Mathematics and Mechanical Engineering, Michigan State University, East Lansing, MI 48824, U.S.A.

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1. Introduction

Vibration of membranes is important in the theory of sound [1]. It is also related to the property of electromagnetic wave guides [2]. The resulting Helmholtz equation, however, has very few exact solutions. Thus, a variety of approximate methods has been proposed for different membrane geometries [3, 4]. Consider a polygonal membrane with a fixed circular core. This problem has been solved before by Laura *et al*. [5] in using conformal mapping and Galerkin approximation. The results for the fundamental frequencies were displayed graphically.

The purpose of the present paper is two-fold. First, we study in detail the asymptotic properties where the center core shrinks to a point, i.e., the membrane is "pinned" at the center. Second, we show that the point match method yields results that differ considerably from those of Laura *et al*. [5], especially when the fixed center core is large.

2. Circular membrane pinned at center

Consider an annular membrane with outer radius 1 and inner radius $\varepsilon \ll 1$. The characteristic equation is [1]

$$Y_0(k)J_0(k\varepsilon) - J_0(k)Y_0(k\varepsilon) = 0, \tag{1}$$

where k is the frequency normalized by length $\cdot \sqrt{\text{density/tension per length}}$. Now when $\varepsilon = 0$, equation (1) shows the fundamental frequency is the first root of $J_0(k)$, or $k = k_0 = 2{\cdot}4048$. We perturb from this value,

$$k = k_0 + \delta(\varepsilon)k_1 + o(\delta(\varepsilon)), \tag{2}$$

where $\delta(\varepsilon)\rightarrow 0$ as $\varepsilon\rightarrow 0$. Thus [6]

$$Y_0(k) = Y_0(k_0 + \delta k_1 + \cdots) = Y_0(k_0) - \delta k_1 Y_1(k_0) + \cdots, \tag{3}$$

$$Y_0(k\varepsilon) = \frac{2}{\pi}\left[\ln\frac{k\varepsilon}{2} + \gamma\right]J_0(k\varepsilon) + 0(\varepsilon^2), \tag{4}$$

where $\gamma = 0{\cdot}5772$. Equation (1) becomes

$$[Y_0(k_0) - \delta k_1 Y_1(k_0) + \cdots] - [J_0(k_0) - \delta k_1 J_1(k_0) + \cdots]\frac{2}{\pi}\left[\ln\varepsilon + \ln\frac{k_0}{2} + \gamma + \cdots\right] = 0. \tag{5}$$

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Balancing the leading orders, one finds

$$\delta(\varepsilon) = \frac{1}{|\ln \varepsilon|} \tag{6}$$

and

$$k_1 = \frac{\pi Y_0(k_0)}{2J_1(k_0)}. \tag{7}$$

Thus

$$k = k_0 + \frac{1}{|\ln \varepsilon|}\frac{\pi Y_0(k_0)}{2J_1(k_0)} + \cdots = 2{\cdot}4048 + 1{\cdot}54288\frac{1}{|\ln \varepsilon|} + \ldots . \tag{8}$$

One sees that as the inner radius ε shrinks to zero, the frequency is k_0, the same as the unrestrained circular membrane. However, for finite radius, the frequency rises quickly with ε since $(dk/d\varepsilon)|_{\varepsilon\to 0} = \infty$. Table 1 shows the comparison of k values as computed from equation (1) and the asymptotic approximation (8).

The analysis of the annular membrane shows that a pinpoint restraint ($\epsilon = 0$) does not affect the frequency of the circular membrane. We expect this property to hold for polygonal membranes as well.

3. RESULTS USING POINT MATCH

The method of eigenfunction expansion and point match has long been used in solving linear boundary value problems [7]. Previous applications of this method include that of Yee and Audeh [8] who considered an eccentric annular membrane. This method will be used on the polygonal membrane with a circular core. Normalize all lengths by the minimal outer radius. The solution to the Helmholtz equation in cylindrical co-ordinates is written as

$$w(r, \theta) = \sum_{n=0}^{\infty} A_n \cos(Mn\theta)\varphi_n(r), \tag{9}$$

where M is the number of sides of the (regular) polygon,

$$\varphi_n(r) \equiv Y_{Mn}(kb)J_{Mn}(kr) - J_{Mn}(kb)Y_{Mn}(kr) \tag{10}$$

and the zero-displacement boundary condition at $r = b$ has been satisfied. We truncate the series to $N+1$ terms and select $N+1$ points on the outer boundary:

$$\theta_j = \frac{(j - 0{\cdot}5)}{(N+1)}\frac{\pi}{M}, \qquad j = 1 \quad \text{to} \quad N+1, \tag{11}$$

TABLE 1

Frequency of the circular membrane pinned at center

ε	k (exact)	Equation (8)
0·1	3·314	3·075
0·01	2·801	2·740
0·001	2·654	2·628
0·0001	2·587	2·572
0·00001	2·548	2·539
0	2·4048	2·4048

TABLE 2
Convergence of k (M = 6)

N	$b = 0{\cdot}1$	$b = 0{\cdot}5$	$b = 0{\cdot}9$
5	3·173	5·759	16·28
10	3·173	5·760	16·36
15	3·173	5·760	16·37
20	3·173	5·760	16·37

TABLE 3
The frequency for small b (M = 6)

b	$1/\lvert\ln b\rvert$	k
0·01	0·2171	2·695
0·001	0·1468	2·556
0·0001	0·1086	2·491
0·00001	0·08086	2·454
0	0	2·32

