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VIBRATION AND BUCKLING OF INITIALLY STRESSED CURVED BEAMS

L.-W. Chen and G.-S. Shen
Department of Mechanical Engineering, National Cheng Kung University, 701 *Tainan, Taiwan, Republic of China*

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Equations of motion for curved beams in a general state of non-uniform initial stresses are derived using the principle of virtual work. The equations are adjusted to a generic expression by using appropriate transformations. The free vibration behaviours of the curved beams subjected to a combination of uniform initial tensile of compressive stresses and uniform initial bending stress are examined. The Galerkin method is employed in obtaining accurate values of free frequencies and initial buckling stresses. The curved beam is assumed to be vibrating in its plane. Natural frequencies and initial buckling stresses for hinged supported curved beams are presented for validation. Effects of arc segment angles, elastic foundation, and initial stresses on the natural frequencies are investigated. Effects of arc segment angles, elastic foundation, and initial bending stresses on the initial buckling stresses are explored in this paper.

1. INTRODUCTION

Beams can be straight or curved. Curved beams are often described as arches when they are open and as rings when they are closed. Open curved beams can have deep or shallow curvature. This paper addresses beams of shallow curvature. Vibration analysis of curved beams was the subject of two survey studies [1, 2]. These studies list more than 200 references on the subject. A recent review article [3] on the subject uncovered more than 400 references. Irie *et al*. [4] studied the natural frequencies of in plane vibration of arcs. Mau and Williams [5] used Green's function to solve the arch vibrations. The effect of an elastic foundation on the free vibration frequency of a shallow arch subjected to a thrust load was analysed by Plaut and Johnson [6]. Perkins [7] formulated the planar vibration of an elastica arch. Chidamparam and Leissa [8] considered the influences of the centreline extensibility on the in-plane free vibrations of a loaded circular arch.

Very little has been done with respect to the formulation and solution of the governing equations for curved beams in an arbitrary state of initial stresses. The theory is more intricate than that for unloaded configurations for one must derive equations of motion based upon small displacements to obtain a linear eigenvalue problem, away from an initial, loaded equilibrium state. Brunelle and Robertson [9] used the average stress method and the variation method to derive the linear equations of motion for a thick plate in an arbitrary state of initial stress. They also considered the effects of initial stress on the natural frequencies of a vibrating thick rectangular plate [10]. Chen *et al*. [11] employed the Galerkin method to solve the thermal stability problems for simply supported transversely thick plates in a state of initial stress where the effects of rotary inertia and

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transverse shear were included. Lin and Soedel [12] developed the equations of motion for the general in-plane motion of rotating thick and thin rings on elastic foundations subjected to initial stresses. They used the nonlinear strain displacement relations to form the energy term. Yang and Shieh [13] derived the equations of motion for the initially stressed thick rectangular orthotropic plates.

There are some works on buckling of arches [14–17]. Yang and Liu [18] studied the buckling and bending behaviour of initially stressed orthotropic thick plates. Smith and Hermann [19] considered the influences of stability of a beam on an elastic foundation by subjecting a follower force. Filipich and Rosales [20] used Rayleigh's method to analyse a Timoshenko beam embedded in a Winkler–Pasternak medium.

In this paper, the virtual work expressions of initially stressed curved beams are derived for a general state of non-uniform initial stress. The characteristic equations for determining natural frequencies and initial buckling stresses are derived using the Galerkin method. The influence of arc segment angles, elastic foundation, and initial stresses on the natural frequencies and effects of arc segment angles, elastic foundation, and initial bending stresses on the initial buckling stresses are investigated.

2. THE VIRTUAL WORK PRINCIPLE

The equations of equilibrium and boundary traction condition in tensor form can be expressed in terms of Trefftz stress components as [21, 22]

$$\lfloor(\delta_{sj} + v_{s,j})t^*_{ij}\rfloor_{,i} + f^*_s = 0 \tag{1}$$

$$P^*_s = (\delta_{sj} + v_{s,j})t^*_{ij}n_i. \tag{2}$$

If relative extensions and shears are small, then the final area and volume are equal to the initial area and volume. It means that $t^*_{ij} = t_{ij}, f^*_s = f_s$ and $P^*_s = P_s$, where t_{ij}, f_s, and P_s are the actual stresses, the body forces, and surface tractions respectively. Using the technique described by Bolotin [23], t_{ij} and v_s are chosen to be the equilibrium large deformation stresses and displacements [i.e. the initial (deformed) state], $t_{ij} + \sigma_{ij}$ and $v_s + u_s$ are chosen to be the final state values after perturbations σ_{ij} and u_s have taken place. Additionally, f_s and P_s are initial large deformation quantities that become, respectively, $f_s + \Delta f_s + x_s - \rho\ddot{u}_s$ and $P_s + \Delta P_s + p_s$ in the final state. It is generally assumed that the terms $\sigma_{ij}u_{s,j}$ and the initial displacement gradient, $v_{s,j}$ are small enough to drop. Thus, equations (1) and (2) can be linearized and simplified as

$$\lfloor t_{ij}u_{s,j} + \sigma_{is}\rfloor_{,i} + \Delta f_s + x_s - \rho\ddot{u}_s = 0 \tag{3}$$

$$\Delta P_s + p_s = (t_{ij}u_{s,j} + \sigma_{is})n_i. \tag{4}$$

Oyibo [24] has derived the virtual work principle by multiplying these two equations by the variation of the displacement components δu_s, and then integrating the resulting expression over the initial volume V [equation (3)] or the surface area σ [equation (4)]. A perturbation strain energy density, $\tilde{u}$, defined by $\sigma_{is} = \partial\tilde{u}/\partial\varepsilon_{is}$, is introduced. Then, by using the product differentiation rules and the divergence theorem, the virtual work principle can be expressed as

$$\delta(\bar{v}+\bar{w})+\int_V \rho\ddot{u}_s\delta u_s\,\mathrm{d}V=\int_\sigma(\Delta P_s+p_s)\delta u_s\,\mathrm{d}\sigma+\int_V(\Delta f_s+x_s)\delta u_s\,\mathrm{d}V \tag{5}$$

where

$$\delta\bar{v}=\delta\int_V \tilde{u}\,\mathrm{d}V=\delta\int_V\left(\frac{1}{2}\sigma_{ij}\varepsilon_{ij}\right)\mathrm{d}V \tag{6}$$

$$\delta\bar{w}=\delta\int_V\frac{1}{2}t_{ij}u_{s,i}u_{s,j}\,\mathrm{d}V \tag{7}$$

and

$$\varepsilon_{ij}=\tfrac{1}{2}(u_{i,j}+u_{j,i}).$$

The virtual work done by elastic foundation is [25]

$$\delta U_f=-\tfrac{1}{2}k_fR\int_\theta w\,\delta w\,\mathrm{d}\theta. \tag{8}$$

This form of virtual work principle will be used subsequently in deriving the needed equations of motion and the associated boundary conditions for curved beams.

