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LETTERS TO THE EDITOR

ON THE EIGENFREQUENCIES FOR MASS LOADED BEAMS UNDER CLASSICAL BOUNDARY CONDITIONS

K. H. Low
School of Mechanical and Production Engineering, *Nanyang Technological University*, *Nanyang Avenue*, *Singapore* 639798, *Republic of Singapore*

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1. INTRODUCTION

Determination of the influence of the parameters characterizing a system on the vibration of the system is of practical interest in engineering applications. Many factors can affect the flexural vibrations of beams, in particular the axial load, intermediate supports and attached masses. Studies of the influences of these individual factors on Bernoulli–Euler beam vibrations for various attachments to the beam have been made from various approaches [1–12].

Kukla [5] investigated the lateral vibration of a loaded beam with intermediate elastic supports and concentrated masses by applying the Green functions method [5], while the optimized Rayleigh–Schmidt approach was used to study the dynamic behaviour of loaded beams with elastic ends [8]. The classical Rayleigh method, with different assumed shape functions, was used to generate the frequency expressions of loaded beam systems [13–15].

In this paper, a frequency analysis of loaded beams is performed for ten classical beams involving guided, fixed, free and/or pinned ends. The explicit frequency equations are obtained by satisfying the differential equation of the eigenvalue problem, the boundary and compatibility conditions. A solution set of the transcendental expression will be compared with the experimental results and also with those obtained by using Rayleigh's method.

2. MODEL CONSIDERED

As shown in Figure 1, the model considered is a beam with a concentrated mass located at $x = a$, where x is the spatial co-ordinate along the beam length of l. Figure 1 illustrates a loaded beam with simply-supported ends. Other beams with guided, fixed, free and/or pinned ends are also considered in this work.

The differential equation associated with the present eigenvalue problem is known as [16]

$$\frac{\mathrm{d}^4 V}{\mathrm{d}x^4} - k^4 V = 0, \tag{1}$$

in which

$$k^4 = \frac{\rho A \omega^2}{EI}, \tag{2}$$

where ρ is the beam's density, A is the cross-sectional area, E is Young's modulus, and I is the moment of inertia of the beam cross-section with respect to the neutral axis of the beam.

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The general solutions of the ordinary differential equation (1) for the loaded beam system are [16, 17]

$$V_1(x) = C_1 \sin kx + C_2 \cos kx + C_3 \sinh kx + C_4 \cosh kx,$$
$$V_2(x) = C_5 \sin kx + C_6 \cos kx + C_7 \sinh kx + C_8 \cosh kx, \quad (3)$$

where V_1 and V_2 are the left and right transverse displacements with respect to the concentrated mass M, and C_i ($i = 1$–8) are constants to be determined.

The compatibility conditions at the location of concentrated mass, which apply to all cases, are given as follows:

$$V_1(a) = V_2(a), \qquad V_1'(a) = V_2'(a), \qquad V_1''(a) = V_2''(a),$$
$$V_1'''(a) - V_2'''(a) + \alpha k^4 V_1(a) = 0, \quad (4)$$

where primes denote differentiation with respect to the spatial variable x, while the mass ratio is defined by $\alpha = M/(\rho A l)$.

To complete the formulation of the boundary-value problem, the boundary conditions must be specified as follows:

$$V' = 0 \quad \text{and } V''' = 0 \quad \text{(guided end)};$$
$$V = 0 \quad \text{and } V' = 0 \quad \text{(fixed end)};$$
$$V'' = 0 \quad \text{and } V''' = 0 \quad \text{(free end)};$$
$$V = 0 \quad \text{and } V'' = 0 \quad \text{(pinned end)}. \quad (5)$$

3. FREQUENCY EQUATIONS

3.1. *Eigenfrequency expressions*

Conditions stated in equations (4) and (5) can be written in terms of C_i by virtue of equation (3),

$$\mathbf{AC} = \mathbf{0}, \quad (6)$$

where $\mathbf{C}^T = \{C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8\}$ and $\mathbf{A}$ is the associated 8×8 matrix.

