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LETTERS TO THE EDITOR

TRANSITION BEHAVIOUR FROM STRING TO BEAM FOR AN AXIALLY ACCELERATING MATERIAL

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1. INTRODUCTION

In a previous paper [1], approximate analytical solutions of a string moving with time dependent velocity have been presented. The axial velocity was assumed to be harmonically varying about a mean velocity. A detailed stability analysis was performed for the cases of principle parametric and combination resonances. After assuming small fluctuations about a mean velocity, stability borders were derived up to the first order of approximation. In this work, small beam effects are further introduced to the problem and the change in stability borders during transition from string to beam are investigated. An application of this transient behaviour may be found in band saw vibrations where the axially moving continua may either be modelled as a strip or a beam.

The dynamic behaviour of axially accelerating strings has been investigated in a number of papers [1–5]. Instead of velocity variations, tension fluctuations were considered for strings by Mockenstrum *et al*. [6]. Constant velocity solutions for strings as well as beams are vast and can be found in the review papers by Ulsoy *et al*. [7] and Wickert and Mote [8]. Recent work for beams moving with constant velocity include those by Wickert and Mote [9] and Wickert [10], the latter being a non-linear analysis.

2. THEORY

Following a similar derivation as given by Wickert [10], it can be shown that the linear, time dependent, dimensionless equation of motion for the axially moving beam problem is

$$\ddot{w} + \dot{v}w' + 2v\dot{w}' + (v^2 - 1)w'' + v_f^2 w^{iv} = 0, \tag{1}$$

where w is the transverse displacement, v is the axial velocity and v_f^2 a constant. When $v_f^2 = 0$, the equation reduces to that of a travelling string. The dot denotes differentiation with respect to time and the primes denote differentiation with respect to the spatial variable x.

Assuming that the velocity is harmonically varying about a constant mean velocity v_0, one writes

$$v = v_0 + \epsilon v_1 \sin \Omega t, \tag{2}$$

where ϵ is a small parameter and ϵv_1, which is also small, represents the amplitude of fluctuations. Ω is the fluctuation frequency. The aim is to investigate the transition behaviour from string to beam and hence one assumes that v_f^2 is small:

$$v_f^2 = \epsilon v_2. \tag{3}$$

Although, with this choice of ordering, the amplitudes of fluctuations and beam effects seem to be related, with the insertion of arbitrary v_1 and v_2 parameters, they can be selected

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independently. Hence $v_1 = 0$, $v_2 \neq 0$ denotes a beam moving with constant velocity whereas $v_1 \neq 0$, $v_2 = 0$ denotes a string moving with harmonically varying velocity.

Substituting equations (2) and (3) into equation (1) and keeping terms up to the first approximation, one has

$$\ddot{w} + 2v_0\dot{w}' + (v_0^2 - 1)w'' + \epsilon\{v_1\Omega\cos\Omega t \cdot w' + 2v_1\sin\Omega t \cdot \dot{w}' + 2v_0v_1\sin\Omega t \cdot w'' + v_2 w^{iv}\} = 0. \tag{4}$$

A direct application of perturbations (direct perturbation method) to the partial differential equation (4) is advantageous over the usual discretization–perturbation method, since the former method does not require conversion of the equations into other forms nor the orthogonalization of eigenfunctions [1]. For higher order perturbation schemes and for finite mode truncations, the direct perturbation method solutions are more accurate than those of the discretization–perturbation method [11–17].

Using the method of multiple scales [18, 19] and assuming a first order expansion, one writes

$$w(x, t; \epsilon) = w_0(x, T_0, T_1) + \epsilon w_1(x, T_0, T_1) + \cdots; \tag{5}$$

where $T_0 = t$ and $T_1 = \epsilon t$ are the usual fast and slow time scales. In terms of the new variables the time derivatives can be written as

$$\mathrm{d}/\mathrm{d}t = D_0 + \epsilon D_1 + \cdots, \quad \mathrm{d}^2/\mathrm{d}t^2 = D_0^2 + 2\epsilon D_0 D_1 + \cdots. \tag{6}$$