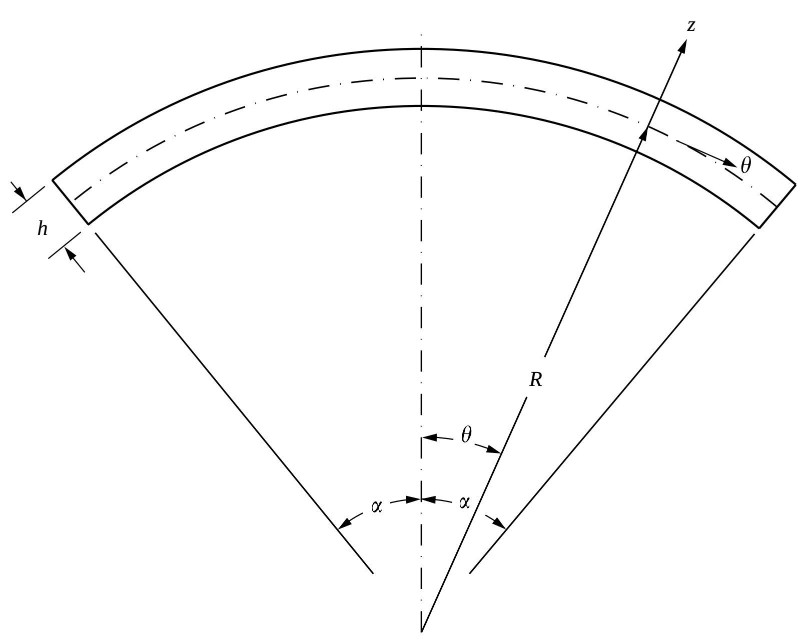

Figure 1. A curved beam

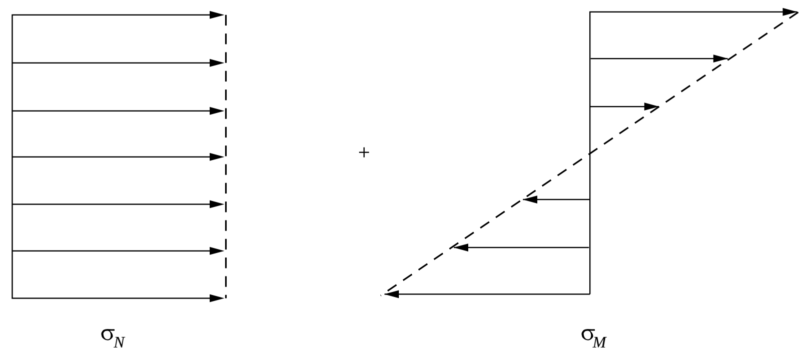

Figure 2. The initial stresses.

3. GOVERNING EQUATIONS FOR INITIALLY STRESSED CURVED BEAMS

A curved beam is illustrated in Figure 1, with the angle θ being measured from the middle of the curved beam. The Reissner–Naghdi type of shell theory will be used in the specialization because, as will be shown, the equations derived using this theory are consistent. Furthermore, the equations derived here are for curved beams subject to in plane loading and/or vibrating in the plane.

The incremental displacement field is assumed to be of the following form:

$$u_1(\theta, z, \tau) = v_0(\theta, \tau) + z\psi(\theta, \tau) \tag{9}$$

TABLE 1

Non-dimensional vibration frequencies of hinged curved beams without initial stresses and elastic foundation

α (degrees)	$\Omega = \omega R^2 \sqrt{\frac{\rho A}{EI}}$ Mode 1	[8]	Mode 2	Mode 3	Mode 4
5	1294·4789	1293·5040	5182·4099	11 662·2848	12 475·7860*
10	322·4973	321·5148	1294·4789	2914·4506	5182·4099
20	79·5094	78·5580	322·4973	727·4890	1294·4789
30	34·5232	33·6261	142·5037	322·4973	574·4919
40	18·7913	17·9641	79·5094	180·7518	322·4973
50	11·5235	10·7761	50·3558	115·1456	205·8608
60	7·5894	6·9268	34·5232	79·5094	142·5037
70	5·2306	4·6534	24·9807	58·0238	104·3024
80	3·7123	3·2179	18·7913	44·0807	79·5094
90	2·6833	2·2667	14·5519	34·5232	62·5125
100	1·9581	1·6133	11·5235	27·6888	50·3558
110	1·4315	1·1515	9·2868	22·6340	41·3624
120	1·0401	0·8179	7·5894	18·7913	34·5232
130	0·7435	0·5724	6·2721	15·8027	29·2019
140	0·5155	0·3891	5·2306	13·4332	24·9807
150	0·3380	0·2505	4·3938	11·5235	21·5764
160	0·1985	0·1447	3·7123	9·9624	18·7913
170	0·0880	0·0632	3·1508	8·6703	16·4842
180	0·0000	0·0000	2·6833	7·5894	14·5519

* It is the θ-direction mode only on $\alpha = 5°$.