The frequency equation det $(\mathbf{A}) = 0$ can be generated by executing the following *Maple* code [18]:

```
> genmatrix({i1, i2, i3, i4, e1, e2, e3, e4}, [C1, C2, C3, C4, C5, C6, C7, C8]);
> det (") = 0;
```
(7)

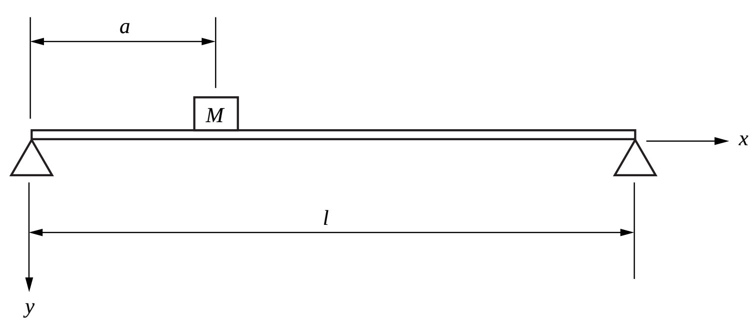

Figure 1. A beam–mass system considered.

where i_p ($p = 1$–4) are the four compatibility conditions at the loading point (see equation (4)), while e_q ($q = 1$–4) are the boundary conditions at both ends for the respective case specified in equation (5). A solution for C_i ($i = 1$–8) and the subsequent substitution into equation (3) constitute the eigenfunctions, V_1 and V_2, which satisfy all the boundary conditions and the differential equation of the eigenvalue problem [19].

The resulting eigenfrequency equations for ten different cases can be written in terms of eigenvalue β ($\beta_i = k_i l$ for mode i) and position parameter η ($= a/l$) as follows:

(i) *guided–guided*

$$2\sin\beta\sinh\beta + \alpha\beta[\sin\beta\sinh\beta(\cos\beta\eta\sin\beta\eta - \cosh\beta\eta\sinh\beta\eta) + \cos\beta\cos^2\beta\eta\sinh\beta + \sin\beta\cosh\beta\cosh^2\beta\eta] = 0; \tag{8a}$$

(ii) *guided–fixed*

$$2(\cos\beta\sinh\beta + \sin\beta\cosh\beta) + \alpha\beta[\cos^2\beta\eta(\cos\beta\cosh\beta - \sin\beta\sinh\beta) + \cosh^2\beta\eta(\cos\beta\cosh\beta + \sin\beta\sinh\beta) - 2\cos\beta\eta\cosh\beta\eta + (\cos\beta\sinh\beta + \sin\beta\cosh\beta)(\cos\beta\eta\sin\beta\eta - \cosh\beta\eta\sinh\beta\eta)] = 0; \tag{8b}$$

(iii) *guided–free*

$$2(\cos\beta\sinh\beta + \sin\beta\cosh\beta) + \alpha\beta[\cos^2\beta\eta\,(\cos\beta\cosh\beta - \sin\beta\sinh\beta) + \cosh^2\beta\eta\,(\cos\beta\cosh\beta + \sin\beta\sinh\beta) + 2\cos\beta\eta\cosh\beta\eta + (\cos\beta\sinh\beta + \sin\beta\cosh\beta)(\cos\beta\eta\sin\beta\eta - \cosh\beta\eta\sinh\beta\eta)] = 0; \tag{8c}$$

(iv) *guided–pinned*

$$2\cos\beta\cosh\beta + \alpha\beta[\cos\beta\cosh\beta(\cos\beta\eta\sin\beta\eta - \cosh\beta\eta\sinh\beta\eta) + \cos\beta\sinh\beta\cosh^2\beta\eta - \sin\beta\cosh\beta\cos^2\beta\eta] = 0; \tag{8d}$$

