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LINEAR DAMPING MODELS FOR STRUCTURAL VIBRATION

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Linear damping models for structural vibration are examined: first the familiar dissipation-matrix model, then the general linear model. In both cases, an approximation of small damping is used to obtain simple expressions for damped natural frequencies, complex mode shapes, and transfer functions. Results for transfer functions can be expressed in the form of very direct extensions of the familiar expression for the undamped case. This allows a detailed discussion of the implications of the various damping models for the interpretation of measured transfer functions, especially in the context of experimental modal analysis. In the case of a dissipation-matrix model, it would be possible in principle to determine all the model parameters from measurements. In the case of the general model, however, there is a fundamental ambiguity which prevents full determination of the model from measurements on a single structure.

1. INTRODUCTION

All structures exhibit vibration damping, but despite a large literature on the subject damping remains one of the least well-understood aspects of general vibration analysis. The major reason for this is the absence of a universal mathematical model to represent damping forces. There are excellent reasons why the stiffness and inertia properties of a general discrete system, executing small vibrations around a position of stable equilibrium, may be approximated via the familiar stiffness and mass matrices. These simply represent the first non-trivial terms which do not vanish when the potential and kinetic energy functions are Taylor-expanded for small amplitudes of motion [1]. Nothing so simple can be done to represent damping, because it is not in general clear which *state variables* the damping forces will depend on. A commonly-used model, originated by Rayleigh [1], supposes that instantaneous generalized velocities are the only relevant state variables. Taylor expansion then leads to a model which encapsulates damping behaviour in a dissipation matrix, directly analogous to the mass and stiffness matrices. This model will be examined in some detail in this paper.

However, it is important to avoid the widespread misconception that this is the *only* linear model of vibration damping. It is perfectly possible for damping forces to depend on values of other quantities, or equivalently to depend, via convolution integrals, on the past history of the motion. Any such model which guarantees that the energy dissipation rate is non-negative can be a potential candidate to represent the damping of a given structure. The dissipation-matrix model is just one among many such models.

The appropriate choice of model depends, of course, on the detailed mechanism(s) of damping. Unfortunately, these mechanisms are more varied and less well-understood than the physical mechanisms governing stiffness and inertia. A very brief review is useful to

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set the scene. In broad terms, structural damping mechanisms can be divided into three classes: (1) energy dissipation distributed throughout the bulk material making up the structure, which can generically be called "material damping"; (2) dissipation associated with junctions or interfaces between parts of the structure, generically "boundary damping"; and (3) dissipation associated with a fluid in contact with the structure, involving either local viscous effects or radiation away into the fluid.

Material damping can arise from a variety of microstructural mechanisms (see e.g., reference [2]) but for small strains it is often adequate to represent it through an equivalent linear, viscoelastic continuum model of the material. Damping can then be taken into account via the "viscoelastic correspondence principle", which leads to the concept of complex moduli [3]. For sinusoidal motion at a given frequency ω, the effects of viscoelasticity can be exactly represented by replacing each of the real elastic constants of the material by a suitable complex value. The number of independent elastic constants for a given material is governed by its microscopic symmetries as usual—an isotropic material requires two, a general orthotropic material requires nine and so on.

It is important to keep in mind that complex moduli must be defined in the frequency domain. Although it is often found empirically that complex moduli are almost independent of frequency in the low audio range, considerations of causality [4] show that it is not possible for the complex moduli to be entirely independent of frequency. If inverse Fourier transformation is used to obtain results in the time domain, this inevitable frequency dependence of the complex moduli must be taken into account. Failure to do this properly leads to the notion of "a differential equation with frequency-dependent coefficients", which is all too often encountered in the literature. As has been pointed out forcibly by Crandall [4], this is a mathematical nonsense because it mixes time-domain and frequency-domain concepts, and is likely to lead to fallacious results. In particular, it disguises the distinction between linear and non-linear models: the cases with constant coefficients and frequency-dependent coefficients are often described as "linear" and "non-linear", respectively, when in reality both are linear models.

"Boundary damping" is less easy to model than material damping, but it is of crucial importance in most engineering structures. When damping is measured on a built-up structure like a ship or a building, it is commonly found to be at least an order of magnitude higher than the intrinsic material damping of the main components of the structure. This difference is attributed to effects such as frictional micro-slipping at joints and air-pumping in riveted seams, but this attribution is usually based on negative evidence ("what else could it be?") rather than on any attempt at detailed modelling. A familiar example would be a window pane in a substantial masonry wall. The intrinsic damping of glass is extremely low, but the damping of an *in-situ* window is far higher. The impedance mismatch to the wall is very high, so the energy is presumably being lost primarily in boundary effects associated with the putty joint holding the glass to the frame, or perhaps the bolts and cement holding the frame to the wall.

In such a system the energy loss mechanism would no doubt be significantly non-linear if examined in detail, and if there is a justification for approximating the behaviour by a linear theory it probably depends on an assumption of "small damping". This issue has been discussed by Heckl [5, 6], who found that linear theory produced acceptable response predictions for panels whose main damping mechanism arose from a bolted-on beam. (He attributed the damping mechanism primarily to air-pumping between beam and panel.)

Is the "dissipation matrix" model of damping suitable for describing any of these effects? For some systems it is certainly appropriate—examples might be vehicle suspensions with "shock absorbers" which are approximately classical dashpots, and other systems in which fluid viscosity is the main energy dissipation mechanism. However, for most physical

mechanisms of material or boundary damping it is far from obvious that instantaneous generalized velocities will be the only state variables determining the rate of energy dissipation, even to a first approximation. In this paper the dissipation-matrix model is considered, and then the results are compared with those from the most general linear model of damping. In both cases, a small-damping approximation is used to obtain simple expressions for complex frequencies and mode shapes, and for transfer functions.

Within the scope of this approximation, it is possible to address some general questions. Given a particular physical system, how could one determine experimentally whether a particular damping model (such as a dissipation-matrix model) is appropriate? If a particular model can be fitted to a set of measurements to satisfactory accuracy, does that mean that the underlying physical mechanisms have been well represented? Will the effect of structural modifications be well predicted? Could an entirely different damping model be fitted equally well to the same set of measurements? Would that matter for the accuracy of prediction of system behaviour, or of the efficacy of vibration-control measures?

A question of particular interest concerns the "complex mode shapes" often revealed by experimental modal analysis [7]. Within the small-damping approximation, it will turn out to be rather easy to examine the significance of such complex shapes. It will be shown that the pole-fitting approach to experimental modal analysis can indeed be used to discover the correct complex modes, even for the most general linear model of damping forces. Modal analysis might thus be able to give an experimental procedure for determining the parameter values of a given damping model, and of testing the validity of different damping models. However, care must be exercised because it is not *a priori* obvious that reciprocity between excitation and observing points will necessarily hold in general.

2. SMALL DAMPING IN THE DISSIPATION-MATRIX MODEL

2.1. COMPLEX FREQUENCIES AND MODE SHAPES

A discrete system with N degrees of freedom, executing small vibrations about a position of stable equilibrium and with damping governed by a dissipation matrix, obeys the governing equations

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{f}, \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, dissipation and stiffness matrices respectively, $\mathbf{y}$ is the vector of generalized co-ordinates, and $\mathbf{f}$ is the vector of generalized forces driving the vibration. By analogy with the kinetic and potential energy functions, Rayleigh's dissipation function is defined by

$$F = \tfrac{1}{2}\dot{\mathbf{y}}^t\mathbf{C}\dot{\mathbf{y}}, \tag{2}$$

and is equal to half the rate of energy dissipation. The dissipation matrix is symmetric and positive semi-definite.

Treatments in the literature of the coupled equations (1) usually follow one of two routes. Either it is assumed that $\mathbf{C}$ is simultaneously diagonalizable with $\mathbf{M}$ and $\mathbf{K}$, so-called *proportional damping*, or else the equations are recast into the form of $2N$ coupled first-order equations (e.g., reference [8]), which allows solutions to be computed readily but which loses much of the intuitive immediacy of the familiar treatment of the undamped case via normal modes. Instead, a route pioneered by Rayleigh himself is followed here, and approximate solutions of equation (1) are considered which assume that the terms of the dissipation matrix are small but not otherwise constrained. The motivation for this lies

in application to engineering systems which one would consider studying by experimental modal analysis—in such systems damping is invariably small, typically of the order of 1% of critical damping.