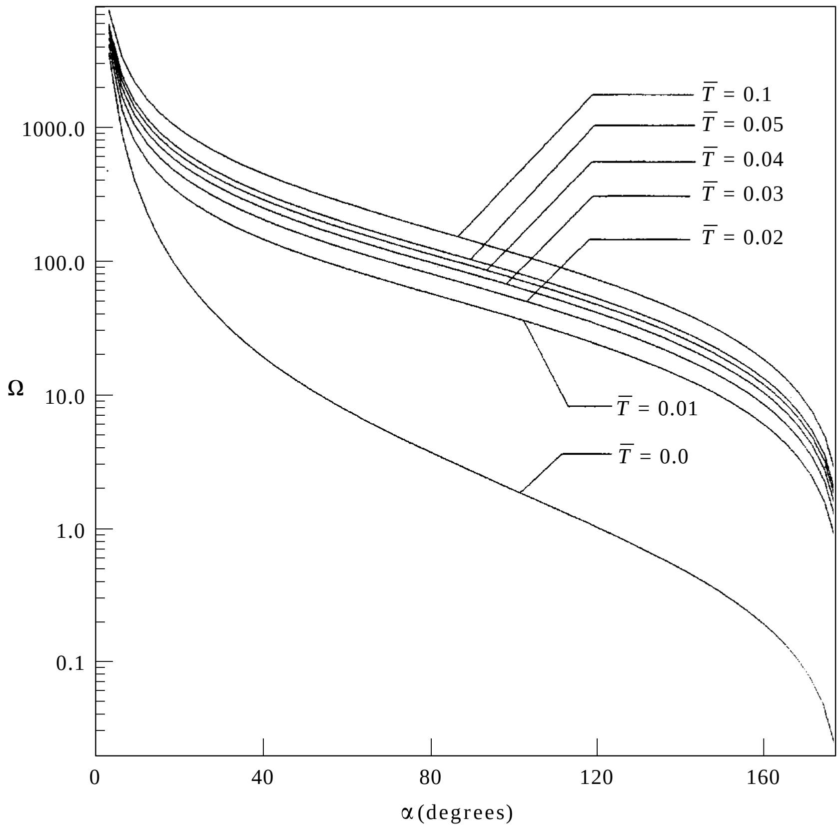

Figure 3. Effects of initial tensile stress parameters $\bar{T}$ and arc segment half angle α on the natural frequency parameter Ω for hinged initially stressed curved beams with $\bar{S} = 0$ and $\tilde{k}_f = 0$.

$$u_3(\theta, z, \tau) = w_0(\theta, \tau) \tag{10}$$

where u_1 is the in plane displacement and u_3 is the lateral deflection. v_0 and w_0 are displacements of the beam middle surface in the θ and z directions, respectively. ψ accounts for the effect of transverse shear.

We follow the Qatu's [26] assumptions the middle surface strain and curvature change are:

$$\varepsilon_{11}^0 = \frac{1}{R}\frac{\partial v_0}{\partial \theta} + \frac{w_0}{R}, \qquad \kappa = -\frac{1}{R^2}\frac{\partial^2 w_0}{\partial \theta^2} + \frac{1}{R^2}\frac{\partial v_0}{\partial \theta} = \frac{1}{R}\frac{\partial \psi}{\partial \theta}\,. \tag{11}$$

By Soedel [27], we assumed that the transverse shear deflection could be neglected. This implies that $\varepsilon_{13}^0 = 0$ and $\psi = (v_0/R) - (1/R)(\partial w_0/\partial \theta)$. However, the integrated effect of the transverse shear stress will not be neglected in this paper. The strain at an arbitrary point can be found from

$$\varepsilon_{11} = \frac{1}{(1+z)/R}\,(\varepsilon_{11}^0 + z\kappa). \tag{12}$$

The term z/R is small in the comparison with unity and can be neglected for thin beams. The well-known Kirchoff hypothesis normals to the middle surface remain straight and normal, and unstretched in length during deformation [28, 29] was used. In the following derivation, the subscript $_0$ will be dropped from the displacement terms for convenience.

The incremental stress–displacement relation is taken to be those of linear elasticity.

$$\sigma_{11} = Eu_{1,1} = E\left(\frac{v' + w}{R} + \frac{z}{R}\psi'\right) \tag{13}$$

where the $(\,)' = \partial(\,)/\partial\theta$.

The force and moment resultants acting on the curved beams are obtained by the integration of the stress through the beam thickness (h) and multiplied through by the width b of the beam.

$$\tilde{N} = b\int_{-h/2}^{h/2} \sigma_{11}\,\mathrm{d}z = \frac{EA}{R}(v' + w), \qquad Q = b\int_{-h/2}^{h/2} \sigma_{13}\,\mathrm{d}z$$

$$\tilde{M} = b\int_{-h/2}^{h/2} \sigma_{11}z\,\mathrm{d}z = \frac{EI}{R}\psi' = \frac{EI}{R^2}(v' - w'')$$

$$(N_x, M_x, M_x^*) = b\int_{-h/2}^{h/2} t_{11}(1, z, z^2)\,\mathrm{d}z. \tag{14}$$

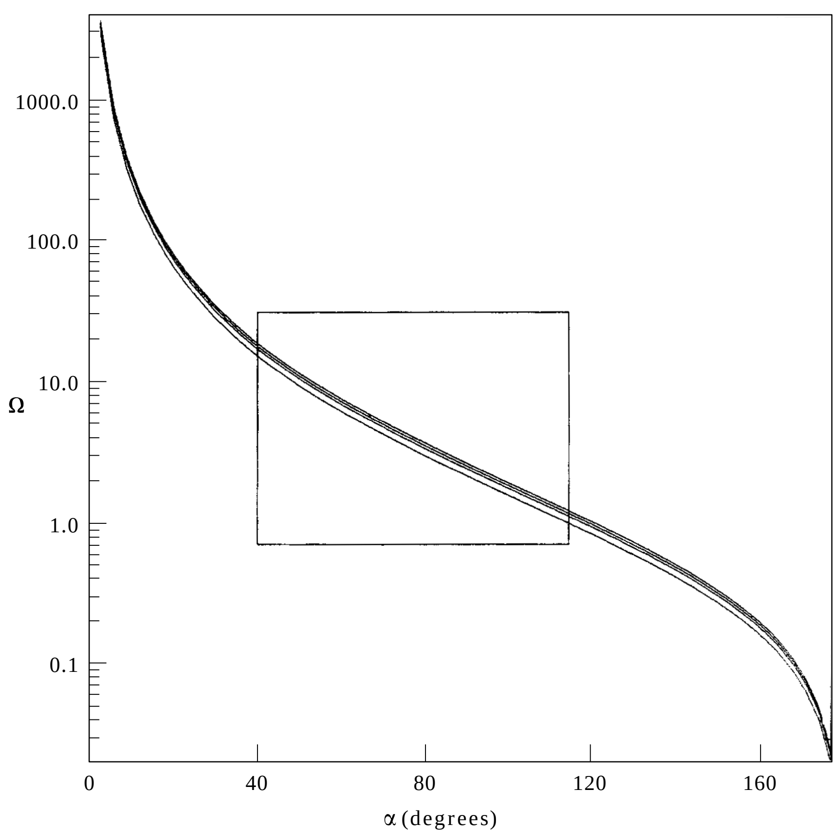

Figure 4. Effects of initial bending stress parameters $\bar{S}$ and arc segment half angle α on the natural frequency parameter Ω for hinged initially stressed curved beams with $\bar{T} = 0$ and $\tilde{k}_f = 0$.

