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NON-LINEAR VIBRATION BY A NEW METHOD

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In this paper, a new method is presented for solving the periodic response of a non-linear system. Both period-one and subharmonic responses can be obtained by the method. The stability and bifurcation of the solutions are discussed. Specifically, the period doubling and symmetry-breaking bifurcation are studied in detail.

1. INTRODUCTION

In references [1, 2], the perturbation method has been expressed in detail, but this method is suited for weak non-linear problems. One of the systematic ways in which to analyze a non-linear system subjected to periodic excitations is the harmonic balance method [3, 4]. The discretization can be performed on the incremental form of the governing equations, and this is called the incremental harmonic balance method [5–7]. In these methods, an approximately known solution and the Fourier coefficients of some non-linear functions are needed. In the present paper, the method in reference [8], which we have used to solve the linear vibration problem, is developed to solve the non-linear vibration problem.

2. THE METHOD OF ANALYSIS

The governing equation considered can be expressed as the following form

$$\ddot{x} + h\dot{x} + \alpha x + f(x, \dot{x}) = q(t), \tag{1}$$

where h and α are system parameters, $f(x, \dot{x})$ is a non-linear function about $\dot{x}$ and x, $q(t)$ is a periodic function with the period T. The solution $x(t)$ of equation (1) satisfies the following periodic conditions

$$x(t) = x(t + \bar{T}) \quad \text{and} \quad \dot{x}(t) = \dot{x}(t + \bar{T}),$$

where $\bar{T}$ is the period of the unknown response $x(t)$. For solving the differential equation (1), the following linear equation is first considered:

$$\ddot{x} + h\dot{x} + \alpha x = \delta(t - t_0), \tag{2}$$

where t and t_0 both belong to the interval $[0, \bar{T}]$, $\delta(t - t_0)$ is a Delta function, that is

$$\delta(t - t_0) = \begin{cases} 0 & t \neq t_0 \\ \infty & t = t_0 \end{cases}$$

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and

$$\int_0^{\bar{T}} \delta(t - t_0)\,\mathrm{d}t = 1.$$

In order to solve equation (2), one discretizes the $x(t)$ and $\delta(t - t_0)$ by Fourier series:

$$x(t) = x_{10} + \sum_{n=1}^{\infty} (x_{1n} \cos n\omega t + x_{2n} \sin n\omega t), \tag{3a}$$

$$\delta(t - t_0) = 1/\bar{T} + \frac{2}{\bar{T}} \sum_{n=1}^{\infty} (\cos n\omega t \cos n\omega\, t_0 + \sin n\omega t \sin n\omega t_0), \tag{3b}$$

where $\omega = 2\pi/\bar{T}$.

Substituting equations (3a) and (3b) into equation (2) and equating the coefficients of $\cos n\omega t$ ($n = 0, 1, 2, \ldots$) and $\sin n\omega t$ ($n = 1, 2, \ldots$) on the left and right sides of equation (2), one obtains the solution of equation (2) denoted as $G(t, t_0)$. $G(t, t_0)$ can be written as

$$G(t, t_0) = a_0 + \sum_{n=1}^{\infty} (\alpha_n \cos n\omega t + b_n \sin n\omega t). \tag{4}$$

where $a_0 = 1/(\bar{T}\alpha)$

$$a_n = c_{1n} \cos n\omega t_0 - c_{2n} \sin n\omega t_0, \qquad b_n = c_{1n} \sin n\omega t_0 + c_{2n} \cos n\omega t_0,$$

here

$$C_{1n} = \frac{2(\alpha - n^2\omega^2)}{\bar{T}(h^2n^2\omega^2 + (\alpha - n^2\omega^2)^2)}, \qquad C_{2n} = \frac{2hn\omega}{\bar{T}(h^2n^2\omega^2 + (\alpha - n^2\omega^2)^2)}.$$

Obviously, $G(t, t_0)$ has the following properties:

$$\frac{\partial G}{\partial t_0} = -\frac{\partial G}{\partial t}, \qquad \frac{\partial^2 G}{\partial t_0^2} = \frac{\partial^2 G}{\partial t^2}. \tag{5a, b}$$

Then one has the following proposition.