Consider first the undamped natural frequencies ω_n and corresponding mode shape vectors $\mathbf{u}^{(n)}$: these satisfy

$$\mathbf{K}\mathbf{u}^{(n)} = \omega_n^2 \mathbf{M}\mathbf{u}^{(n)}, \qquad n = 1, \ldots, N, \tag{3}$$

and the vectors may be normalized as usual by requiring

$$\mathbf{u}^{(n)t}\mathbf{M}\mathbf{u}^{(n)} = 1, \qquad n = 1, \ldots, N. \tag{4}$$

Now for free motion (i.e., $\mathbf{f} = 0$) of the damped system, the complex frequencies will be the roots of

$$\det\left[-\omega^2\mathbf{M} + \mathrm{i}\omega\mathbf{C} + \mathbf{K}\right] = 0. \tag{5}$$

For small damping, one would expect to find roots of this equation close to $\pm\omega_n$ for each value of n, with corresponding displacement vectors $\mathbf{y}$ close to $\mathbf{u}^{(n)}$. (Special cases involving degeneracies will be ignored throughout this paper.) Denote these complex solutions $\bar{\omega}_n$ and $\bar{\mathbf{u}}^{(n)}$. Seek a solution

$$\bar{\mathbf{u}}^{(n)} = \sum_{j=1}^{N} \alpha_j \mathbf{u}^{(j)} \quad \text{where } \alpha_n = 1,\ |\alpha_j| \ll 1\ (j \neq n). \tag{6}$$

Substituting into equation (1), multiplying on the left by the transposed vector $\mathbf{u}^{(k)t}$ and using the orthogonality properties of the undamped modes yields

$$-\omega^2\alpha_k + \mathrm{i}\omega \sum_j C'_{kj}\alpha_j + \omega_k^2\alpha_k = 0, \tag{7}$$

where

$$C'_{kj} = \mathbf{u}^{(k)t}\mathbf{C}\mathbf{u}^{(j)} \tag{8}$$

is the dissipation matrix expressed in normal co-ordinates. Using equation (6), the nth equation of the set (7) yields

$$\omega_n^2 - \bar{\omega}_n^2 + \mathrm{i}\bar{\omega}_n C'_{nn} \approx 0,$$

so that

$$\bar{\omega}_n \approx \pm\omega_n + \mathrm{i}C'_{nn}/2, \tag{9}$$

while the kth equation ($k \neq n$) yields

$$\alpha_k \approx \pm\frac{\mathrm{i}\omega_n C'_{kn}}{(\omega_n^2 - \omega_k^2)},$$

so that

$$\bar{\mathbf{u}}^{(n)} \approx \mathbf{u}^{(n)} + \mathrm{i}\sum_{k \neq n} \frac{\omega_n C'_{kn}\mathbf{u}^{(k)}}{(\omega_n^2 - \omega_k^2)}. \tag{10}$$

These are Rayleigh's results [1, section 102, equations (5) and (6)]. Equation (9) shows that the damped natural frequencies depend (to this order of approximation) only on the diagonal terms of the dissipation matrix in normal co-ordinates. Equation (10) shows that

the off-diagonal terms of the transformed dissipation matrix $\mathbf{C}'$ govern the modified "mode shapes", which are now in general complex. Within this model the complex modes consist approximately of a real part which is the undamped mode shape, and an imaginary part which is a mixture of the other undamped mode shapes. It follows that the imaginary part should be orthogonal to the real part (with respect to the mass or stiffness matrices).

The extra contributions to mode shape are weighted to favour modes close in natural frequency to the one in question because of the term in the denominator. When k and n refer to two adjacent modes, the weighting factor can be rewritten, approximately, using

$$\frac{\omega_n C'_{kn}}{(\omega_n^2 - \omega_k^2)} \approx \tfrac{1}{2}\mu\gamma_{kn}$$

where

$$\gamma_{kn} = C'_{kn}/C'_{nn},$$

and μ is the modal overlap factor of the two modes. Thus, significantly complex modes are to be expected whenever the modal overlap is not small and the off-diagonal damping terms are comparable with the diagonal terms.

2.2. AN EXAMPLE

One useful application of these approximations lies in the fact that the damped frequencies and mode shapes can be deduced readily from the undamped ones. Thus an elastic solution, whether analytic or based on, for example, finite-element computation, can be readily augmented to a damped solution without the necessity to assume proportional damping or to solve a more complicated problem (e.g., using the first-order formulation). It is rather rare in a structural vibration problem to need more accuracy than these equations yield: in practice the damping model is likely to be uncertain, and normally one neither expects nor requires estimates of damping to be more accurate than, perhaps, $\pm 20\%$.

To give a simple illustration, consider an idealised three-degree-of-freedom problem treated by the first-order method by Newland [8, pp. 148–151]. The system is shown in Figure 1. The undamped problem is simple enough to be solved analytically: the normalized modes and natural frequencies are:

$$\frac{1}{2}\begin{bmatrix}1\\ \sqrt{2}\\ 1\end{bmatrix} \text{ at } \omega_1 = \sqrt{2-\sqrt{2}}, \qquad \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 0\\ -1\end{bmatrix} \text{ at } \omega_2 = \sqrt{2}, \qquad \frac{1}{2}\begin{bmatrix}1\\ -\sqrt{2}\\ 1\end{bmatrix} \text{ at } \omega_3 = \sqrt{2+\sqrt{2}}.$$

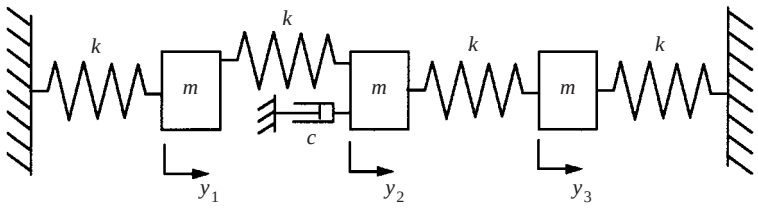

Figure 1. Three degree-of-freedom system as studied by Newland [8]. Parameter values are: $m = 1$ kg, $k = 1$ N/m, $c = 0{\cdot}3$ Ns/m.

The dissipation matrix is

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0{\cdot}3 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so that

$$\mathbf{C}' = \begin{bmatrix} 1/2 & 1/\sqrt{2} & 1/2 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/2 & -1/\sqrt{2} & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0{\cdot}3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 1/\sqrt{2} & 1/2 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/2 & -1/\sqrt{2} & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 0{\cdot}15 & 0 & -0{\cdot}15 \\ 0 & 0 & 0 \\ -0{\cdot}15 & 0 & 0{\cdot}15 \end{bmatrix}.$$

Substitution into equation (9) thus gives the approximate damped frequencies as follows:

$$\bar{\omega}_1 \approx \omega_1 + 0{\cdot}075\mathrm{i}, \qquad \bar{\omega}_2 \approx \omega_2, \qquad \bar{\omega}_3 \approx \omega_3 + 0{\cdot}075\mathrm{i}, \tag{11}$$

with corresponding approximate mode shapes:

$$\bar{\mathbf{u}}^{(1)} \approx \begin{bmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{bmatrix} + \frac{\mathrm{i}\omega_1}{\omega_3^2 - \omega_1^2}(-0{\cdot}15)\begin{bmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{bmatrix} \approx \begin{bmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{bmatrix} - \mathrm{i}0{\cdot}041\begin{bmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{bmatrix}, \tag{12a}$$

$$\bar{\mathbf{u}}^{(2)} \approx \mathbf{u}^{(2)}, \tag{12b}$$

$$\bar{\mathbf{u}}^{(3)} \approx \begin{bmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{bmatrix} + \frac{\mathrm{i}\omega_3}{\omega_1^2 - \omega_3^2}(-0{\cdot}15)\begin{bmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{bmatrix} \approx \begin{bmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{bmatrix} + \mathrm{i}0{\cdot}098\begin{bmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{bmatrix}. \tag{12c}$$

These all agree with Newland's results to excellent accuracy.

2.3. TRANSFER FUNCTIONS

The same small-damping approximation can be used to obtain a surprisingly simple expression for the matrix of transfer functions (receptances, mobilities etc.). Suppose that the forcing vector $\mathbf{f}$ is zero for all entries except the mth, which has harmonic forcing $f\,\mathrm{e}^{\mathrm{i}\omega t}$. Write the response vector

$$\mathbf{y} = \sum_j q_j \mathbf{u}^{(j)}\,\mathrm{e}^{\mathrm{i}\omega t}. \tag{13}$$

Substituting into equation (1) and multiplying on the left by $\mathbf{u}^{(k)t}$ gives

$$-\omega^2 q_k + \mathrm{i}\omega \sum_j C'_{kj} q_j + \omega_k^2 q_k = \mathbf{u}^{(k)t}\mathbf{f} = f u_m^{(k)} \tag{14}$$

or

$$[\mathbf{A} + \mathrm{i}\omega\mathbf{C}'']\mathbf{q} = \mathbf{Q}, \tag{15}$$

where

$$\mathbf{A} = \text{diag}\,[\omega_k^2 + \mathrm{i}\omega C'_{kk} - \omega^2],$$

$$\mathbf{Q} = [f u_m^{(1)} \quad f u_m^{(2)} \quad \cdots]^t$$

and $\mathbf{C}''$ is $\mathbf{C}'$ with the diagonal elements deleted.

