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ON THE APPLICATION OF IHB TECHNIQUE TO THE ANALYSIS OF NON-LINEAR OSCILLATIONS AND BIFURCATIONS

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The main objective of this paper is to introduce certain refinements and alternative formulations, which enhance the applicability and availability of the intrinsic harmonic balancing technique. This is achieved by considering certain illustrative examples concerning non-linear oscillations and dynamic bifurcation phenomena. Indeed, the bifurcation behaviour of a harmonically excited non-autonomous system is analyzed conveniently, with reference to the corresponding autonomous system, by applying the IHB technique, which yields the bifurcation equation as an integral part of the perturbation procedure. A symbolic computer language, namely MAPLE, facilitates the analysis as well as the verification of the ordered approximations to the solutions. The methodology lends itself to MAPLE readily, which in turn, enhances the applicability of the IHB technique.

1. INTRODUCTION

Non-linear oscillations and bifurcation problems can be analyzed via a variety of methods [1–3] such as averaging techniques, multiple time scaling, harmonic balancing, etc. Averaging method, for example, yields a lowest order approximation conveniently, but higher order calculations become lengthy and complicated. The method of harmonic balancing is conceptually simple and it leads to algebraic equations only; however, the results may be inconsistent [3].

The intrinsic harmonic balancing technique was introduced earlier [4–6] in order to overcome certain observed inconsistencies in the application of the conventional harmonic balancing method. The method has been applied effectively to the analysis of non-linear vibrations and dynamic bifurcation problems systematically. In addition, the basic concepts and the methodology of the IHB technique is generalized and adopted for the analysis of non-linear forced oscillations [7, 8].

Nevertheless, it seems that there are a number of issues regarding the perturbation procedure that require further clarification and refinement. This is true for non-linear

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oscillations as well as bifurcation problems. In the former case, for example, the evaluation of perturbation equations for the non-linear system may be referred to the linearized system or directly to the origin of the system in which case the evaluations of the ordered perturbation equations can be performed rapidly and conveniently. In this paper, alternative formulations, certain clarifications, and variations of the methodolgy are discussed with the aid of illustrative examples. The bifurcation analysis of a non-autonomous system is performed through the IHB technique and alternative formulations are discussed. It is observed that the IHB technique, as applied to this bifurcation problem here, provides a very simple treatment compared to other methods. It is expected that the exposition presented in this paper will enhance the application of the IHB technique to a variety of specific problems in many fields.

2. AUTONOMOUS SYTEMS

Consider an autonomous system generally described by

$$\ddot{x} + g(x, \dot{x}, \varepsilon) = 0, \tag{1}$$

where g is a polynomial function of x, $\dot{x}$ and ε; dots on x indicate differentiation with respect to time and $g(x, \dot{x}, 0)$ gives the corresponding linear system. The solution of equation (1) is sought in a parametric form $x = x(t, \varepsilon)$ where ε is a small parameter.

A series of perturbation equations can be obtained by introducing the assumed solution $x = x(t, \varepsilon)$ back into equation (1), differentiating with respect to ε successively, and evaluating these equations at $\varepsilon = 0$:

$$\varepsilon^0\!: \ddot{x} + g = 0,$$

$$\varepsilon^1\!: \ddot{x}' + g_x x' + g_{\dot{x}} \dot{x}' + g_\varepsilon = 0,$$

$$\varepsilon^2\!: \ddot{x}'' + [g_{xx} x' + g_{x\dot{x}} \dot{x}' + 2g_{x\varepsilon}]x' + g_x x'' + [g_{\dot{x}x} x' + g_{\dot{x}\dot{x}} \dot{x}' + 2g_{\dot{x}\varepsilon}]\dot{x}' + g_{\dot{x}} \dot{x}'' + g_{\varepsilon\varepsilon} = 0, \tag{2}$$

etc.,

where the primes and subscripts on g denote differentiation with respect to ε and related variables, respectively, and all perturbation equations are evaluated at $\varepsilon = 0$.

Further, the assumed solution $x = x(t, \varepsilon)$ can be represented by a Fourier series of the form [4–6]

$$x(t, \varepsilon) = \sum_{m=0}^{M} [p_m(\varepsilon) \cos(m\omega(\varepsilon)t) + r_m(\varepsilon) \sin(m\omega(\varepsilon)t)], \tag{3}$$

which is substituted back into the perturbation equations sequentially. At each step, balancing the harmonics, one obtains the derivatives of the amplitudes to construct the Taylor's expansion of the amplitudes to a desired order as

$$p_m(\varepsilon) = p_m^0 + p_m' \varepsilon + \tfrac{1}{2} p_m'' \varepsilon^2 + O(\varepsilon^3),$$

$$r_m(\varepsilon) = r_m^0 + r_m' \varepsilon + \tfrac{1}{2} r_m'' \varepsilon^2 + O(\varepsilon^3),$$

and the solutions are envisaged as

$$x(t, \varepsilon) = x_0(t, 0) + \varepsilon x'(t, 0) + \ldots, \tag{4}$$

where $x_0(t, 0)$ is the solution of the linearized equation.