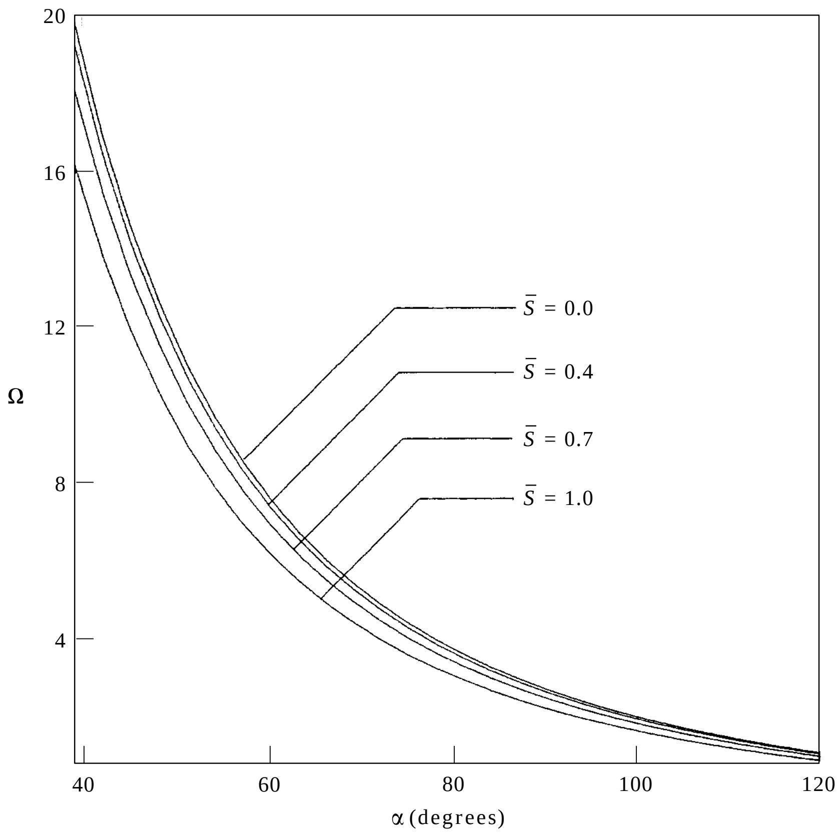

Figure 5. Effects of initial bending stress parameters $\bar{S}$ and arc segment half angle α on the natural frequency parameter Ω for hinged initially stressed curved beams with $\bar{T} = 0$ and $\tilde{k}_f = 0$. (Zoomed in region of Fig. 4.)

By expanding the tensor terms of the virtual work principle in equation (5), using equations (6)–(14) and integrating through the curved beam thickness and multiplying the beam width, the virtual work principle can be obtained.

Consider a curved beam in a general state of initial stress and the initial stress distribution is a function of z only. Then the state of initial stress is

$$t_{11} = \sigma_N + \frac{2z}{h}\sigma_M. \tag{15}$$

It comprises a tension (or compression) stress plus bending stress with all other initial stresses assumed to be zero. It can be shown in Figure 2. From equation (14) the only non-zero initial stresses are

$$N_x = A\sigma_N$$

$$M_x = \frac{bh^2}{6}\sigma_M = \frac{2}{h}I\sigma_M$$

$$M_x^* = \frac{bh^3}{12}\sigma_N = I\sigma_N. \tag{16}$$

By taking variation with respect to the displacement components of the virtual work principle and substituting equations (15) and (16) into the equation of the virtual work principle, the governing equations and the associated boundary conditions for curved beams subjected to initial stress are derived as

$$\tilde{N}' + Q + \frac{1}{R}[N_x(v' + w)]' + \frac{1}{R}N_x(w' - v) + \frac{1}{R^2}[M_x(v' - w'')]' = \rho A R \ddot{v} \qquad (17a)$$

$$\tilde{N} - Q' + \frac{1}{R}N_x(v' + w) + \frac{1}{R^2}M_x(v' - w'') - \frac{1}{R}[N_x(w' - v)]' + \rho A R \ddot{w} + k_f R w = 0 \qquad (17b)$$

$$\tilde{M}' + \frac{1}{R}[M_x(v' + w)]' + \frac{1}{R^2}[M_x^*(v' - w'')]' = RQ \qquad (17c)$$

$$v = 0 \quad \text{or} \quad \tilde{N} + \frac{\tilde{M}}{R} + \frac{1}{R}N_x(v' + w) + \frac{1}{R^2}M_x(2v' + w - w'') + \frac{M_x^*}{R^3}(v' - w'') = 0$$

$$w = 0 \quad \text{or} \quad Q + \frac{1}{R}\tilde{N}(w' - v) = 0$$

$$w' = 0 \quad \text{or} \quad -\left[\frac{\tilde{M}}{R} + \frac{M_x}{R^2}(v' + w) + \frac{M_x^*}{R^3}(v' - w'')\right] = 0. \qquad (18)$$