(v) *fixed–fixed*

$$\begin{aligned}2(1 - \cos\beta\cosh\beta) &+ \alpha\beta[\sin\beta\cosh\beta\cosh^2\beta\eta - \cos\beta\sinh\beta\cos^2\beta\eta \\ &+ \cos\beta\cosh\beta\,(\sin\beta\eta\cosh\beta\eta - \cos\beta\eta\sinh\beta\eta) \\ &+ \cos\beta\sinh\beta\,(\cos\beta\eta\cosh\beta\eta - \sin\beta\eta\sinh\beta\eta) \\ &- \sin\beta\cosh\beta(\cos\beta\eta\cosh\beta\eta + \sin\beta\eta\sinh\beta\eta) \\ &+ \sin\beta\sinh\beta(\sin\beta\eta\cosh\beta\eta + \cos\beta\eta\sinh\beta\eta) \\ &- \sin\beta\sinh\beta\,(\cos\beta\eta\sin\beta\eta + \cosh\beta\eta\sinh\beta\eta) \\ &+ \cos\beta\eta\sinh\beta\eta - \sin\beta\eta\cosh\beta\eta] = 0;\end{aligned} \tag{8e}$$

(vi) *fixed–free*

$$\begin{aligned}2(1 + \cos\beta\cosh\beta) &+ \alpha\beta[\cos\beta\sinh\beta\cos^2\beta\eta - \sin\beta\cosh\beta\cosh^2\beta\eta \\ &+ \cos\beta\cosh\beta(\cos\beta\eta\sinh\beta\eta - \sin\beta\eta\cosh\beta\eta) \\ &+ \cos\beta\sinh\beta(\sin\beta\eta\sinh\beta\eta - \cos\beta\eta\cosh\beta\eta) \\ &+ \sin\beta\cosh\beta(\cos\beta\eta\cosh\beta\eta + \sin\beta\eta\sinh\beta\eta)\end{aligned}$$

$$- \sin\beta \sinh\beta(\cos\beta\eta \sinh\beta\eta + \sin\beta\eta \cosh\beta\eta)$$
$$+ \sin\beta \sinh\beta(\cosh\beta\eta \sinh\beta\eta + \cos\beta\eta \sin\beta\eta)$$
$$+ \cos\beta\eta \sinh\beta\eta - \sin\beta\eta \cosh\beta\eta] = 0; \tag{8f}$$

(vii) *fixed–pinned*

$$2(\cos\beta \sinh\beta - \sin\beta \cosh\beta) + \alpha\beta[\cos^2\beta\eta(\cos\beta \cosh\beta - \sin\beta \sinh\beta)$$
$$- \cosh^2\beta\eta(\cos\beta \cosh\beta + \sin\beta \sinh\beta)$$
$$+ (\cos\beta \sinh\beta + \sin\beta \cosh\beta)(\cos\beta\eta \sin\beta\eta + \cosh\beta\eta \sinh\beta\eta)$$
$$+ 2(\cos\beta \sin\beta\eta - \sin\beta \cos\beta\eta)(\cosh\beta \sinh\beta\eta - \sinh\beta \cosh\beta\eta)] = 0; \tag{8g}$$

(viii) *free–free*

$$2(\cos\beta \cosh\beta - 1) + \alpha\beta[\cos\beta \sinh\beta \cos^2\beta\eta - \sin\beta \cosh\beta \cosh^2\beta\eta$$
$$+ \cos\beta \cosh\beta(\sin\beta\eta \cosh\beta\eta - \cos\beta\eta \sinh\beta\eta)$$
$$+ \cos\beta \sinh\beta(\cos\beta\eta \cosh\beta\eta - \sin\beta\eta \sinh\beta\eta)$$
$$- \sin\beta \cosh\beta(\cos\beta\eta \cosh\beta\eta + \sin\beta\eta \sinh\beta\eta)$$
$$+ \sin\beta \sinh\beta(\cos\beta\eta \sinh\beta\eta + \sin\beta\eta \cosh\beta\eta)$$
$$+ \sin\beta \sinh\beta(\cos\beta\eta \sin\beta\eta + \cosh\beta\eta \sinh\beta\eta)$$
$$+ \cos\beta\eta \sinh\beta\eta - \sin\beta\eta \cosh\beta\eta] = 0; \tag{8h}$$