However, in some specific problems, evaluating the perturbation equations at the origin is more convenient than evaluations at the solutions of the linearized systems, although this may require a further step in the perturbation process. This can be done by introducing appropriate scales such that the solution is expressed in the form of

$$x(t, \varepsilon) = \varepsilon x'(t, 0) + \frac{1}{2!}\varepsilon^2 x''(t, 0) + \cdots, \tag{5}$$

where $\varepsilon = 0$ identifies the *origin*.

Obviously, the non-linear system is now referred to the origin rather than to the solution of the linearized system since $x(t, 0) = 0$ and one has $p_m(0) = r_m(0) = 0$ in the Fourier series (3).

As an example, consider a system described by

$$\mathrm{d}x^2/\mathrm{d}t^2 + x = a + \varepsilon x^2, \tag{6}$$

subject to initial conditions $x(0) = A$, $x_t(0) = 0$. Here ε is a small positive parameter, $\varepsilon = 0$ giving the linearized equation. System (6) was solved before [4, 6, 9] on the basis of the linearized equation.

In order to obtain the solution in the form of equation (5), the perturbation procedure may be facilitated by introducing certain scaling as

$$a = \varepsilon b, \qquad A = \varepsilon B. \tag{7}$$

Further, one may introduce the time scaling $\tau = \omega(\varepsilon)t$, in which case equation (6) takes the form

$$\omega^2 x_{\tau\tau} + x = \varepsilon b + \varepsilon x^2, \tag{8}$$

where the subscript τ indicates differentiation with respect to τ. The periodic solution may be expressed in a parametric form in terms of ε,

$$x = x(\tau; \varepsilon), \qquad \omega = \omega(\varepsilon),$$

and the assumed solution will be in the form of equation (3) with

$$p_m(0) = r_m(0) = 0, \qquad \forall m,$$

since $x(t, 0) = 0$.

The solution is now 2π-periodic in τ, with $\omega(0) = \omega_c = 1$. A sequence of perturbation equations can now be generated by differentiating equation (8) with respect to ε and evaluating the derivatives at $\varepsilon = 0$. Thus, one obtains

$$x'_{\tau\tau} + x' = b \qquad \text{(first order)}, \tag{9}$$

$$4\omega' x'_{\tau\tau} + x''_{\tau\tau} + x'' = 0 \qquad \text{(second order)}, \tag{10}$$

$$6\omega'' x'_{\tau\tau} + 6\omega' x''_{\tau\tau} + 6(\omega')^2 x'_{\tau\tau} + x'''_{\tau\tau} + x''' = 6(x')^2 \qquad \text{(third order)}, \tag{11}$$

etc.

Substituting the assumed solution (3) into equation (9), one obtains

$$\sum_{m=0}^{M} [(1 - m^2)p'_m \cos(m\tau) + (1 - m^2)r'_m \sin(m\tau)],$$

and balancing the harmonics yields the derivatives $p'_0 = b$ and $p'_m = r'_m = 0$, $m \geqslant 2$, which give

$$x'(\tau, 0) = p'_0 + p'_1 \cos(\tau) + r'_1 \sin(\tau).$$

Substituting initial conditions $x'(0, 0) = B$ and $x'_\tau(0, 0) = 0$ yields $p'_1 = B - b$ and $r'_1 = 0$. Using the above information one has

$$x'(\tau, 0) = b + (B - b)\cos\tau$$

and the first order solution can be written as

$$x(\tau, \varepsilon) = \varepsilon b + \varepsilon(B - b)\cos\tau + O(\varepsilon^2).$$

Similarly, substituting the assumed solution and the first order solution $x'(\tau, 0)$ into the second order perturbation equation (10) and balancing all the harmonics, one obtains ω' and all the coefficients except p''_1 and r''_1 as zero. Further, using the initial conditions $x''(0, 0) = 0$ and $x''_\tau(0, 0) = 0$ yields $p''_1 = r''_1 = 0$, which results in $x''(\tau, 0) = 0$.

Keeping in mind the scaling introduced to a and A, for the first order approximation one should go to the third perturbation. Substituting the assumed solution, first and second order solutions ($x'(\tau, 0)$ and $x''(\tau, 0)$) once more into the third order perturbation equation (11) and balancing all the harmonics, one obtains the non-zero coefficients:

$$p'''_0 = 6b^2 + 3(B - b)^2, \qquad p'''_2 = -(B - b)^2, \qquad \omega'' = -2b,$$

and using the third derivative of amplitudes yields

$$x'''(\tau, 0) = 6b^2 + 3(B - b)^2 + p'''_1\cos(\tau) + (B - b)^2\cos(2\tau) + r'''_1\sin(\tau),$$

where p'''_1 and r'''_1 are obtained from the initial conditions $x'''(0, 0) = 0$ and $x'''_\tau(0, 0) = 0$ as

$$p'''_1 = -[6b^2 + 2(B - b)^2] \quad \text{and} \quad r'''_1 = 0,$$

respectively.