Proposition 1. The periodic solution $x(t)$ of the differential equation (1) has the following form:

$$x(t) = -\int_0^{\bar{T}} f(x, \dot{x}) G(t, t_0)\,\mathrm{d}t_0 + \int_0^{\bar{T}} q(t_0) G(t, t_0)\,\mathrm{d}t_0. \tag{6}$$

Proof. Let $x(t)$ be the periodic solution of equation (1). Then, multiplying each side of equation (1) by $G(t, t_0)$ and integrating on t_0 over $[0, \bar{T}]$ yields

$$\int_0^{\bar{T}} (\ddot{x} + h\dot{x} + \alpha x) G(t, t_0)\,\mathrm{d}t_0 = -\int_0^{\bar{T}} f(x, \dot{x}) G(t, t_{,0})\,\mathrm{d}t_0 + \int_0^{\bar{T}} q(t_0) G(t, t_0)\,\mathrm{d}t_0, \tag{7}$$

since

$$\int_0^T \dot{x}G(t, t_0)\,\mathrm{d}t_0 = xG(t, t_0)|_0^T - \int_0^T x\frac{\partial G}{\partial t_0}\,\mathrm{d}t_0 = -\int_0^T x\frac{\partial G}{\partial t_0}\,\mathrm{d}t_0,$$

$$\int_0^T \ddot{x}G(t, t_0)\,\mathrm{d}t_0 = \int_0^T x\frac{\partial^2 G}{\partial t^2}\,\mathrm{d}t_0,$$

by using properties (5a,b) of $G(t, t_0)$, the left side of equation (7) can be changed into the following form:

$$\int_0^T x\left(\frac{\partial^2 G}{\partial t^2} + h\frac{\partial G}{\partial t} + \alpha G\right)\mathrm{d}t_0 = \int_0^T x\delta(t - t_0)\,\mathrm{d}t_0 = x(t)$$

Hence the proposition I is proved. As for the non-linear integral equation (6), substituting expression (4) into equation (6) yields

$$x(t) = p_{10}/2 + \sum_{n=1}^{\infty}(p_{1n}\cos n\omega t + p_{2n}\sin n\omega t), \tag{8}$$

where

$$p_{10} = 2\int_0^T (q(t_0) - f(x, \dot{x}))\alpha_0\,\mathrm{d}t_0, \qquad p_{1n} = \int_0^T (q(t_0) - f(x, \dot{x}))\alpha_n\,\mathrm{d}t_0 \tag{9a, b}$$

$$p_{2n} = \int_0^T (q(t_0) - f(x, \dot{x}))b_n\,\mathrm{d}t_0, \qquad (n = 1, 2, \ldots). \tag{9c}$$

Substituting expression (8) into equations (9a)–(9c) results in the following non-linear algebraic equations with the unknown numbers $p_{10}, p_{1n}, p_{2n} (n = 1, 2, \ldots)$

$$p_{10} = \int_0^T \left[q(t_0) - f\left(\sum_{n=1}^{\infty}(-p_{1n}n\omega\sin n\omega t_0 + p_{2n}n\omega\cos n\omega t_0),\right.\right.$$

$$\left.\left.\frac{p_{10}}{2} + \sum_{n=1}^{\infty}(p_{1n}\cos n\omega t_0 + p_{2n}\sin n\omega t_0)\right)\right]a_0\,\mathrm{d}t_0, \tag{10a}$$

$$p_{1n} = \int_0^T [q(t_0) - f\left(\sum_{n=1}^{\infty}(-p_{1n}n\omega\sin n\omega t_0 + p_{2n}n\omega\cos n\omega t_0),\right.$$

$$\left.\left.\frac{p_{10}}{2} + \sum_{n=1}^{\infty}(p_{1n}\cos n\omega t_0 + p_{2n}\sin n\omega t_0)\right)\right]a_n\,\mathrm{d}t_0, \tag{10b}$$

$$p_{2n} = \int_0^{\bar{T}} \left[q(t_0) - f\left(\sum_{n=1}^{\infty} (-p_{1n} n\omega \sin n\omega t_0 + p_{2n} n\omega \cos n\omega t_0), \right.\right.$$

$$\left.\left. \frac{p_{10}}{2} + \sum_{n=1}^{\infty} (p_{1n} \cos n\omega t_0 + p_{2n} \sin n\omega t_0) \right) \right] b_n \, \mathrm{d}t_0 \qquad (n = 1, 2, \ldots) \tag{10c}$$

In the real calculation, finite unknown numbers p_{1n} and p_{2n} are retained. Then some numerical methods can be applied to solve such non-linear algebraic equations. Therefore, the period solution with the form (8) can be obtained.