(ix) *free–pinned*

$$2(\cos\beta \sinh\beta - \sin\beta \cosh\beta) + \alpha\beta[\cos^2\beta\eta(\cos\beta \cosh\beta - \sin\beta \sinh\beta)$$
$$- \cosh^2\beta\eta(\cos\beta \cosh\beta + \sin\beta \sinh\beta)$$
$$+ (\cos\beta \sinh\beta + \sin\beta \cosh\beta)(\cos\beta\eta \sin\beta\eta + \cosh\beta\eta \sinh\beta\eta)$$
$$+ 2(\cos\beta \sin\beta\eta - \sin\beta \cos\beta\eta)(\sinh\beta \cosh\beta\eta - \cosh\beta \sinh\beta\eta)] = 0; \tag{8i}$$

(x) *pinned–pinned*

$$2 \sin\beta \sinh\beta + \alpha\beta[\sin\beta \sinh\beta(\cosh\beta\eta \sinh\beta\eta - \cos\beta\eta \sin\beta\eta)$$
$$+ \cos\beta \sinh\beta \sin^2\beta\eta - \sin\beta \cosh\beta \sinh^2\beta\eta] = 0. \tag{8j}$$

3.2. *Rayleigh's expression*

The fundamental frequency of three classical beams carrying a mass at various positions was obtained by substituting different shape fucntions, one at a time, into Rayleigh's quotient [12, 13]. A frequency expression for each case was then symbolically written in terms of mass ratio and position parameter as [14]:

$$\omega_1^2 = \frac{KEI}{\rho A l^4} \frac{G_\alpha + G}{H_\alpha + H}, \tag{9}$$

which yields

$$\beta_1^4 = \frac{K(G_\alpha + G)}{(H_\alpha + H)}, \tag{10}$$

since $\beta_1^4 = \rho A l^4 \omega_1^2/(EI)$.

To compare with the corresponding frequency equation in section 3.1, the closed form expression obtained for a fixed–fixed case [14] is reproduced with the associated parameters:

(i) deflection shape of the beam under a concentrated mass M,

$$K = 192, \qquad G = 1, \qquad H = (16/35)\eta(\eta^3 - 2\eta^2 - 2\eta + 3),$$
$$G_\alpha = 0, \qquad H_\alpha = -64\alpha\eta^3(\eta^3 - 3\eta^2 + 3\eta - 1); \tag{11a}$$

(ii) deflection shape in terms of the concentrated mass M and the uniformly distributed beam mass m ($= \rho A l$),

$$K = 192, \qquad G = 1/8, \qquad H = 1/21,$$
$$G_\alpha = -30\alpha^2\eta^3(\eta^3 - 3\eta^2 + 3\eta - 1) + (15/2)\alpha\eta^2(\eta^2 - 2\eta + 1),$$
$$H_\alpha = 1920\alpha^3\eta^6(\eta^6 - 6\eta^5 + 15\eta^4 - 20\eta^3 + 15\eta^2 - 6\eta + 1)$$
$$+ (2/7)\alpha[\eta^2(118\eta^6 - 432\eta^5 + 644\eta^4 - 420\eta^3 + 105\eta^2 - 14\eta + 9)$$
$$- \alpha\eta^4(1728\eta^6 - 8640\eta^5 + 17136\eta^4 - 16704\eta^3 + 7776\eta^2 - 1152\eta - 144)]. \tag{11b}$$