The solution, which is expressed as

$$x(\tau, \varepsilon) = \varepsilon x'(\tau; 0) + \frac{1}{2!}\varepsilon^2 x''(\tau; 0) + \frac{1}{3!}\varepsilon^3 x'''(\tau; 0) + O(\varepsilon^4), \tag{12}$$

now takes the form

$$x(\tau, \varepsilon) = a + (A - a)\cos\tau + \varepsilon\left[\frac{(A - a)^2}{2} + a^2 - \left(a^2 + \frac{1}{3}(A - a)^2\right)\cos\tau - \frac{(A - a)^2}{6}\cos(2\tau)\right] + O(\varepsilon^2), \tag{13}$$

after rescaling (i.e., returning to the original variables). Moreover, approximation to the $\omega(\varepsilon)$ can be written as

$$\omega(\varepsilon) = 1 - a\varepsilon + O(\varepsilon^2).$$

Note that equation (12) is the third order solution of the scaled system and after rescaling, the corresponding solution becomes equation (13).

System (6) is weakly non-linear and a uniformly consistent solution can be obtained by referring to the origin or to the solution of corresponding linear system. However, if the system is strongly non-linear, in other words, if ε is replaced by c which is not small, further scaling will be necessary in order to obtain a solution which is referred to the linearized system or to the origin. If scalings similar to equations (7) are introduced, the non-linear system will be referred to the origin. For example consider the system

$$\mathrm{d}x^2/\mathrm{d}t^2 + x = a + cx^2, \tag{14}$$

subject to initial conditions: $x(0) = A$, $x_t(0) = 0$. Here, c is not small and cannot be treated as a perturbation parameter. A small unidentified parameter should be introduced in order to facilitate the perturbation procedure and similar scalings as in equations (7) are *necessary* for the application of the IHB technique. Let the unidentified small parameter be μ, and introduce the scaling $a = \mu b$, $A = \mu B$, together with the time scaling $\tau = \omega(\mu)t$. Then equation (14) becomes

$$\omega^2 x_{\tau\tau} + x = \mu b + cx^2. \tag{15}$$

Upon writing equation (14) in the first order form $\dot{x} = y$, $\dot{y} = a - x + cx^2$, equilibrium points can be obtained as $y = 0$ and the roots of $cx^2 - x + a = 0$. Periodic solutions may exist if the roots of $cx^2 - x + a = 0$ are real. In addition, attention here is focused on small vibrations in the vicinity of an equilibrium point (centre), so that for an asymptotic solution A and a can be assumed to be small. Then, the procedure of the IHB technique, as described above, is followed and the periodic solution of system (14) is obtained as

$$\begin{aligned} x(\tau, \mu) = {} & [b + (B - b)\cos(\tau)]\mu + [cb^2 + \tfrac{1}{2}c(B-b)^2 - (cb^2 + \tfrac{1}{3}c(B-b)^2)\cos(\tau) \\ & - (\tfrac{1}{6}c(B-b)^2)\cos(2\tau)]\mu^2 + [2c^2b^3 + c^2b(B-b)^2 - c^2b^2(B-b) - \tfrac{1}{3}c^2(B-b)^3 \\ & + \tfrac{1}{6}(-12c^2b^3 - 4c^2b(B-b)^2 + 4c^2b^2(B-b) + \tfrac{29}{24}c^2(B-b)^3)\cos(\tau) \\ & + \tfrac{1}{6}(-2c^2b(B-b)^2 + 2c^2b^2(B-b) + \tfrac{2}{3}c^2(B-b)^3)\cos(2\tau) \\ & + \tfrac{1}{48}c^2(B-b)^3\cos(3\tau)]\mu^3. \end{aligned} \tag{16}$$

Also, the amplitude–frequency relation can be constructed as

$$\omega(\mu) = 1 - cb\mu - \tfrac{1}{2}(3c^2b^2 + \tfrac{5}{6}c^2(B-b)^2)\mu^2. \tag{17}$$

Solution (16) may also be obtained by referring the system to the linearized equation by introducing the additional scaling $x \to \mu z$, so that system (15) may be written as

$$\omega^2 z_{\tau\tau} + z = b + \mu c z^2, \tag{18}$$

and the initial conditions become $z(0) = B$, $z_t(0) = 0$. One obtains the solution of system (18) by substituting the assumed solution (3) sequentially into the zero order, first order, etc., perturbation equations, and an approximate solution $z(\tau, \mu)$ consistent to a desired order can be constructed. The solution of the original system can be written after back scaling (returning to the original variables), $\mu z \to x$, which leads to the same solution (16).