3. STABILITY AND BIFURCATION OF SOLUTION

The Floquet method can be used for the stable analysis of a solution. When the solution $x(t)$ is perturbated by $\delta x(t)$, the increments equation is

$$\delta \ddot{x} + h\delta \dot{x} + \alpha \delta x + \frac{\partial f}{\partial \dot{x}} \delta \dot{x} + \frac{\partial f}{\partial x} \delta x = 0. \tag{11}$$

Equation (11) in state variable form and matrix notation is

$$\dot{Z} = A(t) Z, \tag{12}$$

where

$$Z(t) = \begin{pmatrix} \delta \dot{x} \\ \delta x \end{pmatrix}, \qquad A(t) = \begin{pmatrix} -h - \dfrac{\partial f}{\partial x} & -\alpha - \dfrac{\partial f}{\partial x} \\ 1 & 0 \end{pmatrix}.$$

The stability of equation (12) is checked by evaluating the eigenvalues of the transformation matrix $[B]$ which transforms the state vector $Z(n\bar{T})$ at $t = n\bar{T}$ to $Z((n+1)\bar{T})$ at $t = (n+1)\bar{T}$. If the absolute magnitudes of the eigenvalues are less than unity, the solution is stable. The explicit form $[B]$ can be written as [9]

$$[B] = \prod_{i=1}^{N} \exp\{\Delta t [A(i\Delta t)]\}, \qquad \Delta t = \bar{T}/N,$$

where

$$\exp[A] = I + [A] + [A]^2/2! + [A]^3/3! + \cdots .$$

Instability occurs if a real eigenvalue exceeds unity, this means that there may be a fold or a tangential bifurcation with unchanged period. When instability occurs with a negative real eigenvalue less than -1, there is period doubling. Finally, when two complex conjugate eigenvalues escape the unit circle, one has a Hopf bifurcation point.

4. EXAMPLES

Example 1: *free vibration*. Consider the following system

$$\ddot{x}(t) + \alpha x(t) + b x^3(t) = 0 \qquad (\alpha > 0, b > 0), \qquad x(0) = A, \dot{x}(0) = 0. \tag{13}$$

First the following equation is considered

$$\ddot{x} + \alpha x = \delta(t - t_0).$$

As $\dot{x}(0) = 0$, the solution can be expanded in cosine series, then using the similar method mentioned above gives

$$G(t, t_0) = \frac{1}{\alpha T} + \sum_{n=1}^{\infty} \frac{2}{T(a - n^2\omega^2)} \cos n\omega t_0 \cos n\omega t,$$

where T is the period of the unknown solution $x(t)$ of equation (13). Therefore, the following integral equation is obtained

$$x(t) = \sum_{n=1}^{\infty} x_n \cos n\omega t + x_0/2.$$

Here

$$x_0 = -\frac{2b}{aT} \int_0^T x^3(t_0)\, \mathrm{d}t_0, \tag{14a}$$

$$x_n = -\frac{2b}{T(a - n^2\omega^2)} \int_0^T x^3(t_0) \cos n\omega t_0\, \mathrm{d}t_0 \qquad (n = 1, 2, \ldots). \tag{14b}$$

Let

$$x^3(t_0) = \sum_{n=0}^{\infty} \bar{x}_n \cos n\omega t_0.$$

TABLE 1a

Variation of the cosine coefficients of a 1T period solution with respect to λ

λ	p_{10}	p_{11}	p_{12}	p_{13}	Stability
3·0	−1·9963	−4·9899e − 2	0·0	0·0	s
2·4	−1·9899	−8·3659e − 2	0·0	0·0	s
2·1	−1·9799	−1·1609e − 1	0·0	0·0	u
1·8	−1·9541	−1·7333e − 1	−1·8156e − 3	0·0	u
1·4	−1·8159	−3·3867e − 1	−1·0876e − 2	0·0	u

Note: *u* denotes instability, and *s* stability.

TABLE 1b

Variation of the sine coefficients of a 1T period solution with respect to λ

λ	p_{21}	p_{22}	p_{23}
3·0	1·8703e − 3	0·0	0·0
2·4	4·1961e − 3	0·0	0·0
2·1	7·1020e − 3	0·0	0·0
1·8	1·3611e − 2	0·0	0·0
1·4	4·0934e − 2	3·1278e − 2	0·0

TABLE 2a

Variation of the cosine coefficients of a 2T period solution with respect to λ

λ	p_{10}	p_{11}	p_{12}	p_{13}	Stability
2·1	−1·9662	−0·8486e − 1	−0·1169e − 0	−0·1630e − 2	s
1·84	−1·6689	−0·3751e − 0	−0·1779e − 0	−0·1271e − 1	s
1·7	−1·5084	−0·4396e − 0	−0·2150e − 0	−0·1980e − 1	u
1·5	−1·2873	−0·4870e − 0	−0·2737e − 0	−0·3191e − 1	u
1·3	−1·0579	−0·4960e − 0	−0·3389e − 0	−0·4718e − 1	u