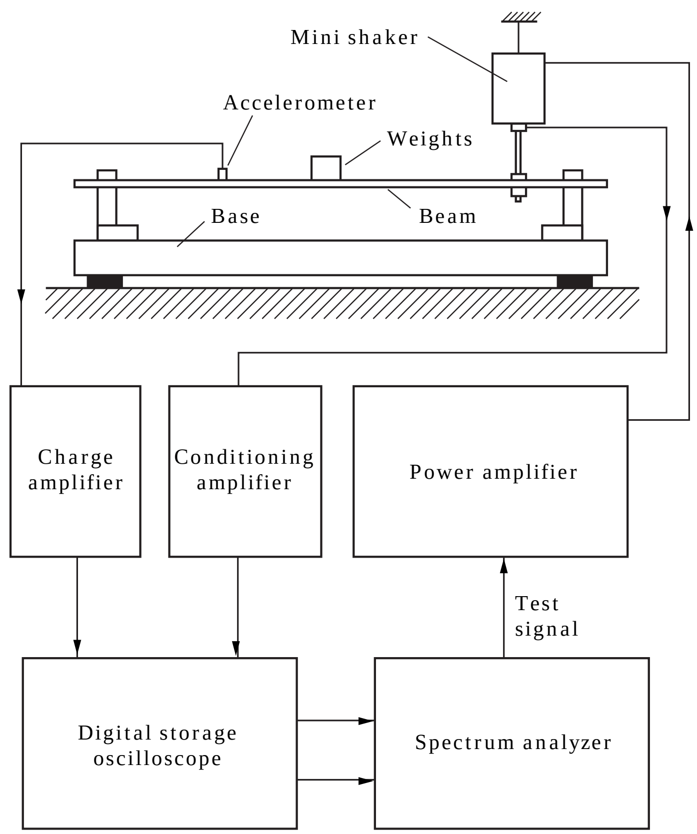

Figure 2. Experimental set-up for the beam-mass system.

Note that the parameters G and H are independent of the mass ratio α. It is obvious that the eigenvalue β_1 can now be calculated by substituting the parameters of equation (11) into the closed-form algebraic equation (10).

4. ANALYTICAL AND EXPERIMENTAL RESULTS

Experimental testing was conducted to obtain the frequencies of different loaded beams with a mass placed at various locations. The experimental set-up for the loaded beam is shown in Figure 2. Analytical results evaluated based on the expressions in section 3 can be compared to those measured experimentally.

Figure 3 shows an analytical-experimental comparison for a fixed–fixed beam with $l = 1$ m, $E = 207$ GPa, $\rho = 7810$ kg/m^3 and $s_r = 727$, where $s_r (= \sqrt{Al^2/I})$ is defined as the slender ratio of the beam. Note that the mode frequency in Hz can generally be calculated as $f_i = \beta_i^2 \sqrt{E/\rho}/(2\pi s_r l)$, where β_i is obtained by solving the corresponding transcendental