Now noting the standard expansion

$$(\mathbf{A} + \mathrm{i}\omega\mathbf{C}'')^{-1} \approx \mathbf{A}^{-1} - \mathrm{i}\omega\mathbf{A}^{-1}\mathbf{C}''\mathbf{A}^{-1}, \tag{16}$$

one obtains

$$q_k \approx \frac{f u_m^{(k)}}{(\omega_k^2 + \mathrm{i}\omega C'_{kk} - \omega^2)} - \mathrm{i}\omega \sum_{j \neq k} \frac{f C'_{kj} u_m^{(j)}}{(\omega_k^2 + \mathrm{i}\omega C'_{kk} - \omega^2)(\omega_j^2 + \mathrm{i}\omega C'_{jj} - \omega^2)}. \tag{17}$$

(The expansion (16) has been previously applied to vibration transfer functions by Bhaskar [9], but with a somewhat different objective). Now the transfer function to the nth generalized co-ordinate is

$$H_{mn} = \frac{y_n}{f} \approx \sum_k \frac{u_m^{(k)} u_n^{(k)}}{(\omega_k^2 + \mathrm{i}\omega C'_{kk} - \omega^2)} - \mathrm{i}\omega \sum_k \sum_{j \neq k} \frac{C'_{kj} u_m^{(j)} u_n^{(k)}}{(\omega_j^2 + \mathrm{i}\omega C'_{jj} - \omega^2)(\omega_k^2 + \mathrm{i}\omega C'_{kk} - \omega^2)}. \tag{18}$$

It is useful to express this as a partial-fraction expansion. The quadratic terms appearing in the denominator can be factorized: for example

$$\omega_k^2 + \mathrm{i}\omega C'_{kk} - \omega^2 = -(\omega - \bar{\omega}_k)(\omega + \bar{\omega}_k^*), \tag{19}$$

where the star denotes the complex conjugate. Ignoring the possibility of degeneracies, the usual "cover-up rule" allows one to write down the residue of the pole at $\omega = \bar{\omega}_k$: it is approximately

$$R_{mn}^{(k)} \approx -\frac{u_n^{(k)} u_m^{(k)}}{\bar{\omega}_k + \bar{\omega}_k^*} - \frac{\mathrm{i}}{2} \sum_{j \neq k} \frac{C'_{jk}(u_n^{(j)} u_m^{(k)} + u_m^{(j)} u_n^{(k)})}{(\omega_k^2 - \omega_j^2)}, \tag{20}$$

where advantage has been taken of the symmetry of $\mathbf{C}'$, and terms involving $\mathbf{C}'$ have been omitted from the denominator of the final term at this order of approximation, because of the term in $\mathbf{C}'$ in the numerator. Thus

$$R_{mn}^{(k)} \approx -\frac{1}{2\omega_k} \left\{ u_m^{(k)} + \mathrm{i}\omega_k \sum_{j \neq k} \frac{C'_{jk} u_m^{(j)}}{(\omega_k^2 - \omega_j^2)} \right\} \left\{ u_n^{(k)} + \mathrm{i}\omega_k \sum_{j \neq k} \frac{C'_{jk} u_n^{(j)}}{(\omega_k^2 - \omega_j^2)} \right\}$$

$$\approx -\frac{1}{2\omega_k} \bar{u}_m^{(k)} \bar{u}_n^{(k)}, \tag{21}$$

using equation (10). Thus finally,

$$H_{mn}(\omega) \approx \sum_{k=1}^{N} \frac{1}{2\omega_k} \left\{ -\frac{\bar{u}_m^{(k)} \bar{u}_n^{(k)}}{(\omega - \bar{\omega}_k)} + \frac{u_m^{(k)*} \bar{u}_n^{(k)*}}{(\omega + \bar{\omega}_k^*)} \right\} \tag{22}$$

(adding in the corresponding terms for the conjugate poles.) This is the simplest conceivable generalization of the familiar expression for undamped systems [10], in which the real mode shapes and natural frequencies are replaced by their complex values at this first order of approximation. It is clear from equation (22) that $H_{nm} = H_{mn}$ so that the

principle of reciprocity between excitation and observing points (or generalized co-ordinates) applies. Reciprocity should not be taken for granted in all linear vibration problems, though. It is certainly not obvious that it will hold for linear damping models other than the one considered so far.

For this case in which the damping is represented by a dissipation matrix, a result very similar to equation (22) can be proved *exactly* using the first-order formalism [11]. The details are given in the Appendix. The term in braces turns out to be exactly correct, provided the exact complex frequencies and mode shapes are used. But the factor $(1/2\omega_k)$ is replaced by a more complicated expression which reduces to this value in the zero-damping limit. The reason for presenting the approximate analysis in this section is that when one proceeds to the case of general linear damping there is no equivalent of the first-order method, but the approximate analysis carries over very straightforwardly.

3. GENERAL LINEAR DAMPING

3.1. DERIVATION AND ENERGY CONSTRAINTS

The next step is to put the dissipation-matrix model in the context of the most general linear model of damping in a discrete system. Suppose that the main stiffness and inertia behaviour of the system has been represented by potential and kinetic energy functions, approximated as quadratic forms for small-amplitude motion and thus described via the usual stiffness and mass matrices. Any other internal forces within the system, including those responsible for dissipation of energy, will now appear as generalized forces. If the generalized co-ordinates are $q_k(t)$, $k = 1, \ldots, N$ then motion of the system corresponding to each co-ordinate will in general result in contributions to all N generalized forces Q_j. For linear behaviour the individual contributions can be expressed as convolution integrals, and the total generalized force is simply a sum of these:

$$Q_j(t) = -\sum_{k=1}^{N} \int_{\tau=-\infty}^{t} g_{jk}(t-\tau)\dot{q}_k(\tau)\,\mathrm{d}\tau. \tag{23}$$

The negative sign and the choice of $\dot{q}_k$ rather than q_k within this expression are purely for convenience—integration by parts would give an expression involving q_k instead, with kernel functions $\dot{g}_{jk}(t)$. The particular case in which the kernel functions are all Dirac delta functions and their matrix of coefficients is symmetric recovers the dissipation-matrix model of the previous section. The kernel functions $g_{jk}(t)$, or others closely related to them, are described under many different names in the literatures of different subjects: examples are "retardation functions", "heredity functions", "after-effect functions" and "relaxation functions". Note that any set of generalized co-ordinates may be used in this formalism, including normal co-ordinates, in which the mass and stiffness matrices have been diagonalised.

The rate of energy dissipation is now given by

$$2F = \sum_j \dot{q}_j Q_j = \sum_j \sum_k \dot{q}_j \int_{\tau=-\infty}^{t} g_{jk}(t-\tau)\dot{q}_k(\tau)\,\mathrm{d}\tau. \tag{24}$$

Any set of kernel functions which guarantee that this function is positive semi-definite gives a model which does not obviously contradict the laws of physics. From this definition, there is no reason in general to expect the matrix $g_{jk}(t)$ to be symmetric. A single counter-example is sufficient to demonstrate this. If the kernels are all delta functions but

with a skew-symmetric matrix of coefficients, the resulting model describes what are usually called "gyrostatic" [1] or "gyroscopic" [12] forces. It is obvious from equation (24) that this is a conservative effect, since the dissipation rate is zero as the terms cancel in pairs. This case is not ruled out by the criterion of non-negative energy dissipation. Indeed, such gyroscopic forces arise occasionally in the modelling of physical phenomena.

This example suggests that one might separate "damping forces" from other internal forces by expressing the matrix of kernels as the sum of a symmetric and a skew-symmetric part. It is easiest to formulate the argument in the frequency domain. Suppose the system is forced in some way at frequency ω, so that the kth generalized co-ordinate responds as $q_k\,\mathrm{e}^{\mathrm{i}\omega t}$. The mean rate of energy dissipation by internal forces is then

$$2F = \frac{1}{2}\sum_j \mathrm{Re}\,(\dot{q}_j^* Q_j) = \frac{\omega^2}{2}\,\mathrm{Re}\left\{\sum_{j,k} q_j^* G_{jk}(\omega) q_k\right\}, \tag{25}$$

where * denotes the complex conjugate, and $G_{jk}(\omega)$ is the Fourier transform of $g_{jk}(t)$. Decompose this matrix $\mathbf{G}$ into the sum of Hermitian-symmetric and Hermitian-antisymmetric parts:

$$\mathbf{G} = \mathbf{G}^{(s)} + \mathbf{G}^{(a)} \equiv \tfrac{1}{2}[\mathbf{G} + \mathbf{G}^{*t}] + \tfrac{1}{2}[\mathbf{G} - \mathbf{G}^{*t}]. \tag{26}$$

It is then clear that $\mathbf{G}^{(a)}$ contributes nothing to the dissipation rate in equation (25), since each pair of terms consists of the difference between a number and its complex conjugate. All "damping" forces are described by $\mathbf{G}^{(s)}$, while $\mathbf{G}^{(a)}$ describes what might be called "generalized gyroscopic forces". Whether this distinction carries any physical significance, however, will depend on the detailed mechanisms of the internal forces.