TABLE 2b

Variation of the sine coefficients of a 2T period solution with respect to λ

λ	p_{21}	p_{22}	p_{23}
2·1	0·3253e − 1	0·8269e − 2	0·7866e − 3
1·84	0·1138e − 0	0·3269e − 1	0·7425e − 2
1·7	0·1227e − 0	0·4286e − 1	0·1132e − 1
1·5	0·1158e − 0	0·5401e − 1	0·1646e − 1
1·3	0·1041e − 0	0·6212e − 1	0·2196e − 1

According to the formula of the coefficients of product of two functions [10], one obtains

$$\bar{x}_n = \frac{x_0^2 x_n}{4} + \frac{x_n}{2}\sum_{m=1}^{\infty} x_m^2 + \frac{1}{4}\sum_{m=1}^{\infty} x_0 x_m (x_{m+n} + x_{m-n})$$

$$+ \frac{1}{4}\sum_{m=1}^{\infty} (x_{m+n} + x_{m-n}) \sum_{p=1}^{\infty} x_p (x_{p+m} + x_{p-m}). \tag{15}$$

Substituting expression (15) into equations (14a) and (14b) gives

$$x_0 = -\frac{b}{a}\left(\frac{x_0^3}{4} + \frac{x_0}{2}\sum_{m=1}^{\infty} x_m^2 + 0{\cdot}5\sum_{m=1}^{\infty} x_0 x_m^2 + 0{\cdot}5\sum_{m=1}^{\infty} x_m \sum_{p=1}^{\infty} x_p (x_{p+m} + x_{p-m})\right)$$

$$x_n = -\frac{b}{\alpha - n^2\omega^2}\left(\frac{x_0^2 x_n}{4} + \frac{x_n}{2}\sum_{m=1}^{\infty} x_m^2 + \frac{1}{4}\sum_{m=1}^{\infty} x_0 x_m (x_{m+n} + x_{m-n})\right.$$

$$\left. + \frac{1}{4}\sum_{m=1}^{\infty} (x_{m+n} + x_{m-n}) \sum_{p=1}^{\infty} x_p (x_{p+m} + x_{p-m})\right) \qquad (n = 1, 2, \ldots),$$

TABLE 3a

Variation of the cosine coefficients of a 4T period solution with respect to λ

λ	p_{10}	p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}	Stability
1·825	−1·6543	−0·1711e − 2	−0·3793e0	−0·5099e − 2	−0·1807e0	−0·1639e − 2	−0·1310e − 1	s
1·826	−1·6440	−0·3233e − 2	−0·3782e0	−0·9253e − 2	−0·1830e0	−0·2930e − 2	−0·1325e − 1	s
1·8035	−1·6410	−0·3978e − 2	−0·3756e0	−0·1161e − 1	−0·1834e0	−0·3841e − 2	−0·1305e − 1	u
1·8	−1·6390	−0·4943e − 2	−0·3739e0	−0·1348e − 1	−0·1837e0	−0·4321e − 2	−0·1304e − 1	u
1·78	−1·6280	−0·6055e − 2	−0·3678e0	−0·1814e − 1	−0·1859e0	−0·5668e − 2	−0·1297e − 1	u
1·75	−1·6121	−0·7451e − 2	−0·3592e0	−0·2436e − 1	−0·1895e0	−0·7579e − 2	−0·1282e − 1	u

TABLE 3b

Variation of the sine coefficients of a 4T period solution with respect to λ

λ	p_{21}	p_{22}	p_{23}	p_{24}	p_{25}	p_{26}
1·825	−0·5034e − 1	0·1150e0	−0·1069e − 1	0·3329e − 1	−0·3222e − 2	0·7534e − 2
1·826	−0·8062e − 1	0·1171e0	−0·1656e − 1	0·3402e − 1	−0·5322e − 2	0·7776e − 2
1·8035	−0·1072e0	0·1183e0	−0·2238e − 1	0·3469e − 1	−0·7327e − 2	0·7972e − 2
1·8	−0·1156e0	0·1177e0	−0·2337e − 1	0·3391e − 1	−0·7639e − 2	0·7699e − 2
1·78	−0·1548e0	0·1186e0	−0·3082e − 1	0·3376e − 1	−0·1029e − 1	0·7653e − 2
1·75	−0·1979e0	0·1206e0	−0·3796e − 1	0·3391e − 1	−0·1328e − 1	0·7707e − 2