The solution obtained by referring the system to the linearized equation reduces the number of perturbations and it seems an advantage. However, the former solution procedure, where the perturbation equations are evaluated at the origin is more convenient since the solution is referred to the origin as mentioned before.

2.1. VERIFICATION OF SOLUTIONS

The solutions obtained above are given in an ordered form of approximations. To verify the solution, one direct way is to substitute this solution back into the original differential equation [10]. In general, if the Nth order approximate solution is substituted back into the original equation and it yields a result in the order $O(\epsilon^{N+1})$, i.e., $x = \cdots + O(\epsilon^{N+1})$, then the Nth order solution contains all possible contributions to this order. In other words, the solution is a consistent approximation. If substitution, which contains lower order terms than $N+1$, the approximation is inconsistent.

The third order asymptotic solution (12) is verified by substituting the solution into equation (8) with the aid of MAPLE, which yields a result of $O(\varepsilon^4)$. Similarly, the first order solution (13), obtained after returning to the original variables (rescaling) is substituted into the original equation (6) and a result of $O(\varepsilon^2)$ is obtained. The approximate solution (16) of the system (15) is verified by substituting the solution (16) and $\omega(\mu)$ back into equation (15) and a result of $O(\mu^4)$ is obtained, confirming the consistency of the approximation (16).

3. NON-AUTONOMOUS SYSTEMS

The IHB technique has been adapted for the analysis of non-linear forced oscillations [7, 8, 11]. Here, the technique will be applied to a bifurcation problem under external harmonic excitation. The analysis will be carried out without the aid of multiple time scaling [11], and a number of important aspects of the procedure will be pointed out. Consider a specific, harmonically excited (non-autonomous) bifurcation problem given by

$$\ddot{x} + \omega_c^2 x - \varepsilon(\eta - x^2)\dot{x} = F \sin(\Omega t), \tag{19a}$$

where $\varepsilon > 0$ is a small parameter. It is assumed that natural frequency ω_c and external frequency Ω satisfy the non-resonance relationship $l_1\omega_c + l_2\Omega \neq 0$ where l_1, l_2 are any positive or negative integers. This system exhibits bifurcations from a periodic solution to a two-frequency quasi-periodic solution. The system is weakly non-linear and the solutions can be obtained without introducing scaling, by referring the non-linear system to the linearized one. If one uses the scale $F \to \varepsilon C$, the solution will be referred to the origin. However, the application here will be carried out by referring the system to the linearized equation without introducing any scaling.

Obviously, $F = 0$ gives the corresponding autonomous system and $x = 0$ is the equilibrium position. System (19a) with $F = 0$ exhibits Hopf bifurcation as η passes through zero. In order to demonstrate this one can write the above second order system as the first order one

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\omega_c^2 x_1 + \varepsilon(\eta - x_1^2)x_2 + F\sin(\Omega t). \tag{19b}$$

The state defined by the origin, $x_1 = x_2 = 0$, is now identified as an equilibrium state of the $F = 0$ system. The Jacobian is evaluated at the origin, and its eigenvalues are given by

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_c^2 & \varepsilon\eta \end{bmatrix}$$

and

$$\lambda_{1,2} = \frac{\varepsilon\eta \pm \sqrt{\varepsilon^2\eta^2 - 4\omega_c^2}}{2},$$

respectively.

Clearly, $\eta < 0$ gives complex conjugate eigenvalues with negative real part and when η is greater than zero real part of eigenvalues become positive. At $\eta = \eta_c = 0$ the pair of complex conjugate eigenvalues becomes an imaginary pair and Hopf bifurcation occurs.

The family of periodic or quasi-periodic solutions is expressed in the parametric form

$$x = x(\tau_1, \tau_2, \varepsilon), \qquad \eta = \eta(\varepsilon), \qquad \omega = \omega(\varepsilon), \tag{20}$$

where $\tau_1 = \Omega t$ and $\tau_2 = \omega(\varepsilon)t$ are introduced so that equation (19a) becomes

$$\Omega^2 x_{11} + 2\Omega\omega x_{12} + \omega^2 x_{22} + \omega_c^2 x - \varepsilon(\eta - x^2)(\omega x_2 + \Omega x_1) = F \sin \tau_1. \tag{21}$$