3.2. MODES AND TRANSFER FUNCTIONS

For harmonic excitation of a single generalized co-ordinate of a discrete system subject to the generalized forces (23), the set of equations of motion equivalent to (14) is

$$-\omega^2 q_k + \mathrm{i}\omega \sum_j G'_{kj} q_j + \omega_k^2 q_k = f u_m^{(k)}, \tag{27}$$

where

$$G'_{kj} = \mathbf{u}^{(k)t}\mathbf{G}\mathbf{u}^{(j)} \tag{28}$$

is expressed in normal co-ordinates, exactly analogous to equation (8). The only differences between equations (27) and (14) are that G'_{kj} may depend on frequency while C'_{kj} did not, and that G'_{kj} may not be symmetric. To obtain the set of transfer functions analogous to equation (18) involves inverting the matrix

$$\mathbf{D} = \mathrm{diag}\,[\omega_k^2 - \omega^2] + \mathrm{i}\omega\mathbf{G}'. \tag{29}$$

Before obtaining an explicit expression for the inverse using the small-damping approximation, it is worth noting an important deduction which is not *a priori* obvious. All transfer functions will have poles corresponding to the zeros of the determinant of $\mathbf{D}$, and these poles are therefore guaranteed to be the same for all possible transfer functions, apart from special cases where a residue happens to be zero. There may also be additional poles associated with the frequency-dependence of the terms of $\mathbf{G}'$, but it is the former set of poles which includes the "damped resonances" if damping is in some sense light.

It will now be assumed that damping is light, in the sense that the terms $|G'_{kj}(\omega)|$ are small for frequencies within the range of interest, covering the undamped resonance

frequencies of the system. We do not rule out the possibility that these terms might not be small at very low or very high frequencies, depending on the particular mechanism(s) responsible for the internal forces. For example, in the experimental characterization of material damping it is common to measure "heredity functions" $g_{kj}(t)$ and approximate them well by a weighted sum of exponential decays [2]. This would correspond to one or more poles on the imaginary ω axis in the corresponding functions $G_{kj}(\omega)$, so that large values might be obtained at very low frequencies.

The calculation of complex frequencies and mode shapes, to first order in the terms of the damping matrix, follows the analysis of section 2.1 very closely. To this order, when an approximation is sought to the nth mode, it is sufficient to evaluate the relevant terms $G_{kj}(\omega)$ at the undamped frequency ω_n. (If the functions $G_{kj}(\omega)$ were to vary rapidly with frequency near the mode in question this approximation might be inadequate, but empirically that eventuality seems unlikely for the damping mechanisms normally operating in structures.) It follows that the complex frequency is given by

$$\bar{\omega}_n \approx \pm\omega_n + \mathrm{i}G'_{nn}(\omega_n)/2, \tag{30}$$

and the corresponding mode shape by

$$\bar{\mathbf{u}}^{(n)} \approx \mathbf{u}^{(n)} + \mathrm{i}\sum_{k\neq n}\frac{\omega_n G'_{kn}(\omega_n)\mathbf{u}^{(k)}}{(\omega_n^2-\omega_k^2)}. \tag{31}$$

In a sense, $G'_{kj}(\omega_j)$ is an "effective dissipation matrix" for this problem, but it is not symmetric because the different rows of the matrix are evaluated at different frequencies. Also, since the values are in general complex the structure of the complex mode shapes is slightly less simple than before. It is no longer the case that the real part is, approximately, the undamped mode shape, with the correction terms being purely imaginary.

The same approximation can be applied to calculate the matrix of transfer functions as in section 2.3. Analogous to equation (15), write equation (27) in the form

$$[\mathbf{B} + \mathrm{i}\omega\mathbf{G}'']\mathbf{q} = \mathbf{Q}, \tag{32}$$

where

$$\mathbf{B} = \mathrm{diag}\,[\omega_k^2 + \mathrm{i}\omega G'_{kk}(\omega_k) - \omega^2],$$

and $\mathbf{G}''$ is $\mathbf{G}'$ with the diagonal terms deleted. Using expression (16) to invert $\mathbf{B} + \mathrm{i}\omega\mathbf{G}''$, the transfer functions corresponding to equation (18) are given by

$$H_{mn} \approx \sum_k \frac{u_m^{(k)}u_n^{(k)}}{(\omega_k^2 + \mathrm{i}\omega G'_{kk}(\omega_k) - \omega^2)}$$

$$-\mathrm{i}\omega\sum_k\sum_{j\neq k}\frac{G'_{kj}(\omega)u_m^{(j)}u_n^{(k)}}{(\omega_j^2 + \mathrm{i}\omega G'_{jj}(\omega_j) - \omega^2)(\omega_k^2 + \mathrm{i}\omega G'_{kk}(\omega_k) - \omega^2)}$$

$$+(\text{terms arising from poles of } \mathbf{G}') \tag{33}$$

and the residue of the pole at $\omega = \bar{\omega}_k$, corresponding to equation (20), is

$$R_{mn}^{(k)} \approx -\frac{u_n^{(k)} u_m^{(k)}}{(\bar{\omega}_k + \bar{\omega}_k^*)} - \frac{\mathrm{i}}{2} \sum_{j \neq k} \frac{G'_{kj}(\omega_k) u_n^{(j)} u_m^{(k)} + G'_{jk}(\omega_k) u_m^{(j)} u_n^{(k)}}{(\omega_k^2 - \omega_j^2)}$$

$$\approx -\frac{1}{2\omega_k} \left\{ u_m^{(k)} + \mathrm{i}\omega_k \sum_{j \neq k} \frac{G'_{kj}(\omega_k) u_m^{(j)}}{(\omega_k^2 - \omega_j^2)} \right\} \left\{ u_n^{(k)} + \mathrm{i}\omega_k \sum_{j \neq k} \frac{G'_{jk}(\omega_k) u_n^{(j)}}{(\omega_k^2 - \omega_j^2)} \right\}. \tag{34}$$

The second bracketed term here is the complex mode component $\bar{u}_n^{(k)}$, but now the first term does not correspond to the component $\bar{u}_m^{(k)}$, unless the matrix $\mathbf{G}'$ is symmetric (as opposed to Hermitian-symmetric). The transfer function matrix now takes the form

$$H_{mn}(\omega) \approx \sum_{k=1}^{N} \left\{ \frac{R_{mn}^{(k)}}{(\omega - \bar{\omega}_k)} - \frac{R_{mn}^{(k)*}}{(\omega + \bar{\omega}_k^*)} \right\}$$

$$+ (\text{terms arising from poles of } \mathbf{G}'). \tag{35}$$

Note that for this general model of damping forces, reciprocity will not necessarily be found.

3.3. AN EXAMPLE

To illustrate the procedure outlined, and to explore the accuracy of the small-damping approximation, a very simple idealized example will be studied in detail. Consider the system shown in Figure 2, in which two oscillators are coupled by a spring, and also in parallel by an element which will provide the damping forces. This element will be modelled as a "Maxwell element", a spring of stiffness C in series with a linear dashpot of rate R. The equations of motion of this system are

$$\left.\begin{aligned} m\ddot{y}_1 + k_1 y_1 + S(y_1 - y_2) + P &= f_1 \\ m\ddot{y}_2 + k_2 y_2 - S(y_1 - y_2) - P &= f_2 \end{aligned}\right\}, \tag{36}$$

where the force P across the viscoelastic element satisfies

$$P = C \int_{-\infty}^{t} (\dot{y}_1(\tau) - \dot{y}_2(\tau))\, \mathrm{e}^{-\lambda(t-\tau)}\, \mathrm{d}\tau, \tag{37}$$

with

$$\lambda = C/R.$$

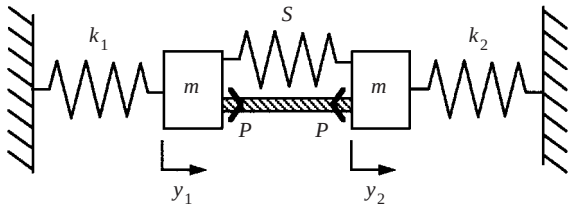

Figure 2. Two-degree-of-freedom model system. The shaded bar represents a viscoelastic element whose behaviour is defined in the text.