TABLE 4a

Variation of the cosine coefficients of a 8T period solution with respect to λ

λ	p_{10}	p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}	p_{17}	p_{18}	Stability
1·8045	−1·64	−0·57e − 2	−0·42e − 2	0·83e − 2	−0·38e0	−0·11e − 2	−0·12e − 1	0·0	−0·18e0	s
1·804	−1·64	−0·94e − 2	−0·48e − 2	0·14e − 1	−0·37e0	−0·14e − 2	−0·13e − 1	0·13e − 2	−0·18e0	u
1·8	−1·64	−0·13e − 1	−0·42e − 2	0·18e − 1	−0·38e0	−0·16e − 2	−0·12e − 1	0·13e − 1	−0·18e0	u
1·78	−1·62	−0·24e − 1	−0·43e − 2	0·33e − 1	−0·38e0	−0·27e − 2	−0·12e − 1	0·28e − 2	−0·19e0	u
1·75	−1·59	−0·27e − 1	−0·55e − 2	0·36e − 1	−0·38e0	0·15e − 2	−0·17e − 1	0·18e − 2	−0·19e0	u

TABLE 4b

Variation of the sine coefficients of a 8T period solution with respect to λ

λ	p_{21}	p_{22}	p_{23}	p_{24}	p_{25}	p_{26}	p_{27}	p_{28}
1·8045	−0·53e − 2	−0·10e0	−0·85e − 2	0·12e0	0·13e − 2	−0·21e − 1	−0·25e − 2	0·34e − 1
1·804	−0·96e − 2	−0·11e0	−0·17e − 1	0·12e0	0·19e − 2	−0·23e − 1	−0·47e − 2	0·34e − 1
1·8	−0·18e − 1	−0·10e0	−0·27e − 1	0·12e0	0·0	−0·21e − 1	−0·70e − 2	0·34e − 1
1·78	−0·28e − 1	−0·11e0	−0·45e − 1	0·12e0	0·38e − 2	−0·22e − 1	−0·12e − 1	0·35e − 1
1·75	−0·34e − 1	−0·14e0	−0·53e − 1	0·12e0	0·49e − 2	−0·28e − 1	−0·14e − 1	0·37e − 1

TABLE 5a

Variation of the cosine coefficients of a 3T period solution with respect to λ

λ	p_{10}	p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}	Stability
2·1	0·0	1·3775	0·0	−0·9258e − 1	0·0	−0·1596e − 1	0·0	s
1·9	0·0	1·4594	0·0	−0·2631e − 1	0·0	−0·8887e − 2	0·0	s
1·7	0·1056e − 2	1·4393	0·1058e − 2	−0·2129e − 1	0·0	−0·6515e − 2	0·0	u
1·3	0·2785e0	1·1494	0·1813e0	−0·2365e − 1	0·5592e − 1	−0·3658e − 1	−0·3393e − 2	u

TABLE 5b

Variation of the sine coefficients of a 3T period solution with respect to λ

λ	p_{21}	p_{22}	p_{23}	p_{24}	p_{25}	p_{26}
2·1	0·6827e0	0·0	0·1539e0	0·0	0·1109e − 2	0·0
1·9	0·3688e0	0·0	0·1272e0	0·0	0·9757e − 2	0·0
1·7	0·2256e0	−0·3885e − 2	0·1048e0	0·0	0·1084e − 1	0·0
1·3	0·3118e0	−0·3280e0	0·9734e − 1	−0·1765e0	−0·2520e − 1	0·1610e − 1

when $m-n<0$, $x_{m-n}=x_{n-m}$. In the above equations, if x_0, x_1 are retained only, and considering $x(0)=A$, one obtains the following equations

$$x_0 = -\frac{b}{a}\left(\frac{x_0^3}{4} + x_0 x_1^2 + 0{\cdot}5\, x_1^2 x_0\right),$$

$$x_1 = -\frac{b}{a-\omega^2}\left(\frac{3x_0^2 x_1}{4} + \frac{3x_1^3}{4}\right),$$

$$\frac{x_0}{2} + x_1 = A.$$

Solving the above equations gives

$$x_0 = 0{\cdot}0, \qquad x_1 = A, \qquad \omega^2 = a + \tfrac{3}{4} b A^2.$$

This agrees with the results obtained by the perturbation and average methods [2].