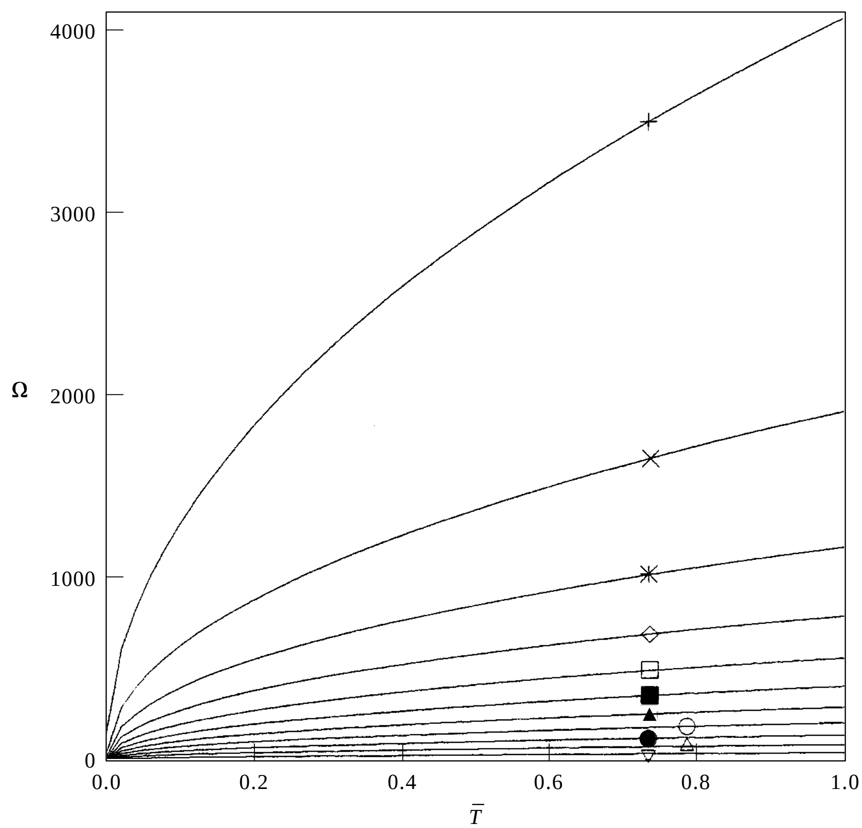

Figure 6. Effects of arc segment half angle α and initial tensile stress parameter $\bar{T}$ on the natural frequency parameter Ω for hinged initially stressed curved beams with $\bar{S} = 0$ and $\bar{k}_f = 0$. $+ \alpha = 15°$; $\times \alpha = 30°$; $* \alpha = 45°$; $\diamond \alpha = 60°$; $\square \alpha = 75°$; $\blacksquare \alpha = 90°$; $\blacktriangle \alpha = 105°$; $\bigcirc \alpha = 120°$; $\bullet \alpha = 135°$; $\triangle \alpha = 150°$; $\triangledown \alpha = 165°$.

TABLE 2

Non-dimensional vibration frequencies of hinged initially stressed curved beams without elastic foundation

$$\Omega = \omega R^2 \sqrt{\frac{\rho A}{EI}}$$

$\alpha =$	15°	45°	75°	105°	135°	165°
			$\bar{T} = 0{\cdot}05$			
$\bar{S} = 0{\cdot}0$	930·74	280·66	140·17	74·34	35·38	9·72
$\bar{S} = 0{\cdot}1$	930·39	280·57	140·13	74·33	35·38	9·72
$\bar{S} = 0{\cdot}2$	929·98	280·48	140·09	74·31	35·37	9·72
$\bar{S} = 0{\cdot}3$	929·49	280·39	140·05	74·29	35·36	9·72
$\bar{S} = 0{\cdot}4$	928·93	280·30	140·01	74·27	35·35	9·72
			$\bar{T} = 0{\cdot}5$			
$\bar{S} = 0{\cdot}0$	2894·65	851·09	410·25	212·36	99·74	27·27
$\bar{S} = 0{\cdot}1$	2893·72	850·88	410·17	212·33	99·73	27·26
$\bar{S} = 0{\cdot}2$	2892·77	850·68	410·10	212·31	99·72	27·26
$\bar{S} = 0{\cdot}3$	2891·79	850·47	410·03	212·28	99·71	27·26
$\bar{S} = 0{\cdot}4$	2890·79	850·26	409·95	212·25	99·70	27·26

If the initial stresses and elastic foundation are neglected, equations (17a)–(17c) will be reduced to the same as the Seodel [27]

$$\tilde{N}' + Q = \rho A R \ddot{v}$$
$$\tilde{N} - Q' + \rho A R \ddot{w} = 0$$
$$\tilde{M}' = RQ. \tag{19}$$

Substitute equations (17c), (14), and (16) into equations (17a) and (17b), and the equations of motion for the initially stressed curved beams become:

$$(v'' + w') + \beta(v'' - w''') + \beta\bar{R}\bar{S}(2v'' + w' - w''') + \beta\bar{T}(v'' - w''') + \bar{T}(v'' + 2w' - v) - \beta\lambda^2\ddot{v} = 0 \tag{20a}$$

$$(v' + w) - \beta(v''' - w'''') - \beta\bar{R}\bar{S}(v''' + 2w'' - v') - \beta\bar{T}(v''' - w'''') + \bar{T}(2v' + w - w'') + \beta\tilde{k}_f + \beta\lambda^2\ddot{w} = 0 \tag{20b}$$

where

$$\beta = \frac{I}{AR^2}, \qquad \bar{R} = \frac{2R}{h}, \qquad \bar{S} = \frac{\sigma_M}{E}, \qquad \bar{T} = \frac{\sigma_N}{E}, \qquad \lambda = \sqrt{\frac{\rho A}{EI}}\,R^2, \qquad \tilde{k}_f = \frac{k_f R^4}{EI}.$$

Since the free vibration and initial buckling stresses under a general state initial stress are governed by differential equation with variable coefficients, exact vibration frequencies and initial buckling stresses are very difficult to obtain. The Galerkin method is therefore employed in computing the frequencies and initial buckling stresses. For linear problems, the method guarantees convergence from the above to the true frequency if one assumes a mathematicaly complete set of trial functions satisfying the geometric boundary conditions of the problem to present the solution of the differential equation. By assuming time-harmonic solutions of frequency ω as before:

$$v(\theta, t) = \tilde{v}(\theta) \sin \omega t, \qquad w(\theta, t) = \tilde{w}(\theta) \sin \omega t. \tag{21}$$

For hinged curved beams, the boundary conditions in equation (18) are

$$w = 0$$

$$\tilde{N} + \frac{\tilde{M}}{R} + \frac{1}{R} N_x(v' + w) + \frac{1}{R^2} M_x(2v' + w - w'') + \frac{M_x^*}{R^3}(v' - w'') = 0$$

$$\frac{\tilde{M}}{R} + \frac{M_x}{R^2}(v' + w) + \frac{M_x^*}{R^3}(v' - w'') = 0.$$

One assumes the displacement field is the following form:

$$\tilde{v} = \sum_{m=1}^{\infty} b_m \cos mk\theta \qquad \tilde{w} = \sum_{m=1}^{\infty} a_m \sin mk\theta \tag{22}$$

where $k = \pi/\alpha$.

