Journal of Sound and Vibration (1998) **215**(3), 577–586
Article No. 981622

ON EXTENSIONAL VIBRATION MODES OF ELASTIC RODS OF FINITE LENGTH WHICH INCLUDE THE EFFECT OF LATERAL DEFORMATION

Shankar Krishnaswamy
Department of Engineering Mechanics, University of Nebraska-Lincoln, Lincoln, NE 68588-0526, U.S.A.

and

R. C. Batra
Department of Engineering Science and Mechanics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061-0219, U.S.A.

(*Received* 28 *July* 1997, *and in final form* 2 *March* 1998)

1. INTRODUCTION

Extensional vibrations are analyzed in the linear theory of an elastic rod of finite length which accounts for lateral deformation. A simplified version of a theory of elastic rods is used; it is based on the concept of a directed curve as developed by Green *et al*. [1–3]. This simplified version is similar to Mindlin and Herrmann's theory [4], and has been used recently by Krishnaswamy and Batra [5] to study wave propagation and pure thickness oscillations in an infinitely long rod.

It may be recalled that for an infinitely long circular rod whose lateral surface is free of traction, the frequencies of extensional vibration are obtained by solving the well-known Pochhammer–Chree equation (see e.g., reference [6]). However, for a finite rod, this equation is invalid as it cannot satisfy the stress-free end boundary conditions. Various approximate solutions for extensional oscillations of a free finite rod have been obtained by Rumerman and Raynor [7], Hutchinson [8, 9], and Rasband [10], all of whom based their work on the classical three-dimensional theory of elasticity. McNiven and Perry [11] presented a solution based on an approximate set of equations which take into account the coupling between longitudinal, axial shear, and radial modes in a rod of infinite length. An experimental study on vibrations of solid cylinders can be found in reference [12].

In the present paper, free extensional vibrations of a finite rod subjected to various boundary conditions are analyzed. It is shown that the analysis can be separated into three cases depending on the range in which the natural frequency lies. Such a situation also arises in the Timoshenko theory of flexural vibrations of a beam (see, e.g., reference [13]). The differential equations of the extensional theory are very similar to those of the Timoshenko theory but there are some differences which we will allude to when appropriate. Our analysis of Case 3 (see section 3) may be regarded as the extensional counterpart of a recent paper by O'Reilly and Turcotte [14] on Timoshenko beams.

2. EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

Consider a uniform straight rod of length L and cross-sectional area A. The rod is referred to a system of Cartesian co-ordinates (x, y, z) with the z-axis along the centreline of the rod and the origin at one end. An elastic rod possessing certain material and geometrical symmetries* and attention is confined to axially symmetric deformations.

* For a discussion of these symmetries, see references [2, 3].

0022–460X/98/330577 + 10 $30.00/0

Denoting time derivatives by a superposed dot and partial derivatives with respect to z by a comma followed by z, and referring the reader to references [2, 3, 5] for details, the governing equations for extensional motions may be written in the form*

$$n_{,z} = \rho_0 \ddot{u}, \qquad m_{,z} - k = \rho_0 y \ddot{\delta}, \tag{1a, b}$$

where $u = u(z, t)$ is the longitudinal displacement of particles on the centreline of the rod, $\delta = \delta(z, t)$ is the displacement associated with a material fibre lying in the cross-section, $y = I/A$, I being the area moment of inertia of the cross-section, and ρ_0 denotes the mass density per unit undeformed length of the rod. The quantity n in equation (1a) is the axial force, and m and k in equation (1b) are generalized forces whose actions account for lateral deformation of the rod. Note that the displacement δ is the kinematical variable associated with this mode of deformation.