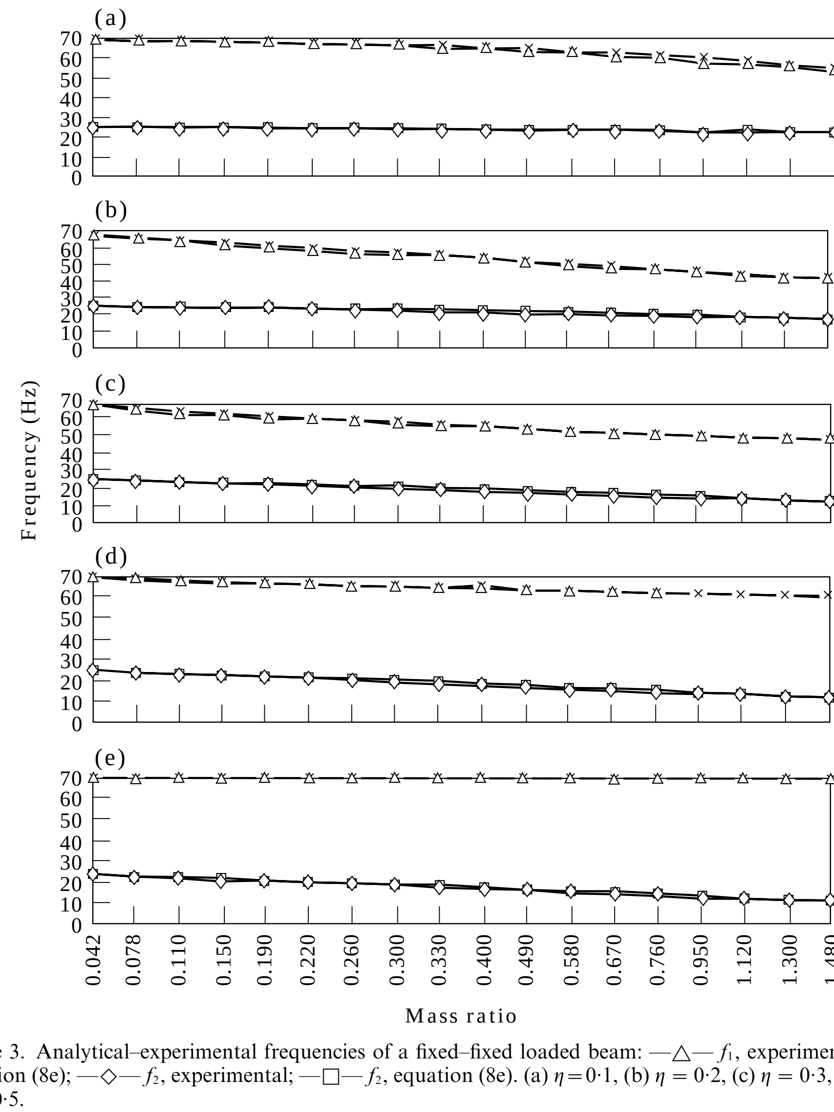

Figure 3. Analytical–experimental frequencies of a fixed–fixed loaded beam: —△— f_1, experimental; —×— f_1, equation (8e); —◇— f_2, experimental; —□— f_2, equation (8e). (a) $\eta = 0{\cdot}1$, (b) $\eta = 0{\cdot}2$, (c) $\eta = 0{\cdot}3$, (d) $\eta = 0{\cdot}4$, (e) $\eta = 0{\cdot}5$.

TABLE 1

Comparison of fundamental frequency parameters obtained by using different expressions

Mass ratio, α	β_{1t} via equation (8e)	β_{1c} via equation (11b)	β_{1W} via equation (11a)	β_{1t} via equation (8e)	β_{1c} via equation (11b)	β_{1W} via equation (11a)	β_{1t} via equation (8e)	β_{1c} via equation (11b)	β_{1W} via equation (11a)
$0(\eta = 0)$	4·730								
	$\eta = 0{\cdot}1$			$\eta = 0{\cdot}3$			$\eta = 0{\cdot}5$		
0·01	4·730	4·738	6·228	4·716	4·723	4·973	4·701	4·709	4·737
0·05	4·728	4·738	6·206	4·660	4·667	4·883	4·591	4·598	4·620
0·10	4·726	4·737	6·178	4·593	4·600	4·781	4·470	4·477	4·492
0·20	4·721	4·738	6·125	4·468	4·473	4·605	4·267	4·270	4·281
0·30	4·717	4·739	6·074	4·355	4·358	4·459	4·102	4·105	4·112
0·40	4·713	4·741	6·024	4·251	4·256	4·331	3·964	3·966	3·971
0·50	4·708	4·744	5·978	4·158	4·161	4·221	3·847	3·847	3·854
0·60	4·704	4·746	5·932	4·072	4·074	4·124	3·745	3·743	3·750
0·80	4·694	4·751	5·846	3·922	3·924	3·959	3·575	3·572	3·579
1·00	4·685	4·755	5·765	3·795	3·796	3·821	3·438	3·432	3·439
1·50	4·660	4·760	5·587	3·545	3·545	3·558	3·182	3·173	3·183
2·00	4·634	4·753	5·432	3·359	3·360	3·367	2·999	2·990	3·000
5·00	4·451	4·559	4·804	2·766	2·766	2·768	2·445	2·434	2·445
10·0	4·123	4·163	4·240	2·355	2·355	2·356	2·074	2·063	2·074
20·0	3·640	3·648	3·669	1·993	1·993	1·993	1·752	1·742	1·752
50·0	2·969	2·971	2·972	1·591	1·591	1·591	1·397	1·389	1·397
80·0	2·654	2·655	2·656	1·416	1·416	1·416	1·243	1·236	1·243
100·0	2·515	2·515	2·516	1·340	1·340	1·340	1·176	1·169	1·176