Substituting equations (5) and (6) into equation (4), separating terms at each order of ϵ, one obtains

$$O(1): \quad D_0^2 w_0 + 2v_0 D_0 w_0' + (v_0^2 - 1)w_0'' = 0, \tag{7}$$

$$O(\epsilon): \quad D_0^2 w_1 + 2v_0 D_0 w_1' + (v_0^2 - 1)w_1'' = -2D_0D_1w_0 - 2v_0D_1w_0' - v_1\Omega\cos\Omega T_0 \cdot w_0' - 2v_1\sin\Omega T_0 \cdot D_0 w_0' - 2v_0v_1\sin\Omega T_0 \cdot w_0'' - v_2 w_0^{iv}. \tag{8}$$

In an approximate sense, one may assume that the initial solution w_0 resembles that of the travelling string problem with boundary conditions $w_0(0, T_0, T_1) = 0$ and $w_0(1, T_0, T_1) = 0$ and hence the required solution is

$$w_0(x, T_0. T_1) = A_n(T_1)\,\mathrm{e}^{\mathrm{i}\omega_n T_0}\psi_n(x) + \overline{A}_n(T_1)\,\mathrm{e}^{-\mathrm{i}\omega_n T_0}\overline{\psi}_n(x), \tag{9}$$

where ω_n are the natural frequencies

$$\omega_n = n\pi(1 - v_0^2), \qquad n = 1, 2, 3, \ldots, \tag{10}$$

and $\psi_n(x)$ are the eigenfunctions (mode shapes) corresponding to ω_n

$$\psi_n(x) = C_n\,\mathrm{e}^{\mathrm{i}\alpha_n x}\sin n\pi x, \qquad \alpha_n = n\pi v_0, \qquad n = 1, 2, 3, \ldots. \tag{11}$$

Substituting equation (9) into equation (8), and arranging, one has

$$\begin{aligned} D_0^2 w_1 + 2v_0 D_0 w_1' + (v_0^2 - 1)w_1'' = {} & D_1 A_n\,\mathrm{e}^{\mathrm{i}\omega_n T_0}(-2\mathrm{i}\omega_n\psi_n - 2v_0\psi_n') \\ & + A_n\,\mathrm{e}^{\mathrm{i}(\Omega+\omega_n)T_0}(-\tfrac{1}{2}v_1\Omega\psi_n' - v_1\omega_n\psi_n' + \mathrm{i}v_0v_1\psi_n'') \\ & + \overline{A_n}\,\mathrm{e}^{\mathrm{i}(\Omega-\omega_n)T_0}(-\tfrac{1}{2}v_1\Omega\overline{\psi}_n' + v_1\omega_n\overline{\psi}_n' + \mathrm{i}v_0v_1\overline{\psi}_n'') \\ & + A_n\,\mathrm{e}^{\mathrm{i}\omega_n T_0}(-v_2\psi_n^{iv}) + \mathrm{cc}, \end{aligned} \tag{12}$$

where cc denotes the complex conjugate of the preceding terms. Three cases arise depending upon the value of fluctuation frequency.

2.1. *Ω away from $2\omega_n$ and 0*

In this case, the solvability condition requires (see reference [18] for details of calculating solvability conditions)

$$D_1 A_n + k_1 A_n = 0, \tag{13}$$

where

$$k_1 = -(\mathrm{i}/2) v_2 n^3 \pi^3 (v_0^4 + 6v_0^2 + 1), \tag{14}$$

The solution is

$$A_n = A_0 \,\mathrm{e}^{-k_1 T_1}. \tag{15}$$

It is obvious that the solutions are bounded and no instabilities arise up to this order of approximation. Secondary instabilities may arise however at higher orders of approximations.

Substituting equation (15) into equation (9) with $T_1 = \epsilon T_0$ yields

$$w_0(x, T_0, T_1) = A_0 \,\mathrm{e}^{(\mathrm{i}\omega_n - \epsilon k_1) T_0} \psi_n(x) + \mathrm{cc}, \tag{16}$$

and one can easily see that the approximate natural frequency for a moving beam is $\omega_n + \epsilon \mathrm{i} k_1$, or

$$(\omega_b)_n = n\pi(1 - v_0^2) + \tfrac{1}{2}\epsilon v_2 n^3 \pi^3 (v_0^4 + 6v_0^2 + 1). \tag{17}$$

This analytical result is valid for small $v_f^2 = \epsilon v_2$ values: that is, the equation represents the transient behaviour from strip to beam.