The mass and stiffness matrices are thus

$$\mathbf{M} = m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{K} = \begin{bmatrix} k_1 + S & -S \\ -S & k_2 + S \end{bmatrix}, \tag{38}$$

and the matrix of kernel functions is

$$\mathbf{g}(t) = C\,\mathrm{e}^{-\lambda t}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \tag{39}$$

with Fourier transform

$$\mathbf{G} = \frac{C}{(\mathrm{i}\omega + \lambda)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{40}$$

Since this matrix $\mathbf{G}$ is symmetric, reciprocity will hold for this system. The matrices $\mathbf{G}^{(s)}$ and $\mathbf{G}^{(a)}$ in this case are simply the real and imaginary parts of $\mathbf{G}$. The dissipation is associated with the "resistive" real part, while the "reactive" imaginary part describes conservative effects. This is a case where there seems to be no very interesting physical significance in this separation. Note that any model of a system in which the internal forces were represented by a sum of forces like P here, acting between pairs of points with equal and opposite forces whose line of action is along the line joining the points, will give a symmetric matrix. One might guess that many types and distributions of internal force could be modelled in this way, so that symmetric matrices might occur quite commonly.

If the applied forces are harmonic at frequency ω, the (exact) response is the solution of the equations

$$\begin{bmatrix} -m\omega^2 + k_1 + S + \mathrm{i}\omega C/(\mathrm{i}\omega + \lambda) & -S - \mathrm{i}\omega C/(\mathrm{i}\omega + \lambda) \\ -S - \mathrm{i}\omega C/(\mathrm{i}\omega + \lambda) & -m\omega^2 + k_2 + S + \mathrm{i}\omega C/(\mathrm{i}\omega + \lambda) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}. \tag{41}$$

The determinant of this matrix, after slight manipulation, is

$$\frac{1}{(\mathrm{i}\omega + \lambda)}\{(k_1 - m\omega^2)(k_2 - m\omega^2)(\mathrm{i}\omega + \lambda) + (k_1 + k_2 - 2m\omega^2)[(S + C)\mathrm{i}\omega + S\lambda]\}. \tag{42}$$

Poles of the transfer functions will arise at the zeros of this determinant. The numerator is a 5th-order polynomial in $\mathrm{i}\omega$ with real coefficients, so as well as the two pairs of complex-conjugate roots corresponding to the damped resonance frequencies of this two-degree-of-freedom system, there must be a fifth root which is real. This corresponds to a pole on the imaginary ω-axis associated with the internal behaviour of the damping model.

In terms of this determinant, the matrix of transfer functions can be trivially written down. At first sight there appears to be an additional pole where $\mathrm{i}\omega + \lambda = 0$, from the terms of the matrix involving C, but this is cancelled by a corresponding factor in the denominator of the determinant. Since the determinant is a quintic expression, it is not possible to find the roots analytically and thus work out the exact expansion in partial fractions corresponding to the approximate version equation (35). Instead, it is easy to compute the approximate transfer functions from equation (35), in which the addition pole due to the damping model is simply ignored. These can then be compared with the exact transfer functions deduced from the matrix inversion.

The undamped problem is trivial to solve. The undamped natural frequencies satisfy

$$2m\omega^2 = k_1 + k_2 + 2S \pm \sqrt{(k_1 - k_2)^2 + 4S^2}, \tag{43}$$

and in terms of these frequencies the mode shapes have the displacement ratios

$$\frac{x_1}{x_2} = \frac{(k_1 + S - m\omega^2)}{S}. \tag{44}$$

These mode shapes and frequencies exhibit the familiar "veering" behaviour when the two springs k_1 and k_2 are close to equality, provided $S \neq 0$ [13]. The modes must be mass-normalized, they can then be used to transform the matrix **G** into normal co-ordinates according to equation (28). Evaluating the rows of this matrix at the two undamped natural frequencies, the approximate complex frequencies and complex modes can be calculated. Since **G** is symmetric, the residues $R_{mn}^{(k)}$ are given directly in terms of these complex modes, and the approximate transfer functions can thus be computed by equation (35).

For this very simple example the results could be worked out in analytic form, but they would be too complicated to be very illuminating. Instead, the main features of the behaviour will be illustrated by numerical computations. As one would anticipate, the approximate theory is most accurate when C is small, and increasing C while keeping all other parameters fixed eventually leads to unacceptable errors. This behaviour is illustrated in Figure 3. This shows a contour plot of the maximum deviation (in dB) between the exact and approximate calculations of $|H_{12}(\omega)|$, as S and C are varied over quite wide ranges

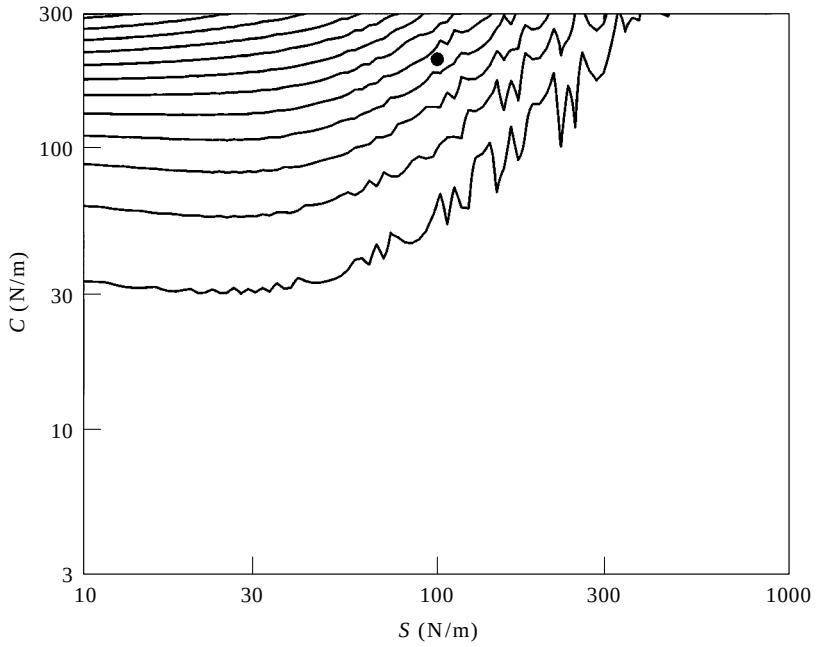

Figure 3. Contour plot of the maximum deviation of $|H_{12}(\omega)|$ between the exact result and the approximation developed here, applied to the model system of Figure 2, for a range of values of the stiffness S and the "viscoelastic stiffness" C defined in the text. Other parameter values are: $k_1 = 1000$ N/m, $k_2 = 1100$ N/m, $\lambda = 100\ \text{s}^{-1}$. Contours are logarithmically spaced at 1-dB intervals. The large blank space in the figure corresponds to agreement between the two theories better than 1 dB, the first (jagged) contour is the 1-dB level, and the contour levels are monotonically increasing into the top left-hand corner of the plot. The solid circle marks the case treated in Figure 4.

while k_1, k_2 and λ are held fixed at values given in the caption. This deviation gives a reasonable measure of the adequacy of the approximate theory: when the prediction is within 1 dB of the exact result for all frequencies the approximate theory is presumably accurate enough for all practical purposes, but once the deviation grows to a few dB it becomes less convincing. The jagged form of the lower contours in Figure 3 arises from numerical resolution, together with the fact that the measure calculated is a maximum deviation, and as parameter values change the frequency at which the maximum deviation occurs can jump.

The detailed behaviour for a case near the threshold of acceptability is shown in Figure 4. It corresponds to the point marked in Figure 3, close to the 5-dB contour. It can be seen that the deviation of $|H_{12}(\omega)|$ is indeed a good measure of the general behaviour of this case. The phase deviation and the deviations of H_{11} and H_{22} all follow similar patterns. The detailed modal results for this case are as follows: the undamped frequencies and mode vectors are

$$\omega_1 = 32{\cdot}2211, \qquad \mathbf{u}_1 = \begin{bmatrix} 0{\cdot}8507 \\ 0{\cdot}5257 \end{bmatrix},$$

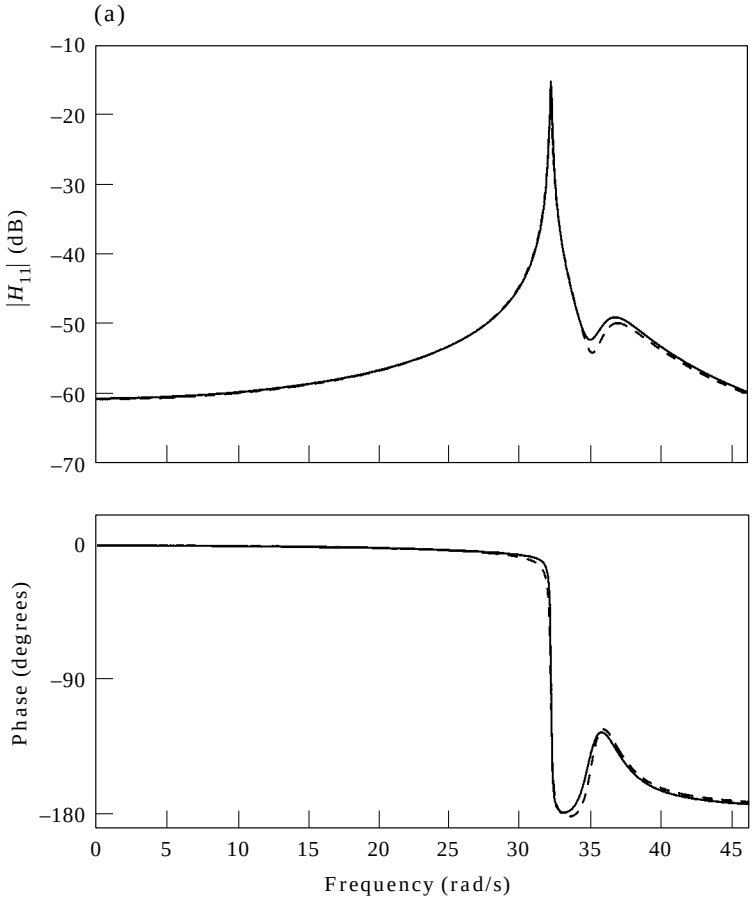

Figure 4(a)—(*Caption page* 562)