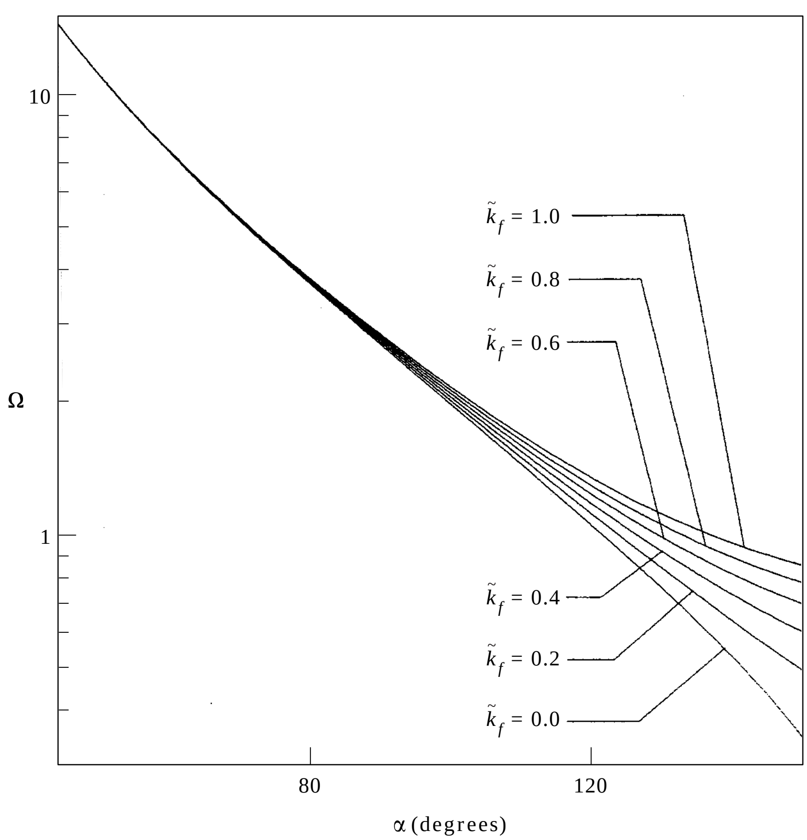

Figure 7. Effects of elastic foundation parameter $\tilde{k}_f$ and arc segment half angle α on the natural frequency parameter Ω for hinged initially stressed curved beams with $\bar{T} = 0$ and $\bar{S} = 0$.

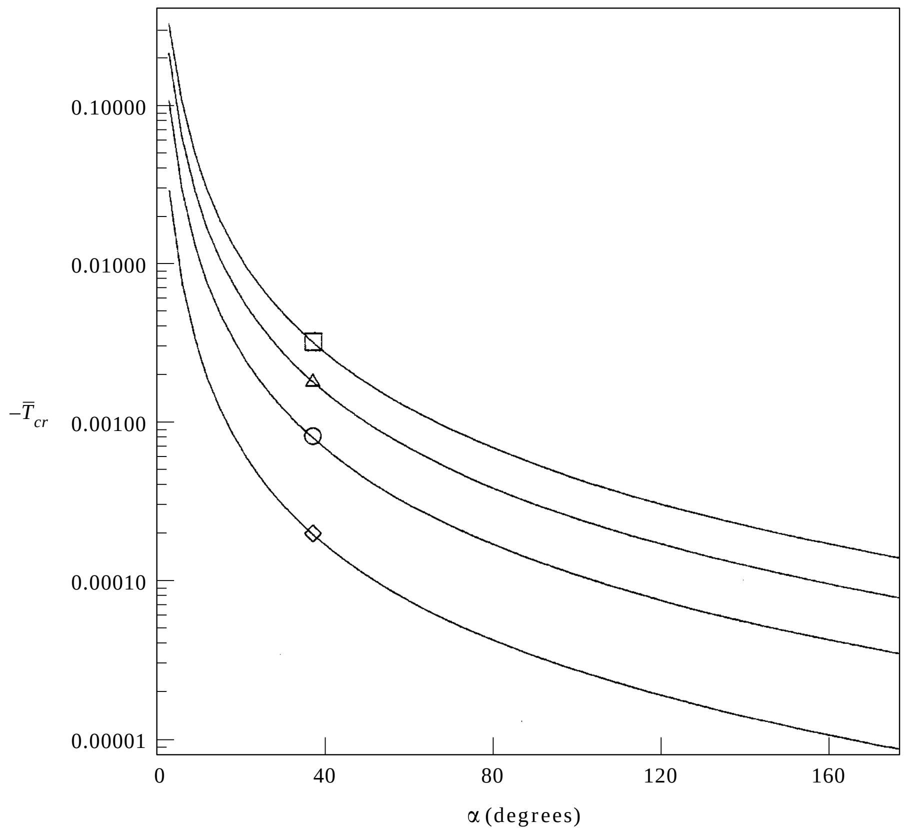

Figure 8. Effects of arc segment half angle α on initial buckling stress parameter $\overline{T}_{cr}$ for hinged initially stressed curved beams with $\overline{S}=0$ and $\tilde{k}_f=0$. □ $m=4$; △ $m=3$; ○ $m=2$; ◇ $m=1$.

If one assumes that

$$\tilde{v}=\sum_{m=1}^{\infty} b_m \sin mk\theta \quad \text{and} \quad \tilde{w}=\sum_{m=1}^{\infty} a_m \cos mk\theta$$

in equation (22), then the problem of curved beam having simple supports is solved. This will yield a frequency determinant that is the same as that of hinged curved beams, and consequently the same natural frequencies whatever initial stresses are neglected.

A four term Galerkin method is employed to obtain the following characteristic equations for the determination of the natural frequencies and initial buckling stresses.

4. RESULTS AND DISCUSSION

The material properties and the geometric dimensions of the curved beam in the following computations are $E=2{\cdot}0\times10^5$ N/mm^2, $\rho=7{\cdot}75\times10^{-6}$ kg/mm^3, $b=20{\cdot}0$ mm, $h=10{\cdot}0$ mm, and $R=1000{\cdot}0$ mm.

The free vibration frequencies of a curved beam were computed with inclusion of initial stresses and elastic foundation. The non-dimensional natural frequency parameter Ω is defined as a function of the arc segment half angle α of the curved beam with no initial stresses and elastic foundation. The lowest four non-dimensional natural frequencies of the hinged curved beams without initial stresses and elastic foundation and the lowest non-dimensional frequencies shown in [8] are presented in Table 1. The differences between the computed results shown in Table 1 and the lowest non-dimensional frequencies from reference [8] are insignificant in the small arc segment half angle α. The fourth vibration mode of the hinged curved beams has been converted from the z-direction mode to the θ-direction mode as the arc angle $\alpha = 5^\circ$. Figure 3 shows the effects of initial tensile stress parameter $\bar{T}$ and arc segment half angle α on the non-dimensional frequencies when initial bending stress and elastic foundation are ignored. Clearly, the initial tensile stress inspires the increases in the frequencies of the curved beam as the initial bending stresses and elastic foundation are ignored. It also shows the effect of arc segment half angle α on the natural frequency parameter Ω of the initially stressed curved beam. It is clear that the non-dimensional natural frequency parameter Ω decreases rapidly as the arc segment half angle α increases. As has been shown in Figures 4 and 5, the existence of initial bending stress reduces the natural frequency parameter Ω, and the effects of initial bending stress on the non-dimensional natural frequency parameter Ω are ambiguous. The effects of initial tensile stress on the non-dimensional natural frequency parameter Ω with varied arc angle α are shown in Figure 6. As the arc angle α enlarges, the influence of initial tensile stress on the non-dimensional natural frequency parameter Ω is reduced. The frequency parameter Ω increases rapidly as the initial tensile stress parameter $\bar{T}$ enlarges on the region of $\alpha \leqslant 30^\circ$. The frequency parameter Ω is almost unaffected as the initial tensile stress