2. *Ω close to $2\omega_n$*

In this case, to represent the nearness, one writes

$$\Omega = 2\omega_n + \epsilon\sigma, \tag{18}$$

where σ is a detuning parameter. The solvability condition requires

$$D_1 A_n + k_0 \overline{A_n} \,\mathrm{e}^{\mathrm{i}\sigma T_1} + k_1 A_n = 0, \tag{19}$$

where k_1 is defined previously and k_0 is

$$k_0 = (v_1/4)[\sin 2\alpha_n - \mathrm{i}(1 - \cos 2\alpha_n)]. \tag{20}$$

To perform a stability analysis, one introduces the transformation

$$A_n = B_n \,\mathrm{e}^{\mathrm{i}(\sigma/2) T_1} \tag{21}$$

and obtains

$$D_1 B_n + (\mathrm{i}(\sigma/2) + k_1) B_n + k_0 \overline{B}_n = 0. \tag{22}$$

Separating each term into its real and imaginary parts,

$$B_n = B_{nR} + \mathrm{i} B_{nI}, \qquad k_1 = -\mathrm{i} k_{1I}, \qquad k_0 = k_{0R} - \mathrm{i} k_{0I}, \tag{23}$$

one obtains the coupled equations

$$B'_{nR} - (\sigma/2 - k_{1I} + k_{0I}) B_{nI} + k_{0R} B_{nR} = 0, \qquad B'_{nI} + (\sigma/2 - k_{1I} - k_{0I}) B_{nR} - k_{0R} B_{nI} = 0. \tag{24}$$

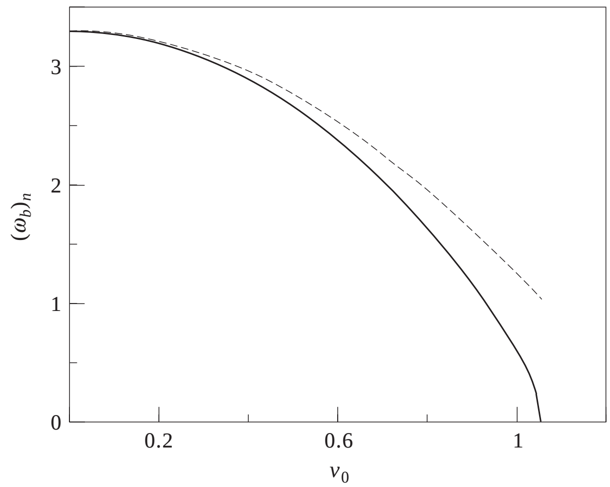

Figure 1. Comparison of exact and approximate natural frequency values for a moving beam. Approximate (- - -); exact (—); $n=1$, $v_f = 0{\cdot}1$.

Assuming solutions of the form

$$B_{nR} = b_{nR}\,\mathrm{e}^{\lambda_1 T_1}, \qquad B_{nI} = b_{nI}\,\mathrm{e}^{\lambda_1 T_1} \tag{25}$$

and substituting, for non-trivial solutions yields

$$\lambda_1 = \mp\sqrt{-((\sigma/2) - k_{11})^2 + k_{0R}^2 + k_{0I}^2}. \tag{26}$$

For

$$-\sqrt{k_{0R}^2 + k_{0I}^2} < ((\sigma/2) - k_{1I}) < \sqrt{k_{0R}^2 + k_{0I}^2} \tag{27}$$

the response is unstable whereas it is stable outside this region. Hence the stability boundaries are determined by

$$\sigma = 2k_{1I} \mp 2\sqrt{k_{0R}^2 + k_{0I}^2}. \tag{28}$$

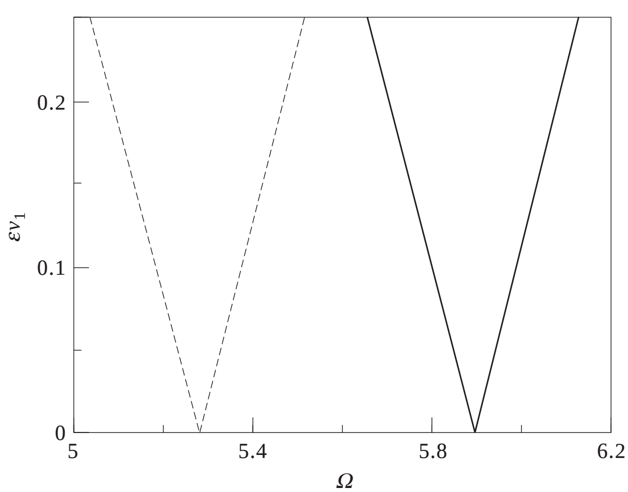

Figure 2. Stability borders for the string and beam. String (- - -); beam (——); $n=1$, $v_f = 0{\cdot}1$).