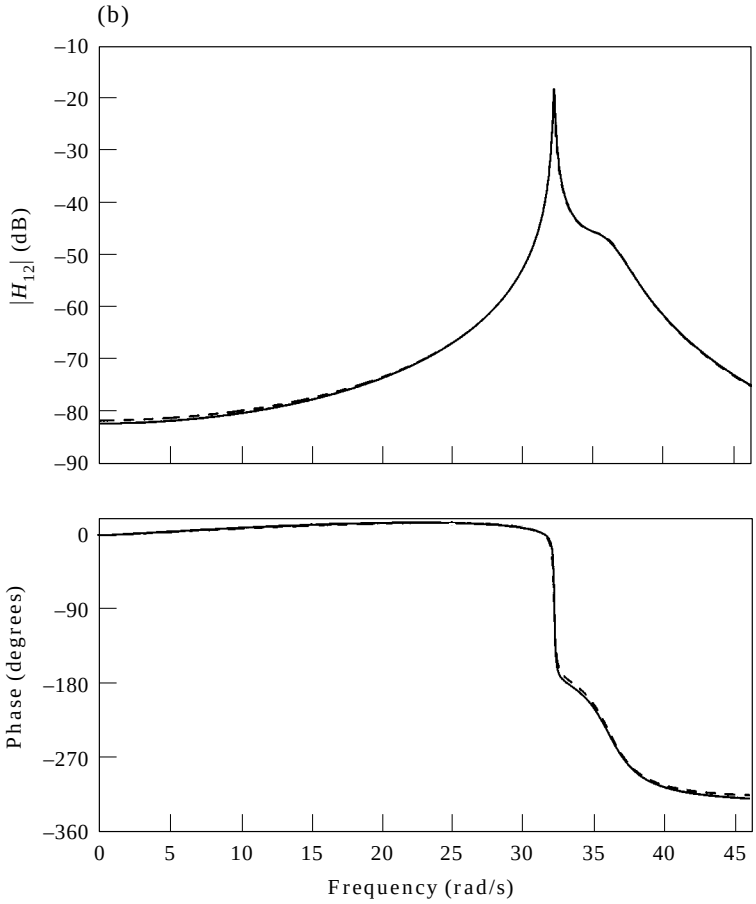

Figure 4(b)—(*Caption overleaf*)

$$\omega_2 = 35{\cdot}5219, \qquad \mathbf{u}_2 = \begin{bmatrix} -0{\cdot}5257 \\ 0{\cdot}8507 \end{bmatrix}, \tag{45}$$

and the approximate complex values from equations (30) and (31) are

$$\bar{\omega}_1 \approx 32{\cdot}2519 + 0{\cdot}0956\mathrm{i}, \qquad \bar{\mathbf{u}}_1 \approx \begin{bmatrix} 0{\cdot}8309 - 0{\cdot}0614\mathrm{i} \\ 0{\cdot}5577 + 0{\cdot}0993\mathrm{i} \end{bmatrix},$$

$$\bar{\omega}_2 \approx 36{\cdot}1194 + 1{\cdot}6822\mathrm{i}, \qquad \bar{\mathbf{u}}_2 \approx \begin{bmatrix} -0{\cdot}5639 - 0{\cdot}1073\mathrm{i} \\ 0{\cdot}8271 - 0{\cdot}0663\mathrm{i} \end{bmatrix}. \tag{46}$$

The two modal damping factors are quite different in this case: in terms of Q-factors, given for small damping by

$$Q_j \approx \frac{\mathrm{Re}\,(\bar{\omega}_j)}{2\,\mathrm{Im}\,(\bar{\omega}_j)}, \tag{47}$$

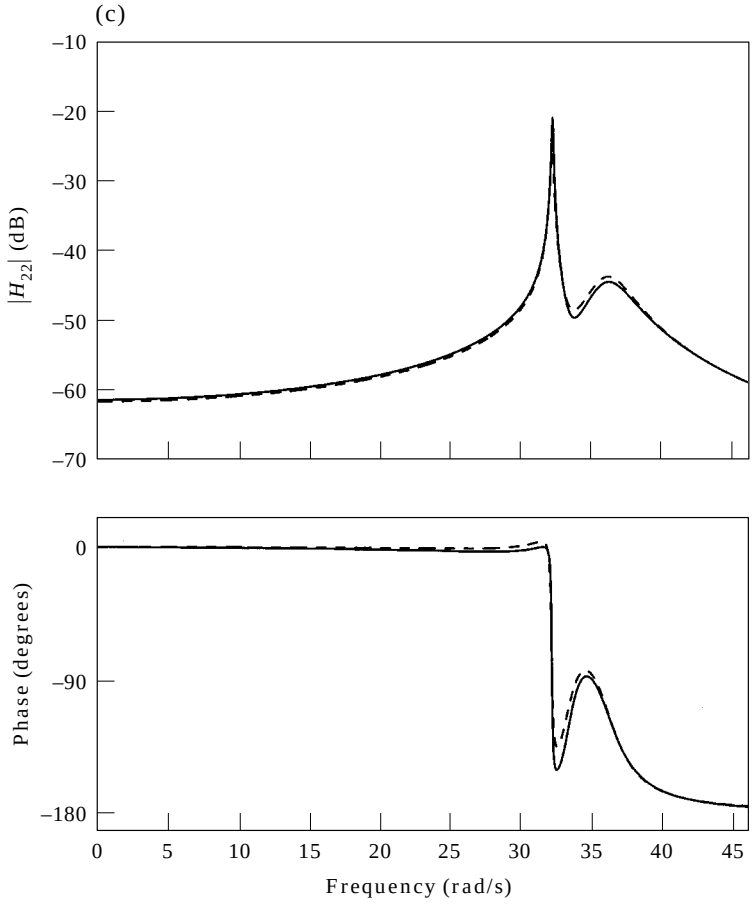

Figure 4(c)

Figure 4. Comparisons between exact theory (solid lines) and approximate theory (dashed lines) for the three transfer functions of the system shown in Figure 2: (a) $H_{11}(\omega)$; (b) $H_{12}(\omega)$; (c) $H_{22}(\omega)$. Parameter values are: $k_1 = 1000$ N/m, $k_2 = 1100$ N/m, $S = 100$ N/m, $C = 200$ N/m, $\lambda = 100$ s^{-1}.

they are

$$Q_1 \approx 169, \qquad Q_2 \approx 10{\cdot}7. \tag{48}$$

The reason is that the coupling spring S is sufficiently strong that the system is within the "veering range", with modes involving significant motion of both masses, as shown by equation (45). The lower of these two modes involves motion of the two masses in the same direction, with little extension of the "viscoelastic" element and hence little damping, but the higher mode involves motion of the masses in opposite directions, and thus much more damping. Notice that the corresponding mode shape is also changed quite substantially from the undamped case: one element has an imaginary part which is 19% of its real part, a significantly complex mode. The approximate theory is near the edge of acceptability when one mode has a Q-factor of 10, very high damping for most structural vibration applications. This gives a strong indication that the approximate theory might be useful for a wide range of practical problems. Indeed, it is questionable whether there are many real systems with modal Q-factors this low in which the damping obeys a linear model to the required accuracy.

The general features of Figure 3 are now easy to explain. For all values of S, the deviation increases with increasing C as expected. Less immediately obvious is that for a given value of C, decreasing S increases the deviation. The reason lies in the "veering" phenomenon. The mode shapes of this system are very sensitive to the strength of coupling between the two masses: for weak coupling, towards the left of the figure, the modes consist (approximately) of separate oscillation of the two masses, but for stronger coupling, towards the right of the figure, they tend towards symmetric and antisymmetric motion of the masses, as described for the case shown in Figure 4. This strength of coupling arises from the parallel combination of the spring S and the viscoelastic element. If the spring is strong, a relatively large force from the viscoelastic element still represents a small perturbation, and the approximate theory works well. However, if the spring S is weak, the net coupling strength is influenced strongly by the viscoelastic element, and it is not surprising that the perturbation approximation is less good.