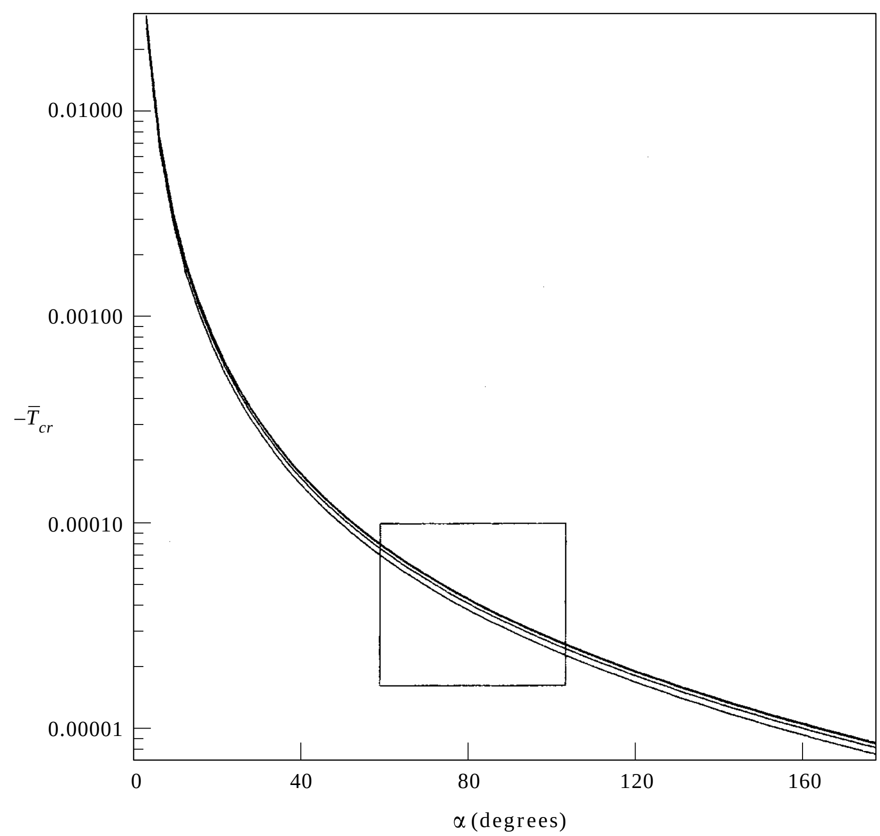

Figure 9. Effects of initial bending stress parameter $\bar{S}$ and arc segment half angle α on initial buckling stress parameter $\bar{T}_{cr}$ for hinged initially stressed curved beams with $\tilde{k}_f = 0$.

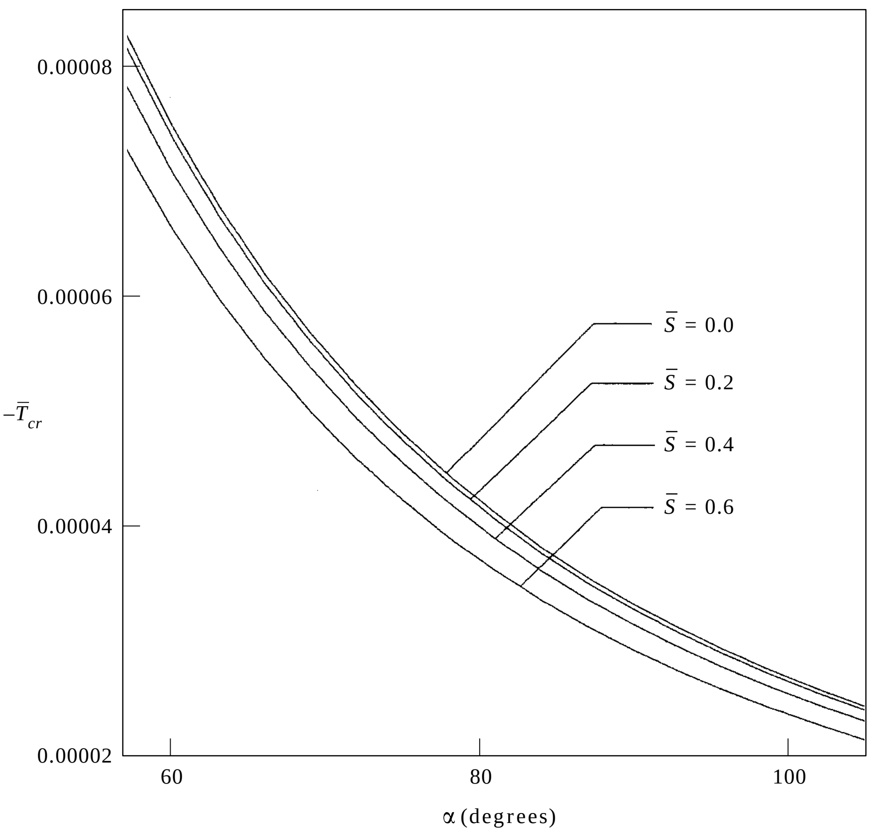

Figure 10. Effects of initial bending stress parameter $\bar{S}$ and arc segment half angle α on initial buckling stress parameter $\bar{T}_{cr}$ for hinged initially stressed curved beams with $\tilde{k}_f = 0$. (Zoomed in region of Fig. 9.)

parameter $\bar{T}$ increases on the region of $\alpha \geqslant 120°$. The non-dimensional frequencies of the initially stressed curved beam without elastic foundation are shown in Table 2. The frequency parameter Ω decreases as the initial bending stress parameter $\bar{S}$ increases whether the initial tensile stress parameter $\bar{T} = 0{\cdot}05$ or $\bar{T} = 0{\cdot}5$. The influence of initial bending stress on the frequency parameter Ω is gradually unclear as the arc angle α increases. Figure 7 displayed the effects of elastic foundation parameter $\tilde{k}_f$ on natural frequency parameter Ω as the arc angle α varies. The elastic foundation raises the natural frequencies as $\alpha \geqslant 80°$ and it is especially obvious as $\alpha \geqslant 120°$.