Subscripts 1 and 2 on x indicate differentiation with respect to τ_1 and τ_2, respectively. Further, one may assume a solution in the form of a generalized Fourier series (with two frequencies),

$$x(\tau_1, \tau_2; \varepsilon) = \sum_{\substack{m=0 \\ m_1+m_2=m}}^{M} p_{m_1,m_2}(\varepsilon) \cos(m_1\tau_1 + m_2\tau_2) + r_{m_1,m_2}(\varepsilon) \sin(m_1\tau_1 + m_2\tau_2), \tag{22}$$

where m_1 and m_2 may be chosen positive or negative; and M is an arbitrary positive integer. It is noted that equation (22) reduces to the ordinary Fourier series in the case of $m_1 \equiv 0$ or $m_2 \equiv 0$. $m_1 \equiv 0$ describes periodic solutions of the associated autonomous system while $m_2 \equiv 0$ denotes periodic solutions which are purely excited by the external force $F \sin(\Omega t)$. As a matter of fact, assumption (22) embraces equilibrium, and periodic solutions as well as quasi-periodic motions, thus enabling one to identify bifurcations from one solution to the other.

A series of perturbation equations is obtained by substituting equation (20) into equation (21), differentiating with respect to ε and evaluating at $\varepsilon = 0$. It is noted that $\varepsilon = 0$ is the corresponding linear system so that one obtains zeroth, first, second, etc., order perturbation equations as

$$\Omega^2 x_{11} + 2\Omega\omega_c x_{12} + \omega_c^2 x_{22} + \omega_c^2 x = F \sin(\tau_1) \quad \text{(0th order)}, \tag{23}$$

$$\Omega^2 x'_{11} + 2\Omega\omega' x_{12} + 2\Omega\omega_c x'_{12} + 2\omega_c\omega' x_{22} + \omega_c^2 x'_{22} + \omega_c^2 x' - (\eta - x^2)(\omega_c x_2 + \Omega x_1) = 0 \quad \text{(1st order)}, \tag{24}$$

etc.,
where primes indicate differentiation with respect to ε evaluated at $\varepsilon = 0$.

Substituting the series solution (22) into equation (23) and balancing the harmonics yields the non-zero coefficients as

$$p^0_{01} = p_{01}(0) \neq 0, \qquad r^0_{01} = r_{01}(0) \neq 0, \qquad r^0_{10} = r_{10}(0) = F/(\omega_c^2 - \Omega^2),$$

which yield

$$x^0(\tau_1, \tau_2; 0) = p^0_{01} \cos \tau_2 + r^0_{01} \sin \tau_2 + \frac{F}{(\omega_c^2 - \Omega^2)} \sin \tau_1.$$

Here, it is understood that the solution is envisaged as

$$x(\tau_1, \tau_2, \varepsilon) = x^0(\tau_1, \tau_2, 0) + \varepsilon x'(\tau_1, \tau_2, 0) + O(\varepsilon^2).$$

Similarly, substituting the assumed solution and the zeroth order solution $x^0(\tau_1, \tau_2, 0)$ into the first order perturbation equation (24) and balancing the harmonics one obtains the following non-zero coefficients.

$$\cos(2\tau_1 - \tau_2) \Rightarrow p'_{2-1} = \frac{F^2 r^0_{01}(\omega_c - 2\Omega)}{4(\omega_c^2 - \Omega^2)^2(4\Omega\omega_c - 4\Omega^2)},$$

$$\sin(2\tau_1 - \tau_2) \Rightarrow r'_{2-1} = \frac{F^2 p^0_{01}(\omega_c - 2\Omega)}{4(\omega_c^2 - \Omega^2)^2(4\Omega\omega_c - 4\Omega^2)},$$

$$\cos(-\tau_1 + 2\tau_2) \Rightarrow p'_{-12} = \frac{F[(p^0_{01})^2 - (r^0_{01})^2](2\omega_c - \Omega)}{4(\omega_c^2 - \Omega^2)(4\omega_c\Omega - \Omega^2 - 3\omega_c^2)},$$

$$\sin(-\tau_1+2\tau_2) \Rightarrow r'_{-12} = \frac{2Fp_{01}^0 r_{01}^0(2\omega_c-\Omega)}{4(\omega_c^2-\Omega^2)(4\omega_c\Omega-\Omega^2-3\omega_c^2)},$$

$$\cos(\tau_1) \Rightarrow p'_{10} = \frac{\eta\Omega F}{(\omega_c^2-\Omega^2)^2} - \frac{F^3\Omega}{4(\omega_c^2-\Omega^2)^4} - \frac{[(p_{01}^0)^2+(r_{01}^0)^2]\Omega F}{2(\omega_c^2-\Omega^2)^2},$$

$$\cos(2\tau_1+\tau_2) \Rightarrow p'_{21} = \frac{F^2 r_{01}^0(2\Omega+\omega_c)}{4(\omega_c^2-\Omega^2)^2(-4\Omega^2-4\Omega\omega_c)},$$

$$\sin(2\tau_1+\tau_2) \Rightarrow r'_{21} = \frac{-F^2 p_{01}^0(2\Omega+\omega_c)}{4(\omega_c^2-\Omega^2)^2(-4\Omega^2-4\Omega\omega_c)},$$