Finally, it should be noted that the omission of the contribution associated with the additional pole of the damping model has made very little difference to the accuracy of prediction. Even at very low frequencies, the exact and approximate calculations of all three transfer functions remain in good agreement in amplitude and phase.

4. IMPLICATIONS FOR EXPERIMENTAL MODAL ANALYSIS

4.1. THE DISSIPATION-MATRIX MODEL

Equations (10) and (22) generalize readily from discrete to continuous systems, the mode vectors becoming continuous functions of position in the system and the number of modes becoming infinite. Thus, any system with small damping which can be described by a Rayleigh dissipation function should have transfer functions which are well approximated by equation (22), and which are described exactly by equation (A9). This fact can be used to predict the results of measurements designed to find mode shapes.

The most common method of experimental modal analysis involves measuring many transfer functions, usually with a fixed observation point and exciting the structure at many different points using an impulse hammer. Various computational schemes may be used to identify the poles of these transfer functions, and then to calculate the spatial variation of the residue of a given pole as the excitation point moves. It is immediately clear from equation (22) that if damping forces are well-represented by a dissipation matrix, the result of such a measurement does indeed reveal the complex mode corresponding to the chosen pole, as given by equation (10). Also, for this case the principle of reciprocity between excitation and observing points (or generalized co-ordinates) applies: it does not matter whether the excitation point or the observation point is moved around.

If a set of measurements satisfies the requirement of reciprocity, a dissipation-matrix model of damping is a possible candidate, and one might try to extract the parameters of such a model from the measurements. A natural approach is to determine the complex mode shapes (and the diagonal damping terms) by pole-fitting, then use the simple expression (10) to determine the off-diagonal terms of the dissipation matrix $\mathbf{C}'$. This would involve determining the "undamped" modes from the real part of the measured shapes, then expressing the imaginary part of a given complex mode as a linear combination of these undamped modes. The coefficients of such an expression would directly give values for C'_{kn}. If the measurements are truly reciprocal, the matrix $\mathbf{C}'$ deduced in this way will automatically be symmetric. Carrying this procedure through in practice would, of course, raise questions of numerical methods and accuracy which are not explored further here.

Often, complex modes are regarded as simply a nuisance when they appear in a modal test. However, the analysis of section 2 shows that for a general dissipation matrix,

complex modes appear at the same order of approximation as complex (i.e., damped) natural frequencies. Only for the special case of proportional damping do the modes remain real in the presence of damping. Necessary and sufficient conditions on the dissipation matrix for this to occur have been given by Caughey and O'Kelly [14]. However, the concept of proportional damping seems to be entirely a matter of mathematical convenience. There is no obvious justification for expecting physical systems to exhibit this effect, and complex modes should be regarded as the norm.

Tracking the poles of transfer functions is not the only approach to experimental determination of mode shapes. A common alternative is to use a "full-field" visualization technique. Typically the structure would be driven at one or more points, with a sinusoidal excitation tuned to match a resonant peak. Efforts may be made to suppress nearby modes by making the excitation pattern orthogonal to them. For example, in a system having a plane of symmetry one might use symmetrically-placed drive points driven in antiphase to excite antisymmetric modes but not symmetric ones. The response field would then be observed by, for example, holographic means or scanning laser vibrometry.

Not surprisingly, such a method sometimes shows phase differences between different points on the system. These are often described as "complex modes", but one can now see easily that they are *not* in general the same as the complex modes defined earlier. A simple counter-example is sufficient. Consider a long, uniform, one-dimensional system with uniformly distributed viscous damping, such as a long beam undergoing flexural vibration when submerged in oil. Such a system has a dissipation matrix with the same diagonal form as the mass matrix, so this is a system with proportional damping [1]. The modes, defined as the free motion of the system corresponding to a given (complex) natural frequency, are still purely real in this case. A particular mode involves a certain vibration pattern in the rod which decays exponentially in time without phase difference between different points.

If this system is driven at a point with a *real* frequency ω, in order to make a full-field measurement, the response pattern will show phase differences. If the rod is long, one would expect to see traces of outgoing waves from the excitation point, which are not entirely matched by reflected waves to form pure standing waves. The reason can be simply stated in terms of travelling-wave components of the motion: the modes have real wavenumbers but (slightly) complex frequencies. If driven at a real frequency, the wavenumber inevitably becomes slightly complex. The two are directly related via the group velocity of the system. This problem could only be circumvented if the excitation was distributed over the entire system in such a way as to be orthogonal to all the other modes. But to do that would involve knowing the mode shapes in detail in advance, which defeats the point of the measurement.

The conclusion seems to be that such full-field methods, although very powerful for showing the general character of modes, would require considerable care in interpretation if used to study complex modes and damping models. As an example, consider the practice of using arrangements of driving to eliminate contributions from nearby modes. Equation (10) shows that, in the presence of off-diagonal damping terms in a dissipation matrix model, the main perturbation to a given mode shape comes from an admixture of the other modes, predominantly those closest in frequency. These contributions are in quadrature with the original "undamped" component of the mode. So according to this model, the driving arrangement must eliminate any in-phase contribution of the neighbouring mode but not remove a quadrature component. Apart from the practical difficulties of doing this reliably, there is a hint of circular logic here: to use this fact deliberately in the design of the driving is to assume that a dissipation-matrix model of damping is appropriate at the outset.

4.2. GENERAL DAMPING MODELS

What happens if the methodology of experimental modal analysis is applied to a system described by the more general damping model of section 3? The first comment to be made is that the basic methodology of experimental modal analysis should still be applicable to this rather general class of systems, since it is based on fitting a fixed set of complex pole frequencies to a set of measured transfer functions, and it was shown from equation (29) that the set of poles associated with structural resonances will be the same in all transfer functions.

One consequence of equation (35) for experimental modal analysis is that for this general model of damping forces, reciprocity will not necessarily be found. However, if the spatial variation of the residue of one pole is tracked as the *observation point* (but not the excitation point) is varied, the result will accurately reproduce the form of the complex mode corresponding to that pole. No assumption has been made about the form of $\mathbf{G}'$ except that its terms are small, so this conclusion remains true in the presence of generalized gyroscopic forces as well as truly dissipative forces. If the internal forces in a given system were to behave as in the example of section 3.3, then reciprocity would hold, and the measurement could be made in the usual way by moving the excitation point.

A thorough experimental study aimed at establishing which damping model was appropriate to a given system, and determining the relevant parameter values, might proceed by the following steps. First, tests would be made to check for linearity. If non-sinusoidal response to sinusoidal driving, or amplitude-dependence of measured transfer functions, were seen, then there would be little point in trying to fit the fine details of a linear model. If the system passes this test, a set of transfer functions would be measured. Enough measurements must be made to check carefully for failures of reciprocity. If reciprocity is verified to sufficient accuracy, the simpler version of the general damping model with a symmetric matrix should be expected to hold, and it might even transpire that a dissipation-matrix model will be adequate.

In any case, the next step is to identify the set of poles, and fit the residues of these poles in all the measured transfer functions. If all the modal Q-factors exceed 20 or so, the results of the simple example from section 3.3 suggest that the small-damping approximate theory will be sufficiently accurate to interpret the results with some confidence. In that case the complex mode shapes can be deduced, bearing in mind that if reciprocity does not hold then results must be deduced from variation of the observation point rather than the driving point. The final stage is to attempt to invert equations (10) or (31) to deduce the parameters of the damping model.

If a dissipation-matrix model is adequate, then there is in principle just enough data available to determine the full set of parameters. The diagonal terms of the dissipation matrix are deduced from the complex frequencies, via equation (9). To determine the off-diagonal terms, it would be necessary to express the complex modes as a linear combination of the undamped mode shapes, each coefficient of this expansion giving the value of one term of the dissipation matrix in normal co-ordinates. The undamped mode shapes would be approximated by the real parts of the complex mode shapes. Ideally, these would be checked for orthogonality with respect to the mass matrix, then advantage could be taken of this fact to carry out the decomposition of the imaginary parts into a modal sum. To carry out this sequence of steps would be a tall order in terms of computational accuracy, but it is certainly conceivable. The final check on whether the dissipation-matrix model is valid would be made by checking whether the matrix C'_{jk} deduced was symmetric. Recall that this question is unrelated to the issue of reciprocity: the derivation of equation (10) makes no use of symmetry.