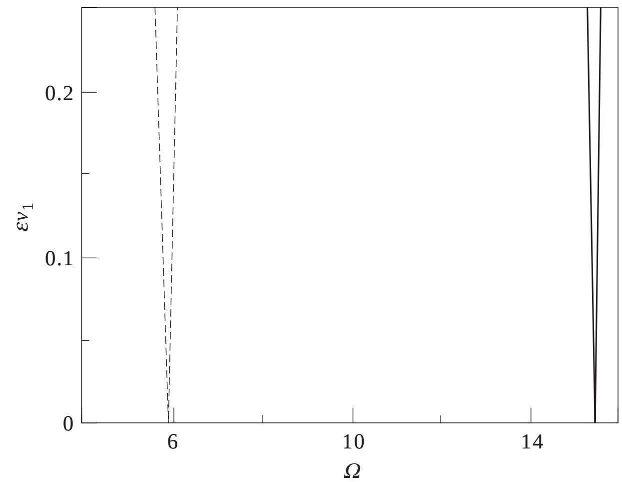

Figure 3. The stability borders for a beam for the first two modes. $n=1$ (- - -); $n=2$ (——); $v_f=0{\cdot}1$).

Substituting for k_{0R}, k_{0I} and k_{1I}, one has

$$\sigma = v_2 n^3\pi^3(v_0^4 + 6v_0^2 + 1) \mp v_1 \sin \alpha_n. \tag{29}$$

Inserting σ further into equation (18) gives the final result as

$$\Omega = 2n\pi(1 - v_0^2) + \epsilon[v_2 \mathrm{n}^3\pi^3(v_0^4 + 6v_0^2 + 1) \mp v_1 \sin(n\pi v_0)]. \tag{30}$$

When $v_2 = 0$, equation (30) reduces to equation (59) of reference [1] for strings. The two values of Ω denote the stability boundaries for small ϵ. Numerical solutions for equation (30) will be given in the next section.

Note that for a beam with constant velocity $v_2 \neq 0$, $v_1 = 0$ and hence $k_0 = 0$ from equation (20). This reduces equation (19) to equation (13) and no instabilities arise for this case.

2.3. *Ω close to* 0

A similar calculation yields stable solutions up to the first order of approximation for this case.

3. NUMERICAL EXAMPLES

In this section, numerical plots for the natural frequencies (equation (17)) and stability borders (equation (29)) will be presented.

By employing the method given by Wickert [10], exact natural frequencies were calculated and compared with those of approximate ones (equation (17)) as shown in Figure 1. For $v_f = 0{\cdot}1$ ($\epsilon v_2 = 0{\cdot}01$), the agreement is very good for small v_0 values. As v_0 increases gradually, the approximate and exact solutions diverge, the former being always higher than the latter. As one increases v_f (i.e. if $v_f = 0{\cdot}2$), the range of v_0 where reasonable agreement exists between the exact and approximate solution shrinks.

Next, by using equation (30), the approximate stability borders were plotted for parametric excitation frequency versus amplitude. In Figure 2, for the first mode, the stability borders for the string ($v_f = 0$) and beam are compared ($v_f = 0{\cdot}1$ or $\epsilon v_2 = 0{\cdot}01$) (unstable solutions in between the lines). It can be seen that, when beam effects are introduced, the stability borders shift to higher Ω values without any change in their slopes.

Finally, Figure 3 shows the stability borders for a beam for the first two modes ($v_f = 0{\cdot}1$).

4. CONCLUDING REMARKS

The transient behaviour from strip to beam is investigated for axially moving continua. An approximate analytical expression for the natural frequency is given for the problem. For velocity profiles harmonically varying about a mean velocity, stability borders are determined analytically for fluctuation frequency versus amplitudes. Beam effects cause the stability boundaries to shift to higher frequency values.

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