The following linear constitutive equations relate the forces $\{n, m, k\}$ to the displacements $\{u, \delta\}$ and their derivatives:

$$n = 2\alpha_8 \delta + \alpha_3 u_{,z}, \qquad m = (\alpha_{10} + \alpha_{17})\delta_{,z}, \qquad k = (\alpha_2 + \alpha_7)\delta + \alpha_8 u_{,z}, \tag{2a–c}$$

where the α's are material constants which, for a circular rod of radius a, have been recently determined by Krishnaswamy and Batra [5] to be

$$\alpha_2 = \rho_0 y \bar{\omega}^2 - \alpha_7, \qquad \alpha_3 = EA + \frac{2\alpha_8^2}{\alpha_2 + \alpha_7}, \qquad \alpha_7 = \alpha_8 = \frac{EA\nu}{(1+\nu)(1-2\nu)}, \tag{3a–c}$$

$$\alpha_{10} = \frac{(0{\cdot}615 + 0{\cdot}792\nu)^2}{(1+\nu)^3} EI, \qquad \alpha_{17} = 0. \tag{4a, b}$$

In equations (3) and (4), E is Young's Modulus, ν is Poisson's ratio, $A = \pi a^2$, $I = \pi a^4/4$, and the quantity $\bar{\omega}$ is the frequency of pure radical oscillations of the rod given by†

$$\bar{\omega}^2 = c_D^2 \beta^2, \qquad c_D^2 = \frac{EA(1-\nu)}{(1+\nu)(1-2\nu)\rho_0}, \tag{5a, b}$$

where c_D is the speed of propagation of dilatational waves in a linearly elastic body of infinite extent, and β satisfies

$$(1-2\nu)J_1(\beta a) = (1-\nu)\beta a J_0(\beta a), \tag{6}$$

J_n being the Bessel function of order n. Also note that the constants in equations (3) and (4a) are non-zero. The following coupled linear differential equations governing the displacements u and δ are obtained by substituting equations (2a–c) into equations (1a, b):

$$2\alpha_8 \delta_{,z} + \alpha_3 u_{,zz} = \rho_0 \ddot{u}, \qquad \alpha_{10}\delta_{,zz} - (\alpha_2 + \alpha_7)\delta - \alpha_8 u_{,z} = \rho_0 y \ddot{\delta}, \tag{7, 8}$$

where equation (4b) has been used in obtaining equation (8).‡ It may be remarked that the differential equations (7) and (8) bear a striking resemblance to those governing flexural vibrations of a rod according to Timoshenko theory.

For free vibrations of the rod, we seek solutions to equations (7) and (8) of the form

$$u(z, t) = U(z)\,\mathrm{e}^{\mathrm{i}\omega t}, \qquad \delta(z, t) = \Delta(z)\,\mathrm{e}^{\mathrm{i}\omega t}, \tag{9}$$

* These equations are valid only for the case when the three-dimensional rod is free of traction on its lateral surface. However, in its original form, the theory can accommodate other types of conditions imposed on this surface.

† This mode of vibration is one in which $u(z, t)$ vanishes for all z and t.

‡ Note that equations (7) and (8) can be decoupled into two fourth order differential equations for u and δ; since these equations are not explicitly used, they are not recorded.

ω being the radian frequency, so that equations (7) and (8) reduce to

$$2\alpha_8 \frac{\mathrm{d}\Delta}{\mathrm{d}z} + \alpha_3 \frac{\mathrm{d}^2 U}{\mathrm{d}z^2} + \rho_0 \omega^2 U = 0, \tag{10}$$

$$\alpha_{10} \frac{\mathrm{d}^2 \Delta}{\mathrm{d}z^2} - \alpha_8 \frac{\mathrm{d}U}{\mathrm{d}z} + (\rho_0 y \omega^2 - \alpha_2 - \alpha_7)\Delta = 0. \tag{11}$$

It is clear from equations (2a–c), (4b) and (9) that $\{n, m, k\}$ have the forms $\{N(z), M(z), K(z)\}\, \mathrm{e}^{\mathrm{i}\omega t}$, where

$$N = 2\alpha_8 \Delta + \alpha_3 \frac{\mathrm{d}U}{\mathrm{d}z}, \qquad M = \alpha_{10} \frac{\mathrm{d}\Delta}{\mathrm{d}z}, \qquad K = (\alpha_2 + \alpha_7)\Delta + \alpha_8 \frac{\mathrm{d}U}{\mathrm{d}z}. \tag{12a–c}$$