TABLE 4
Fundamental frequencies k

b/M	4	6	8	∞
0	2·221	2·317	2·353	2·405
0·001	2·45	2·56	2·60	2·65
0·1	3·01	3·17	3·23	3·31
0·3	3·88	4·17	4·27	4·41
0·5	5·09	5·76	5·97	6·25
0·7	6·77	8·89	9·63	10·46
0·9	9·51	16·37	21·36	31·41
1	11·86	27·15	47·79	∞

where the remaining boundary condition on the polygon is to be satisfied:

$$0 = \sum_{n=0}^{N} A_n \cos(Mn\theta_j)\varphi_n\left(\frac{1}{\cos\theta_j}\right), \qquad j = 1 \quad \text{to} \quad N + 1. \tag{12}$$

The determinant of the coefficients of A_n is set to zero. A simple root search program yields the eigenvalue k. The accuracy can be determined by increasing N. Consider, for example, the hexagonal membrane ($M = 6$). Table 2 shows that convergence occurs when N is about 15.

Guided by the analysis of the annular membrane, values of k versus $1/|\ln b|$ were computed and are shown in Table 3.

Extrapolating to zero, one finds $k_0 = 2{\cdot}32$ for $b = 0$ which is close to the value of 2·317 for the simply connected hexagonal membrane obtained by Conway [9]. The slope or value of k_1 is 1·54.

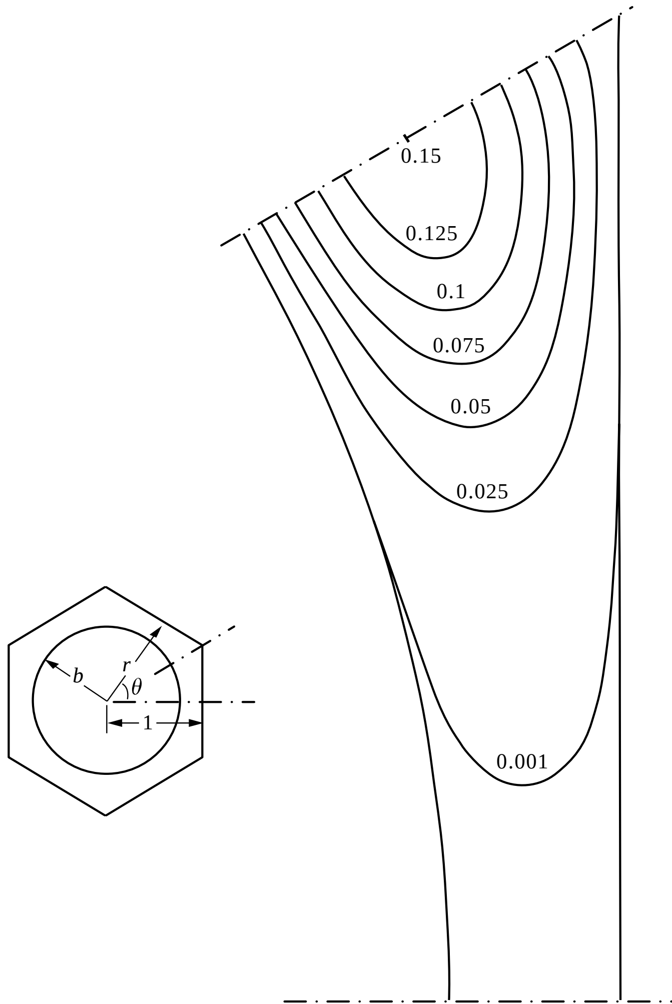

Figure 1. The level curves for $M = 6$, $b = 0{\cdot}9$. Only 1/12 of the domain is shown. Inset shows the full geometry.

In contrast, reference [5] gave a value of 2·7852 for $b = 0$ which is 20% higher. The graphs of reference [5] also show $k = 3{\cdot}25$ for $b = 0{\cdot}1$, $k = 5{\cdot}6$ for $b = 0{\cdot}5$, and $k = 21{\cdot}3$ for $b = 0{\cdot}9$, the latter has a difference of 30% as compared with our computed values.

Another test is how well the eigenfunction satisfies the governing equation and the boundary conditions. Using the k value found, A_0 was set to equal 1 in equation (12) and N equations and unknowns for A_n were solved. The only place equation (9) is not exact is on the polygonal boundary. For $M = 6$, $N = 15$ and $b = 0{\cdot}9$, the maximum error in w was found to be less than 1%. Figure 1 shows the detailed level curves.

Instead of a graph, our results are tabulated for ease of possible comparison with those of other researchers (Table 4).

For $b = 0$, the $k = k_0$ values for $M = 4$, ∞ are from exact theory, while those of $M = 6$, 8 agree with point match [9] and conformal mapping [10]. For $b \approx 0$, one can use the asymptotic formula $k = k_0 + k_1/|\ln b|$, where k_1 are all about 1·54. The $M = \infty$ column is the value for the annular membrane obtained from equation (1). Also note for $b = 1$ (inner circle touching outer polygon) the value of k is finite for finite M, since the corner pieces have non-zero area.

4. CONCLUSIONS

For certain problems the point-match method may not converge [11]. But for the problems where convergence does occur, the method is efficient and accurate, as demonstrated in this paper. It also yields detailed field profiles. The fundamental frequency of a polygonal membrane with a circular core is found successfully for all ranges of the core radius ratio b. Our results, however, differ from those of Laura *et al*. [5] for both large and small values of b.

The singular nature of the effect of a point constraint on a vibrating membrane is noted. If the constraint size is infinitesimally small, the frequency is surprisingly the same as the unconstrained membrane. But for any finite increase in constraint size, the frequency increases very rapidly at first, then more slowly.

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