$$\cos(\tau_1+2\tau_2) \Rightarrow p'_{12} = \frac{F[-(p_{01}^0)^2+(r_{01}^0)^2](2\omega_c+\Omega)}{4(\omega_c^2-\Omega^2)(-4\omega_c\Omega-\Omega^2-3\omega_c^2)},$$

$$\sin(\tau_1+2\tau_2) \Rightarrow r'_{12} = \frac{-2Fp_{01}^0 r_{01}^0(2\omega_c+\Omega)}{4(\omega_c^2-\Omega^2)(-4\omega_c\Omega-\Omega^2-3\omega_c^2)},$$

$$\cos(3\tau_1) \Rightarrow p'_{30} = \frac{\Omega}{4(\omega_c^2-9\Omega^2)}\left(\frac{F}{(\omega_c^2-\Omega^2)}\right)^3,$$

$$\cos(3\tau_2) \Rightarrow p'_{03} = \frac{-(r_{01}^0)^3+3(p_{01}^0)^2 r_{01}^0}{32\omega_c},$$

$$\sin(3\tau_2) \Rightarrow r'_{03} = \frac{-(p_{01}^0)^3+3p_{01}^0(r_{01}^0)^2}{32\omega_c}.$$

Note that, amplitudes $p_{01}(\varepsilon)$ and $r_{01}(\varepsilon)$ are envisaged in the form of Taylor's expansions given by

$$p_{01}(\varepsilon) = p_{01}^0 + p'_{01}\varepsilon + \frac{1}{2!}p''_{01}\varepsilon^2 + O(\varepsilon^3), \qquad r_{01}(\varepsilon) = r_{01}^0 + r'_{01}\varepsilon + \frac{1}{2!}r''_{01}\varepsilon^2 + O(\varepsilon^3).$$

Depending on the ordered form of the results, $p_{01}(\varepsilon)$ and as $r_{01}(\varepsilon)$ may be represented by the first, second, etc., order terms in the above expansions. The important point here is to keep the consistency of approximations with regard to the ordered form of the solutions.

Thus, upon using the above information, the first perturbation gives

$$\begin{aligned} x'(\tau_1,\tau_2;0) = {} & p'_{2-1}\cos(2\tau_1-\tau_2) + r'_{2-1}\sin(2\tau_1-\tau_2) + p'_{-12}\cos(-\tau_1+2\tau_2) \\ & + r'_{-12}\sin(-\tau_1+2\tau_2) + p'_{10}\cos(\tau_1) + p'_{21}\cos(2\tau_1+\tau_2) \\ & + r'_{21}\sin(2\tau_1+\tau_2) + p'_{12}\cos(\tau_1+2\tau_2) + r'_{12}\sin(\tau_1+2\tau_2) \\ & + p'_{30}\cos(3\tau_1) + p'_{03}\cos(3\tau_2) + r'_{03}\sin(3\tau_2), \end{aligned}$$

and the first order approximation for the solution of equation (19a) can be expressed as

$$\begin{aligned} x(\tau_1,\tau_2;\varepsilon) = {} & p_{01}\cos(\tau_2) + r_{01}\sin(\tau_2) + r_{10}^0\sin(\tau_1) \\ & + \varepsilon\{p'_{2-1}\cos(2\tau_1-\tau_2) + r'_{2-1}\sin(2\tau_1-\tau_2) + p'_{-12}\cos(-\tau_1+2\tau_2) \\ & + r'_{-12}\sin(-\tau_1+2\tau_2) + p'_{10}\cos(\tau_1) + p'_{21}\cos(2\tau_1+\tau_2) + r'_{21}\sin(2\tau_1+\tau_2) \\ & + p'_{12}\cos(\tau_1+2\tau_2) + r'_{12}\sin(\tau_1+2\tau_2) + p'_{30}\cos(3\tau_1) + p'_{03}\cos(3\tau_2) \\ & + r'_{03}\sin(3\tau_2). \end{aligned} \tag{25}$$

On the other hand, comparing the coefficients of $\cos(\tau_2)$ and $\sin(\tau_2)$ leads to

$$\cos(\tau_2) \Rightarrow \frac{1}{2}\frac{F^2 r_{01}^0}{(\omega_c^2 - \Omega^2)^2} - \eta r_{01}^0 + \frac{1}{4}(p_{01}^0)^2 r_{01}^0 + \frac{1}{4}(r_{01}^0)^3 - 2\omega' p_{01}^0 = 0, \tag{26}$$

$$\sin(\tau_2) \Rightarrow -\frac{1}{2}\frac{F^2 p_{01}^0}{(\omega_c^2 - \Omega^2)^2} + \eta p_{01}^0 - \frac{1}{4}p_{01}^0(r_{01}^0)^2 - \frac{1}{4}(p_{01}^0)^3 - 2\omega' r_{01}^0 = 0, \tag{27}$$

which can be used to construct

$$\frac{1}{2\sqrt{(p_{01}^0)^2 + (r_{01}^0)^2}} * \{-p_{01}^0 * [\text{equation (26)}] - r_{01}^0 * [\text{equation (27)}]\} = 0,$$

$$\frac{1}{2\sqrt{(p_{01}^0)^2 + (r_{01}^0)^2}} * \{-r_{01}^0 * [\text{equation (26)}] + p_{01}^0 * [\text{equation (27)}]\} = 0.$$