If the more general linear model of damping is relevant, then things are much more difficult: there simply is not enough information available, even from a perfect measurement, to determine the model completely. The same sequence of steps could be followed, but there are two problems in the final stage. The first problem is that there is no longer a direct *experimental* way to deduce the undamped mode shapes. It is conceivable that this could be overcome by a combination of theoretical modelling and careful data analysis, but it would be a very tall order. The second problem is even more serious—even if the decomposition of complex modes into a linear combination of undamped modes could be carried out, the results would only give very sketchy information about the damping model. The reason is that each element G'_{jk} is an unknown function of frequency, and all one can deduce from equation (31) is the value of this function sampled at the particular frequency ω_k. It seems clear that many different damping models might happen to give the same complex value at this particular frequency, and thus be indistinguishable by this measurement method.

Admittedly, this argument has omitted one consideration. If it were possible to find reliably the additional poles associated with the damping model, then the theory of complex functions might in principle allow the frequency-dependent behaviour to be deduced on the assumption that the functions G'_{jk} are all analytic except at isolated poles. However, this seems a very slender hope in practice. Probably, one would need to have some knowledge of the relevant damping model based on knowledge of the physical mechanisms operating to resolve the serious ambiguity outlined in the previous paragraph.

It thus appears that measurements confined to a certain frequency bandwidth could be fitted equally well by a variety of different damping models. Since all these models fit the full set of transfer functions well, it is reasonable to ask whether the ambiguity over damping models is important. Under some circumstances, it might not matter at all. In that case, one might as well fit a dissipation-matrix model if the experimentally-determined matrix is symmetric within the bounds of experimental accuracy. However, for some purposes of engineering design the ambiguity about damping models might be quite significant, especially if the purpose of the measurements on a structure were to predict the effect of structural modifications. If the damping model is wrong, the spatial variation of the energy dissipation is probably not represented correctly. Recall that the proposed measurements are made in normal co-ordinates, and have to be transformed back to give spatial information. That might lead to quite misleading predictions of the effect of a localized change to the structure. This issue would merit further study.

5. CONCLUSIONS

The vibration behaviour of systems with linear damping has been analyzed in some detail, both for the case when the damping forces can be expressed through a dissipation matrix and for the more general case where nothing is assumed beyond linearity. In both cases, an approximation of "small damping" was made, and this allowed very simple expressions to be calculated for the damped natural frequencies, complex mode shapes and transfer functions. The formulae for transfer functions follow very closely the familiar result for undamped systems, giving them considerable intuitive appeal. The approximate forms can be found by simple post-processing of the results of an undamped calculation, such as a finite-element computation. A simple numerical example was presented, for which it was found that the approximate results gave satisfactory accuracy over a very wide range of parameter values.

The theory has useful implications for the interpretation of results in experimental modal analysis. It has been shown that, within the limits of this approximation, it should be

possible to determine reliably the correct complex modes of any structure for which the mechanism of damping is linear to sufficient accuracy. Furthermore, these complex mode shapes can be analyzed to give information about the damping model. It has been shown that it is not in general possible to determine the correct damping model for a structure by purely empirical means, but the class of possible models can in principle be identified. The problem is that a damping model is specified by a matrix of frequency-dependent functions, but for a given structure each of these functions can only be observed at the frequency of one particular normal mode, via the residue of the corresponding pole in a transfer function. To resolve this fundamental ambiguity would presumably require detailed modelling of the physical mechanisms of damping.

ACKNOWLEDGMENTS

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APPENDIX

Equations (1) can be recast in first-order form (see e.g., reference [8]) as

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{F}, \tag{A1}$$

in terms of the state vector and force vector

$$\mathbf{z} = \begin{bmatrix} \mathbf{y} \\ \dot{\mathbf{y}} \end{bmatrix}, \qquad \mathbf{F} = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{f} \end{bmatrix},$$

and the $2N \times 2N$ matrix

$$\mathbf{A} = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}.$$

Assuming the eigenvalues $\lambda_j (= \mathrm{i}\bar{\omega}_j)$ of this matrix are distinct, each is associated with a right eigenvector $\mathbf{v}_r^{(j)}$ and a left eigenvector $\mathbf{v}_l^{(j)}$:

$$\mathbf{A}\mathbf{v}_r^{(j)} = \lambda_j \mathbf{v}_r^{(j)}, \qquad \mathbf{v}_l^{(j)t}\mathbf{A} = \lambda_j \mathbf{v}_l^{(j)t}$$

or

$$\mathbf{AR} = \mathbf{R}\boldsymbol{\Lambda}, \qquad \mathbf{LA} = \boldsymbol{\Lambda}\mathbf{L} \tag{A2}$$

where $\mathbf{R}$ has the right eigenvectors as its columns, $\mathbf{L}$ has the left eigenvectors as its rows, and $\boldsymbol{\Lambda} = \mathrm{diag}\,(\lambda_j)$.

For distinct eigenvalues λ_j, λ_k, the left and right eigenvectors are orthogonal:

$$\mathbf{v}_l^{(j)t}\mathbf{v}_r^{(k)} = 0 \ (j \neq k).$$

If one normalizes so that

$$\mathbf{v}_l^{(j)t}\mathbf{v}_r^{(j)} = 1, \tag{A3}$$

then

$$\mathbf{L} = \mathbf{R}^{-1}. \tag{A4}$$

Thus, from equation (A2)

$$\mathbf{A} = \mathbf{R}\boldsymbol{\Lambda}\mathbf{L},$$

and so from equation (A1)

$$\mathbf{z} = [\mathrm{i}\omega\mathbf{I} - \mathbf{A}]^{-1}\mathbf{F} = [\mathrm{i}\omega\mathbf{RL} - \mathbf{R}\boldsymbol{\Lambda}\mathbf{L}]^{-1}\mathbf{F} = \mathbf{R}\,\mathrm{diag}\left(\frac{1}{\mathrm{i}\omega - \lambda_j}\right)\mathbf{LF} = \left(\sum_{j=1}^{2N} \frac{\mathbf{v}_r^{(j)}\mathbf{v}_l^{(j)t}}{\mathrm{i}\omega - \lambda_j}\right)\mathbf{F}. \tag{A5}$$

This expresses the matrix of transfer functions in partial fraction form, which must now be related to equation (22). Note that

$$\mathbf{v}_r^{(j)} = \begin{bmatrix} \bar{\mathbf{u}}^{(j)} \\ \lambda_j \bar{\mathbf{u}}^{(j)} \end{bmatrix}$$

in terms of the complex modes of the second-order system as defined earlier. Now let

$$\mathbf{v}_l^{(j)} = \begin{bmatrix} \mathbf{v}_1^{(j)} \\ \mathbf{v}_2^{(j)} \end{bmatrix}.$$

Then from equation (A1),

$$\begin{cases} \lambda_j \mathbf{v}_1^{(j)t} = -\mathbf{v}_2^{(j)t}\mathbf{M}^{-1}\mathbf{K} \\ \lambda_j \mathbf{v}_2^{(j)t} = \mathbf{v}_1^{(j)t} - \mathbf{v}_2^{(j)t}\mathbf{M}^{-1}\mathbf{C} \end{cases} \tag{A6}$$

from which it follows that

$$[\lambda_j^2\mathbf{M} + \lambda_j\mathbf{C} + \mathbf{K}](\mathbf{M}^{-1}\mathbf{v}_2^{(j)}) = 0,$$

so that

$$\mathbf{M}^{-1}\mathbf{v}_2^{(j)} = \alpha_j \mathbf{v}_r^{(j)} \tag{A7}$$

for some scalar multiple α_j. The value is determined by the normalization condition (A3). Substitution from (A6) yields

$$\alpha_j = \frac{1}{\lambda_j \bar{\mathbf{u}}^{(j)t}\mathbf{M}\bar{\mathbf{u}}^{(j)} - \lambda_j^{-1}\bar{\mathbf{u}}^{(j)t}\mathbf{K}\bar{\mathbf{u}}^{(j)}}. \tag{A8}$$

Using the second-order governing equation for this mode, this can be re-written

$$\alpha_j = \frac{1}{2\lambda_j \bar{\mathbf{u}}^{(j)t}\mathbf{M}\bar{\mathbf{u}}^{(j)} + \bar{\mathbf{u}}^{(j)t}\mathbf{C}\bar{\mathbf{u}}^{(j)}}$$

$(=1/2\mathrm{i}\omega_j$ for the undamped case with the usual normalization.)

Combining equations (A5) and (A7), and recalling that the first-order modes occur in complex-conjugate pairs with eigenvalues $\mathrm{i}\bar{\omega}_k$, $-\mathrm{i}\bar{\omega}_k^*$, one obtains

$$H_{mn}(\omega) = \sum_{k=1}^{N} \mathrm{i}\alpha_k \left\{ -\frac{\bar{\mathbf{u}}_m^{(k)}\bar{\mathbf{u}}_n^{(k)}}{(\omega - \bar{\omega}_k)} + \frac{\bar{\mathbf{u}}_m^{(k)*}\bar{\mathbf{u}}_n^{(k)*}}{(\omega + \bar{\omega}_k^*)} \right\}. \tag{A9}$$

This is written in a form immediately comparable with equation (22).