The buckling of initially stressed curved beam is studied by letting $\Omega^2 = 0$ and seeking values of initial buckling stress for various values of other parameters. The lowest four initial buckling stresses with variant arc angle α are shown in Figure 8. It is obvious that the initial buckling stress parameter $\bar{T}_{cr}$ approaches zero as arc segment half angle α increases. The minus sign of the initial buckling stress is indicative that the initial buckling stress is initial compress stress. The effects of initial bending stress on the initial buckling stress with variant arc angle α are sketched in Figures 9 and 10. As initial bending stress increases, the initial buckling stress approaches zero. The effects of initial bending stress on the initial buckling stress are unclear. Figure 11 shows the effects of elastic foundation on the initial buckling stress. It indicates that the elastic foundation increases the initial buckling stress. When arc angle increases, the influence of elastic foundation on initial buckling stress is gradually obvious, especially as arc angle $\alpha \geqslant 120°$.

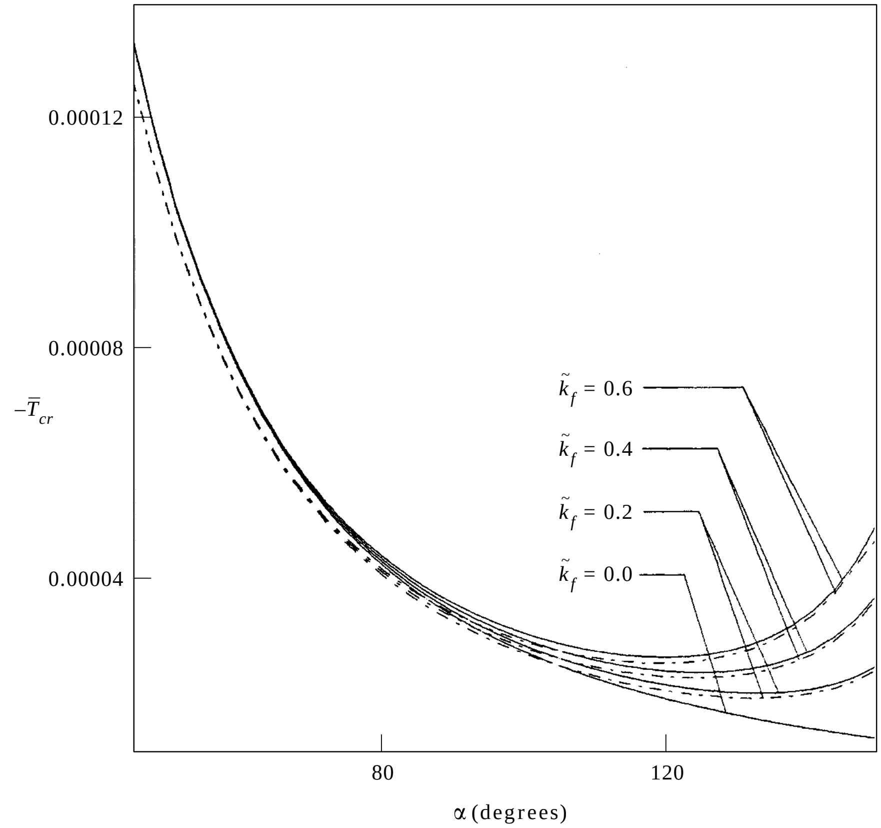

Figure 11. Effects of initial bending stress parameters $\overline{S}$, elastic foundation $\tilde{k}_f$, and arc segment half angle α on initial buckling stress parameter $\overline{T}_{cr}$ for hinged initially stressed curved beams. —— $\overline{S} = 0{\cdot}0$; —·— $\overline{S} = 0{\cdot}4$.

It is obvious that the incidence of initial bending stress reduces the influence of elastic foundation on the initial buckling stress.

5. CONCLUSION

The initial tensile stress will strengthen the natural frequency on the curved beam. With the initial bending stress present, the natural frequency of the curved beam decreases. Whether the initial stress exists or not, the natural frequency of the curved beam decreases as the arc segment half angle α increases. The increases in arc segment half angle α also reduce the influence of the initial tensile stress on the natural frequency of the curved beam. The influence of initial tensile stress on the natural frequency is evident as arc angle $\alpha \leqslant 30°$. However, it is almost unaffected for arc angle $\alpha \geqslant 120°$. The initial bending stress decreases the natural frequency of the curved beam, but the influence is obscure. It also reduces the effects of initial tensile stress on the natural frequency of the curved beam. The elastic foundation increases the natural frequency of the curved beam for arc angle. The influence of elastic foundation on the natural frequency of the curved beam is particularly obvious for arc angle $\alpha \geqslant 120°$. The increases in arc angle α reduce the initial buckling stress. The minus sign of the initial buckling stress is indicative that the initial buckling stress is compressed. As the initial bending stress increases, the initial buckling stress approaches zero. However, the effects are unclear. It also decreases the influence of elastic foundation on initial buckling stress. The elastic foundation gradually strengthens the initial buckling stress of the curved beam as arc angle α increases. The effects are particularly obvious for arc angle $\alpha \geqslant 120^{\circ}$.

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APPENDIX A: NOMENCLATURE

A	cross-section area of the curved beam
b	width of the curved beam
h	thickness of the curved beam
k_f, $\tilde{k}_f$	elastic foundation stiffness and elastic foundation parameter
L	the length of the curved beam
m	number of the Galerkin method term
p_s, f_s	non-conservative traction forces and body-forces
ΔP_s, Δf_s	perturbation traction forces and body-forces
R	radius of curvature of the curved beam
t_{ij}, σ_{ij}	initial stress and perturbing stress
$\bar{T}$, $\bar{S}$	initial tensile stress parameter and initial bending stress parameter
$\bar{T}_{cr}$	initial buckling stress parameter
u_s, $u_{s,j}$	perturbing displacement and displacement gradients, respectively
$\tilde{u}$	perturbation strain energy density
$\bar{v}$, $\bar{w}$	perturbation and initial stress virtual work, respectively
α	arc segment half angle
ρ	density of the curved beam
$\rho\ddot{u}_s$	inertia force due to the perturbation where the superior double dot denotes the second partial derivative with respect to time
σ_N, σ_M	initial tensile and bending stress, respectively
ω	natural frequency of the curved beam
Ω	non-dimensional natural frequency parameter
ψ	angular changes of lines initially normal to the neutral surface.