Example 2: *Duffing system*. The governing equation of the system is

$$\ddot{x} + 0{\cdot}1\dot{x} - 0{\cdot}5x + 0{\cdot}5x^3 = 0{\cdot}4 \cos \lambda t. \tag{16}$$

From now on, it is assumed that $T=2\pi/\lambda$. In our computation, a $1T(\bar{T}=1T)$ period solution is found. The Fourier coefficients of the solution are shown in Tables 1(a) and (b). For this solution, when $\lambda > 2{\cdot}1$, it is stable, when $\lambda = 2{\cdot}1$, a negative real eigenvalue of $[B]$ is less than -1, so there is period doubling. Two $2T$ period solutions occur after the period doubling. The Fourier coefficients of one of them are illustrated in Tables 2(a) and (b), and those of the others are only different in the sign of even harmonic components. When $\lambda > 1{\cdot}825$, the solution is stable, while $\lambda = 1{\cdot}825$, a negative real eigenvalue of $[B]$ is less than -1. Therefore the second period doubling occurs, and four $4T$ period solutions are born. Only one is listed in Tables 3(a) and (b). The four $4T$ period solutions are stable when $\lambda > 1{\cdot}8045$. While $\lambda = 1{\cdot}8045$, the third period doubling happens, and eight $8T$ period solutions are born. The Fourier coefficients of only one of them are listed in Tables 4(a) and (b). From these tables, one can see that the stable region of the $8T$ period solutions is very small. A $3T$ period solution is obtained, whose Fourier coefficients are listed in Tables 5(a) and (b). When $\lambda > 1{\cdot}7$, the coefficients of the even terms are zero, and so it has symmetry; when $\lambda \geqslant 1{\cdot}7$, it becomes unsymmetric, and therefore a symmetry-breaking bifurcation occurs. The above results obtained by the present method agree with those depicted in Figure 5(d) of reference [11].

TABLE 6

Time histories for $\omega = 1$ obtained with different numbers of harmonic terms (*NH*)

NH *t*:	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π	Source of results
3	5·352	4·813	2·979	0·534	−2·049	−4·176	−5·352	Present paper
	5·352	4·813	2·980	0·534	−2·048	−4·176	−5·352	Reference [12]
5	5·359	4·793	2·984	0·529	−2·042	−4·172	−5·359	Present paper
	5·359	4·793	2·985	0·529	−2·041	−4·172	−5·359	Reference [12]
8	5·360	4·793	2·984	0·528	−2·043	−4·173	−5·360	Present paper
	5·360	4·793	2·985	0·529	−2·043	−4·173	−5·360	Reference [12]

Example 3. Consider the single-degree-of-freedom system consisting of a mass, a viscous damper, and a piecewise-linear spring. The equation of motion can be written as

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+c\frac{\mathrm{d}x}{\mathrm{d}t}+kx+F(x)=\cos\omega t,$$

where m, c, k, t and x denote the mass, viscous damping coefficient, linear spring constant, time and displacement, respectively. ω is the frequency of the external force. The non-linear restoring force $F(x)$ can be expressed as follows:

$$F(x)=k_1h(x-e_1)(x-e_1)+k_{-1}h(e_{-1}-x)(x-e_{-1}),$$

TABLE 7a

Variation of the cosine coefficients of a 1*T period solution* u_1 *with respect to f*

f	p_{10}	p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}	u_1 $(t=0)$
1·0	0·0	1·184	0·0	5·295e − 2	0·0	0·0	0·0	1·23
2·1	0·0	1·462	0·0	1·504e − 1	0·0	1·092e − 2	0·0	1·62
2·15	−1·571e − 2	1·471	0·0	1·543e − 1	0·0	1·134e − 2	0·0	1·63
2·5	3·086e − 1	1·399	−1·292e − 1	7·188e − 2	1·308e − 2	−1·991e − 2	6·298e − 3	1·75
3·0	4·662e − 1	1·426	−2·738e − 1	6·473e − 2	−2·734e − 3	−2·892e − 2	8·828e − 3	1·85
6·6	6·636e − 1	1·813	−4·461e − 1	5·495e − 1	−9·899e − 2	9·719e − 2	−2·864e − 2	2·32

TABLE 7b

Variation of the sine coefficients of a 1*T period solution* u_1 *with respect of f*

f	p_{21}	p_{22}	p_{23}	p_{24}	p_{25}	p_{26}
1·0	3·942e − 1	0·0	5·986e − 2	0·0	4·456e − 3	0·0
2·1	2·708e − 1	0·0	6·613e − 2	0·0	9·457e − 3	0·0
2·15	2·685e − 1	3·922e − 2	6·630e − 2	6·1051e − 3	9·6340e − 3	0·0
3·0	3·606e − 1	−6·951e − 1	7·841e − 2	−1·081e − 1	2·423e − 2	−7·208e − 3
6·6	2·559e − 1	−2·184e − 1	3·779e − 2	−8·317e − 2	3·097e − 2	−2·478e − 2