expression (8) or by equation (10). Only results of $0 \leqslant \eta \leqslant 0{\cdot}5$ are required owing to the symmetrical ends of fixed–fixed beams. The comparison in Figure 3 shows that the transcendental expression (8e) can well predict the frequencies of the beam carrying a mass at various positions. Note that the first two frequencies for the unloaded beam ($m_b = 556{\cdot}2$ g) are: $f_1 = 25{\cdot}15$ Hz (experimental), $f_1 = 25{\cdot}22$ Hz (analytical); $f_2 = 69{\cdot}25$ Hz (experimental), $f_2 = 69{\cdot}51$ Hz (analytical).

In another comparison, as shown in Table 1, the fundamental frequency parameters obtained by Rayleigh's method, equation (10), are listed with respect to those generated by solving the transcendental equation (8e).

The following observations can be made by virtue of the results in Figure 3 and Table 1: (1) As shown in Figure 3 for $\eta = 0{\cdot}1$, magnitude of the load is insignificant to the fundamental frequency if the mass is placed near the beam end (an anti-node of the mode shape). (2) The fundamental frequencies increase as the mass is placed away from the beam's centre, but not for the second frequencies. (3) The second mode frequency remains constant for different central concentrated masses. In fact, the trends just stated can in fact be associated with the corresponding mode shapes. (4) In general, the frequencies predicted by the transcendental expression, equation (8e), is slightly higher than the measured values. (5) As shown in Table 1, the frequency expression (11a), which is generated by using the shape function involving the load (M) only, gives very poor results if $\alpha < 10$ and $\eta = 0{\cdot}1$, a location very near to the beam end. For example, the eigenvalues for $\alpha = 0{\cdot}01$ via equations (8e) and (11b) are both close to the fundamental eigenvalue of the unloaded beam with fixed ends, 4·730 [16, 17], but the value of β_{1W} is totally out. (6) The values of β_{1W} approach those of β_{1t} for cases with $\eta = 0{\cdot}5$ and $\alpha \geqslant 10$. On the other hand, all the three expressions would give similar eigenvalues if $M \gg m$. (7) The values of β_{1W} are always the highest among the three. In most cases, the values of β_{1c} are close to those solutions of the transcendental equation, β_{1t}, while the former values are higher. (8) As concluded in Figure 3 and Table 1, the transcendental expression for β_{1t} can best predict the experimental frequencies, but the closed-form algebraic solution for β_{1W} enables a *quick* estimation of the fundamental frequency of beams carrying a mass at various postions.

5. CONCLUDING REMARKS

In this paper a frequency analysis of a Euler–Bernoulli beam carrying a concentrated mass at any arbitrary location is presented. The dimensionless eigenfrequency equations for the boundary value problem have been derived for ten different sets of boundary conditions, involving guided, fixed, free and/or pinned ends. The expressions were generated by satisfying the differential equations of motion and by imposing the corresponding boundary and compatibility conditions. On the other hand, approximate results are given using Rayleigh's method with two static deflection shape functions. The effects of the position and magnitude of the mass, as well as comparisons of the different results obtained experimentally and analytically, have been determined.