The first equation above yields

$$a\omega' = 0, \tag{28}$$

and the second equation gives the important result

$$\frac{a}{2}\left[\eta - \frac{1}{4}a^2 - \frac{1}{2}\left(\frac{F}{(\omega_c^2 - \Omega^2)}\right)^2\right] = 0, \tag{29}$$

where $a = \sqrt{(p_{01}^0)^2 + (r_{01}^0)^2}$.

As noted earlier, here p_{01}^0 and r_{01}^0 represent $p_{01}(\varepsilon)$ and $r_{01}(\varepsilon)$, respectively, keeping the consistency of approximations. Equation (29) yields two distinct, steady state solutions:

$$\text{Solution (I)}, \qquad a = 0$$

and

$$\text{Solution (II)}, \qquad \eta = \frac{1}{4}a^2 + \frac{1}{2}\left(\frac{F}{(\omega_c^2 - \Omega^2)}\right)^2.$$

A careful inspection of equation (25), with the aid of the derivatives of the amplitudes listed above, reveals that $a \equiv 0$ represents a periodic solution (with frequency Ω) and Solution (II) represents quasi-periodic motions on an invariant torus (with frequencies Ω and ω). Indeed, Solution (I) is described by

$$x(\tau_1; \varepsilon) = r_{10}^0 \sin(\tau_1) + \varepsilon\{\bar{p}'_{10}\cos(\tau_1) + p'_{30}\cos(3\tau_1)\},$$

where $\bar{p}'_{10}$ is p'_{10} with $p_{01}^0 = r_{01}^0 = 0 (a = 0)$ and Solution (II) by equation (25) with $a \neq 0$. On a plot of a versus η (see Figure 1), one clearly observes a bifurcation phenomenon, Solution (II) bifurcating from Solution (I) at the critical point

$$\eta_c = \frac{1}{2}\left(\frac{F}{(\omega_c^2 - \Omega^2)}\right)^2. \tag{30}$$

On the other hand, equation (28) yields $\omega' = 0$ for $a \neq 0$, and one has to carry out higher order perturbations if ω'', ω''', etc., are needed in constructing

$$\omega(\varepsilon) = \omega_c + \omega'\varepsilon + \tfrac{1}{2}\omega''\varepsilon^2 + O(\varepsilon^3).$$

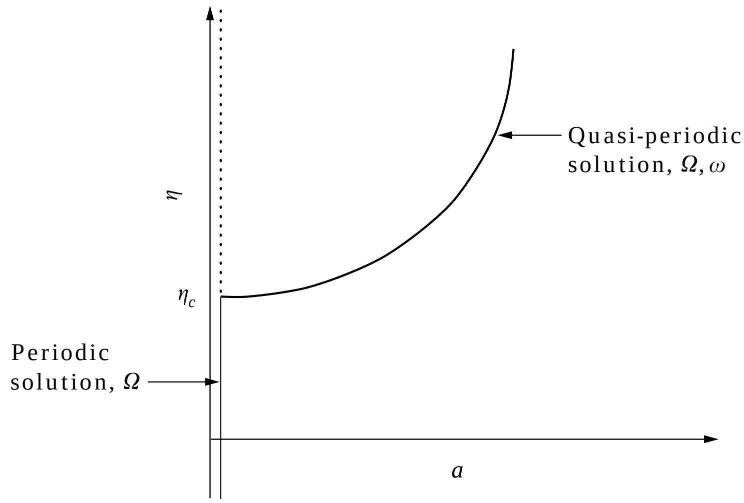

Figure 1. Plot of a versus η: $\eta_c = 1/2\ (F/(\omega_c^2 - \Omega^2))^2$.

It is noted that equation (29) is a bifurcation equation, providing valuable information about the behaviour of the system, and it has been obtained through the application of the IHB technique only. A formal stability analysis is not among the objectives of this paper, but it can be shown (see the Appendix) that Solution (I) is *stable* for $\eta < \eta_{cr}$ and *unstable* for $\eta > \eta_{cr}$. On the other hand, Solution (II), bifurcating from Solution (I) at $\eta = \eta_{cr}$, represents a *stable* family of quasi-periodic motions.