TABLE 7c

Variation of the cosine coefficients of a 1*T period solution* u_2 *with respect to f*

f	p_{10}	p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}	u_1 $(t=0)$
1·0	0·0	1·185	0·0	5·295e − 2	00	0·0	0·0	1·12
2·1	0·0	1·419	0·0	1·257e − 1	0·0	5·931e − 3	0·0	1·55
2·15	1·039e − 2	1·429	−7·479e − 2	1·302e − 1	0·0	6·453e − 3	0·0	1·56
3·0	1·588e − 1	1·480	9·297e − 2	1·409e − 1	4·944e − 2	−2·436e − 3	9·114e − 3	1·42
6·6	5·400e − 1	1·797	−3·222e − 1	5·419e − 1	−5·575e − 2	8·536e − 2	−1·322e − 2	2·317

TABLE 7d

Variation of the sine coefficients of a 1*T period solution* u_2 *with respect to f*

f	p_{21}	p_{22}	p_{23}	p_{24}	p_{25}	p_{26}
1·0	5·598e − 1	0·0	7·144e − 2	0·0	3·600e − 3	0·0
2·1	4·250e − 1	0·0	9·549e − 2	0·0	1·226e − 2	0·0
2·15	4·219e − 1	−0·980e − 3	9·613e − 2	−1·928e − 3	1·262e − 2	0·0
3·0	4·302e − 1	−4·477e − 1	8·324e − 2	−6·842e − 2	2·898e − 3	−5·502e − 3
6·6	4·241e − 1	−2·883e − 1	6·302e − 2	−1·097e − 1	4·055e − 2	−3·431e − 2

TABLE 8

The error of the 1T period solution

t	0·0	$T/5$	$2T/5$	$3T/5$	$4T/5$	T
ER	−1·33e − 2	1·44e − 2	−5·31e − 3	−2·25e − 3	9·42e − 3	−1·33e − 2

where $h(x - e_1)$ is the step function, which satisfies the following conditions

$$h(x - e_1) = \begin{cases} 1 & \text{when } x \geqslant e_1 \\ 0 & \text{when } x < e_1 \end{cases}.$$

when $m = 1$, $k = 1$, $k_1 = k_{-1} = 9$, $e_1 = 5$, $e_{-1} = -5$, $c/2\sqrt{mk} = 0{\cdot}01$, the fundamental response of the symmetric piecewise linear system is obtained. To compare with the results of reference [12] and show the convergence of the solution obtained, this problem has also been computed with three, five and again eight odd harmonic terms (i.e., up to cos $(15\,\omega t)$ and sin $(15\,\omega t)$). The time histories obtained for $\omega = 1$ are listed in Table 6. Comparisons with the results of reference [12] show good agreement. The fundamental solution obtained by the present method is

$$\begin{aligned} &5{\cdot}21211 \cos t + 0{\cdot}108543 \cos 3t + 0{\cdot}0274508 \cos 5t + 0{\cdot}00877528 \cos 7t \\ &\quad + 0{\cdot}00266933 \cos 9t + 0{\cdot}552429 \sin t + 0{\cdot}0344767 \sin 3t + 0{\cdot}0157503 \sin 5t \\ &\quad + 0{\cdot}00788707 \sin 7t + 0{\cdot}00367589 \sin 9t + 0{\cdot}00137932 \sin 11t. \end{aligned}$$

Example 4. Consider a system of two simply supported beam with immovable ends linked by a linear spring. If only one mode of vibration is considered, according to reference [13], the governing equations of the system are

$$\ddot{u}_1 + 2\mu_1\dot{u}_1 + (k_1 + k_s)u_1 + \alpha_1 u_1^3 - k_s u_2 = f_1 \cos t,$$

$$\ddot{u}_2 + 2\mu_2\dot{u}_2 + (k_2 + k_s)u_2 + \alpha_2 u_2^3 - k_s u_1 = f_2 \cos t,$$

where μ_i, k_i, α_i, k_s and f_i are the coefficients of damping, linear stiffness, cubic stiffness, linking stiffness and excitation amplitude of the system, respectively. In the following discussion, the system parameters are assumed to be $k_1 = k_2 = 0{\cdot}5$, $\alpha_1 = \alpha_2 = 1$, $\mu_1 = 0{\cdot}1$, $\mu_2 = 0{\cdot}2$, $k_s = 0{\cdot}3$, $f_1 = f_2 = f$. Using the above method, the solution of period one is obtained. Its Fourier coefficients are listed in Tables 7(a)–(d). When $f = 2{\cdot}15$, the coefficients of even terms are not equal to zero again, and so the symmetry-breaking occurs. At $f = 6{\cdot}6$, a eigenvalue of $[B]$ is less than -1, and so doubling bifurcation happens. These results agree with those of reference [13].