It can be seen from the comparisons that the eigenfrequencies of the beam–mass system can be accurately predicted by the solutions of transcendental equations. However, the closed-form Rayleigh expressions would be used for a quick estimation of fundamental frequency.

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REFERENCES

1. C. Kameswara Rao and S. Mirza 1989 *Journal of Sound and Vibration* **130**, 453–465. A note on vibrations of generally restrained beams.
2. C. Levy 1991 *Journal of Sound and Vibration* **150**, 111–118. An iterative technique based on the Dunkerley method for determining the natural frequencies of vibrating systems.
3. M. J. Maurizi and P. M. Belles 1991 *Journal of Sound and Vibration* **150**, 330–334. Natural frequencies of the beam–mass system: comparison of the two fundamental theories of beam vibrations.
4. M. U. Jen and E. B. Magrab 1993 *Journal of Vibration and Acoustics*, *Transactions of the ASME* **115**, 202–209. Natural frequencies and mode shapes of beams carrying a two degree-of-freedom spring–mass system.
5. S. Kukla 1994 *Journal of Sound and Vibration* **172**, 449–458. Free vibrations of axially loaded beams with concentrated masses and intermediate elastic supports.
6. M. Gurgoze 1996 *International Journal of Mechanical Science* **38**, 1295–1306. On the eigenfrequencies of cantilevered beams carrying a tip mass and spring-mass in-span.
7. M. Gurgoze and H. Batan 1996 *Journal of Sound and Vibration* **195**, 163–168. On the effect of an attached spring–mass system on the frequency spectrum of a cantilevered beam.
8. M. A. De Rosa, C. Franciosi and M. J. Maurizi 1996 *Computers and Structures* **58**, 1145–1159. On the dynamic behaviour of slender beams with elastic ends carrying a concentrated mass.
9. P. K. Sarkar 1996 *Journal of Sound and Vibration* **195**, 229–240. Approximate determination of the fundamental frequency of a cantilevered beam with point masses and restraining springs.
10. E. Ozkaya, M. Pakdemirli and H. R. Oz 1997 *Journal of Sound and Vibration* **199**, 679–696. Non-linear vibrations of a beam–mass system under different boundary conditions.
11. K. H. Low 1997 *Computers and Structures* **65**, 97–107. An analytical–experimental comparative study of vibration analysis for loaded beams with variable boundary conditions.
12. K. H. Low 1997 *Journal of Sound and Vibration* **201**, 528–533. Closed-form formulas for fundamental vibration frequency of beams under off-centre load.
13. K. H. Low and R. N. Dubey 1997 *Journal of Sound and Vibration* **202**, 134–138. A note on the fundamental shape function and frequency for beams under off-center load.
14. K. H. Low 1997 *Journal of Sound and Vibration* **207**, 132–135. A further note on closed-form formulas for fundamental vibration frequency of beams under off-centre load.
15. M. A. De Rosa and C. Franciosi 1996 *Journal of Sound and Vibration* **191**, 795–808. The optimized Rayleigh method and *Mathematica* in vibrations and buckling problems.
16. S. S. Rao 1995 *Mechanical Vibrations*. Reading, MA: Addison-Wesley, third edition.
17. F. S. Tse, I. E. Morse and R. T. Hinkle 1978 *Mechanical Vibrations*: *Theory and Applications*. Boston, MA: Allyn and Bacon, second edition.
18. K. M. Heal, M. L. Hansen and K. M. Rickard 1996 *Maple V*: *Learning Guide*. New York: Springer-Verlag.
19. L. Meirovitch 1967 *Analytical Methods in Vibrations*. New York: Macmillan.