It was demonstrated earlier that the corresponding autonomous system, described by equation (19a) with $F = 0$, exhibits a Hopf bifurcation as η passes through zero. It is now observed that, the introduction of the external harmonic excitation results in a shift of bifurcation point along the η-axis to $\eta = \eta_c$ defined by equation (30), and the bifurcation takes place from a family of periodic motions to a family of quasi-periodic motions, as compared to Hopf bifurcation from an equilibrium path to a family of periodic solutions, associated with the corresponding autonomous system.

Finally, it is noted that the consistency of solutions (I) and (II) as well as that of the general solution (25), up to first order approximations, has been verified by substituting these results into equation (21) and following the procedure described earlier with the aid of MAPLE.

4. CONCLUSIONS

The intrinsic harmonic balancing technique has been applied successfully to many bifurcation problems associated with autonomous systems and non-linear oscillations. In this paper, certain refinements and alternative formulations, enhancing the applicability and availability of the method, are discussed. It is demonstrated that selecting appropriate scaling in advance, and evaluating the perturbation equations at the origin may simplify the analysis considerably. An appropriately modified formulation can also yield ordered approximate solutions for a strongly non-linear system, as demonstrated in section 2.

The scope of the method is further expanded to embrace bifurcation analyses of non-autonomous systems. For the Van de Pol oscillator under harmonic forcing, it is demonstrated that the technique yields a bifurcation equation as an integral part of the perturbation procedure. This equation shows clearly that a family of quasi-periodic motions bifurcates from a family of periodic motions at a critical value of the parameter. It is also observed that the external harmonic excitation results in a shift of the bifurcation point along the parameter axis, compared to the Hopf bifurcation associated with the corresponding autonomous system. A formal stability analysis is not the objective of this paper; however, a brief discussion concerning the stability properties of the solution is presented in the Appendix for completeness.

A verification scheme is outlined and applied to ascertain the validity and consistency of the ordered approximations. A symbolic computer language, namely MAPLE, is used extensively to obtain and verify various solutions. It is observed that the method lends itself conveniently to this process.

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APPENDIX

It can be shown that the local dynamics in the vicinity of the critical point is governed by the first order differential equations

$$\frac{da}{dt} = \frac{\varepsilon a}{2}\left[\eta - \frac{1}{2}\left(\frac{F}{(\omega_c^2 - \Omega^2)}\right)^2 - \frac{1}{4}a^2\right]$$

and

$$a\frac{d\theta}{dt} = a(\omega_c + \varepsilon\omega') = a\omega_c, \qquad \omega' = 0,$$

which are based on the perturbation equations.

In order to prove the above relations, one considers perturbing the general solution (25) ($x(\tau_1, \tau_2; \varepsilon)$) in the vicinity of the critical point. All amplitudes in general solution (25) are in terms of P_{01} and r_{01}, as indicated on the list in the text. To this end, consider the first order system (19b), where the solution of the first order system can be written as

$$x_1 = x(\tau_1, \tau_2; \varepsilon)$$

and

$$x_2 = \Omega \frac{\mathrm{d}x(\tau_1, \tau_2; \varepsilon)}{\mathrm{d}\tau_1} + \omega_c \frac{\mathrm{d}x(\tau_1, \tau_2; \varepsilon)}{\mathrm{d}\tau_2}.$$

Here, for simplicity, r_{01} can be assumed to be zero, as in reference [11]. Suppose now that time-invariant constant p_{01} is a function of time denoted by a. Further, τ_2 is also assumed to be $\theta(t)$. After these transformations, the solution takes the form

$$x_i = f_i(a, \theta, \Omega t). \tag{A1}$$

By substituting equation (A1) into equation (19b) and solving for $\mathrm{d}a/\mathrm{d}t$ and $\mathrm{d}\theta/\mathrm{d}t$, the rate equations are obtained after truncating the higher order terms. This procedure has been described in a number of earlier papers for autonomous systems (see for example, Appendix D in reference [12]) and now it is applied to the non-autonomous system considered in this paper.

The stability of the steady states can be examined by considering the Jacobian of $\mathrm{d}a/\mathrm{d}t$,

$$J = \frac{\mathrm{d}}{\mathrm{d}a}\left(\frac{\mathrm{d}a}{\mathrm{d}t}\right) = \frac{\varepsilon}{2}\left[\eta - \frac{1}{2}\left(\frac{F}{(\omega_c^2 - \Omega^2)}\right)^2 - \frac{1}{4}a^2\right] - \frac{\varepsilon}{4}a^2.$$

Thus, evaluating the Jacobian for Solution (I) ($a = 0$) it is concluded that Solution (I) is stable (unstable) for $\eta < \eta_c$ ($\eta > \eta_c$). Similarly, evaluating the Jacobian for Solution (II) indicates that this solution is stable for $\eta > \eta_c$, which is the case here.