TABLE 9

The error of the 1T period solution

t	0·0	$T/5$	$2T/5$	$3T/5$	$4T/5$	T
ER	−4·84e − 4	3·00e − 4	−4·96e − 5	−2·20e − 4	4·01e − 4	−4·87e − 4

5. THE CONVERGENCE AND ERROR

Consider the $1T$ period solution shown in Tables 1(a) and (b), when $\lambda = 2{\cdot}0$, of p_{10}, p_{11}, p_{21} are retained as the unknown numbers in the algebraic equations (10a,b,c) one obtains

$$-1{\cdot}973900600248/2 - 0{\cdot}13124110735487 \cos 2t + 0{\cdot}0086481375894509 \sin 2t. \quad (17)$$

Substituting expressions (17) into the left side of equation (16) and subtracting the right side of equation (16) gives the error denoted by ER. The results are listed in Table 8. If $p_{10}, p_{11}, p_{12}, p_{21}, p_{22}$ are taken as the unknown numbers, the solution has the following form:

$$-1{\cdot}9738340460742/2 - 0{\cdot}131382794580 \cos 2t - 8{\cdot}4214983507991e - 4 \cos 2t$$

$$+ 8{\cdot}668843073719e - 3 \sin 2t + 1{\cdot}3434170978144e - 4 \sin 4t.$$

The errors described in the above are listed in Table 9. Therefore, one can see that the new method converges rapidly and the solutions obtained by the present method are very accurate.

6. CONCLUSION

In the present paper, a new approach is given for non-linear vibration. This method can be widely used to solve non-linear vibration problems, and can also be used to study the bifurcation phenomenon, especially the symmetry-breaking and period doubling bifurcation. Moreover, it has good convergence and accuracy.

REFERENCES

1. A. H. NAYFEH 1973 *Perturbation Methods*. New York: John Wiley.
2. A. H. NAYFEH and D. T. MOOK 1979 *Nonlinear Oscillations*. New York: John Wiley.
3. M. URABE 1965 *Archives for Rational Mechanics and Analysis* **20**, 120–152. Galerkin's procedure for nonlinear periodic systems.
4. M. URABE and A. REITER 1966 *Journal of Mathematics Applications and Analysis* **14,** 107–140. A numerical computation of nonlinear forced oscillations by Galerkin procedure.
5. S. L. LAU and Y. K. CHEUNG 1981 *Journal of Applied Mechanics* **48**, 959–964. Amplitude incremental variational principle for nonlinear vibration of elastic system.
6. S. L. LAU, Y. K. CHEUNG and S. Y. WU 1983 *Journal of Applied Mechanics* **50**, 871–816. Incremental harmonic balance method with multiple time scales for aperiodic vibration of nonlinear system.
7. A. A. FERRI 1986 *Journal of Applied Mechanics* **53**, 455–457. On the equivalence of the incremental harmonic balance method and the harmonic balance–Newton Raphson Method.
8. M. XU and D. CHENG 1994 *Journal of Sound and Vibration* **177**, 565–571. A new approach to solving a type of vibration problem.
9. P. FERIEDMANN, C. E. HAMMOND and T. H. WOO 1977 *International Journal for Numerical Method in Engineering* **11**, 1117–1136. Efficient numerical treatment of periodic system with application to stability problem.
10. YAN ZONG DA 1989 *The Fourier Series Method in Structure Mechanics*. Tian Jin: Tian Jin University Press (in Chinese).
11. K. L. JANICKI and W. SZEMPLIN SKA-STUPNICKA 1995 *Journal of Sound and Vibration* **180**, 253–269. Subharmonic resonances and criteria for escape and chaos in a driven oscillator.
12. S. L. LAU and W.-S. ZHANG 1992 *Journal of Applied Mechanics* **59**, 153–160. Nonlinear vibrations of piecewise-linear system by incremental harmonic balance method.
13. A. Y. T. LEUNG and S. K. CHUI 1995 *Journal of Sound and Vibration* **181**, 619–633. Nonlinear vibration of coupled Duffing oscillators by an improved incremental harmonic balance method.