The differential equations (10) and (11) constitute an eigenvalue problem with eigenvalue ω^2 and eigenfunction pair $\{U(z), \Delta(z)\}$. Attention is confined to certain common boundary conditions; a simply-supported end for which $U = M = 0$, a clamped end for which $U = \Delta = 0$, and a free end for which $N = M = 0$. For various combinations of these end conditions, it can be shown using standard techniques that the pair $\{U(z), \Delta(z)\}$ satisfies the orthogonality relation

$$\int_0^L \{\rho_0 U_i U_j + \rho_0 y \Delta_i \Delta_j\}\, \mathrm{d}z = 0, \tag{13}$$

where $\{U_i, \Delta_i\}$ and $\{U_j, \Delta_j\}$ are eigenfunction pairs corresponding to distinct frequencies ω_i and ω_j, respectively. The eigenfunctions may be normalized by the condition $\int_0^L \{\rho_0 U_i^2 + \rho_0 y \Delta_i^2\}\, \mathrm{d}z = 1$.

3. FREQUENCY EQUATIONS AND EIGENFUNCTIONS

Solutions to the ordinary differential equations (10) and (11) may be obtained by standard techniques. First, assume that $U(z) = U_0\, \mathrm{e}^{\lambda z}$ and $\Delta(z) = \Delta_0\, \mathrm{e}^{\lambda z}$ and note that the following characteristic equation must be satisfied in order for non-trivial solutions to exist:

$$\lambda^4 + b\lambda^2 + c = 0, \tag{14}$$

where

$$b = \{2\alpha_8^2 + \alpha_3(\rho_0 y \omega^2 - \alpha_2 - \alpha_7) + \alpha_{10}\rho_0\omega^2\}/\alpha_3\alpha_{10}, \tag{15}$$

$$c = \{\rho_0\omega^2(\rho_0 y \omega^2 - \alpha_2 - \alpha_7)\}/\alpha_3\alpha_{10}. \tag{16}$$

There are several possible cases associated with the characteristic equation (14) depending on the range in which the frequency ω lies. Each case is treated separately below.

3.1. *Special case*: $\omega = 0$

This case corresponds to static extensional deformations of the rod and the four roots of equation (14) are 0, 0, $\pm\sqrt{b^*}$, with $b^* = \{\alpha_3(\alpha_2 + \alpha_7) - 2\alpha_8^2\}/\alpha_3\alpha_{10}$. It may be readily verified from equations (3) and (4) that b^* is positive. The solutions for the displacements U and Δ are

$$U(z) = \frac{-2\alpha_8}{\alpha_3\sqrt{b^*}} A_0 \sinh\sqrt{b^*}z - \frac{2\alpha_8}{\alpha_3\sqrt{b^*}} B_0 \cosh\sqrt{b^*}z + \left(1 + \frac{2\alpha_8^2}{\alpha_3^2}\right) C_0 z + D_0, \tag{17}$$

$$\Delta(z) = A_0 \cosh \sqrt{b^*}z + B_0 \sinh \sqrt{b^*}z - \frac{\alpha_8 C_0}{\alpha_3 b^*}. \tag{18}$$

The constants A_0, B_0, C_0 and D_0 are to be determined from the two boundary conditions at each end of the rod. It may be remarked that the corresponding case for flexural vibrations in a Timoshenko beam (i.e., the $\omega = 0$ case) admits polynomial solutions for the field variables in contrast to the solutions (17) and (18) (see, e.g., reference [15]).

The remaining three cases correspond to ω^2 being less than, equal to, or greater than, the quantity $(\alpha_2 + \alpha_7)/\rho_0 y$. Once again, there is a similarity between the present development and the Timoshenko theory; solutions to the latter are also separable into three cases depending on the value of ω^2 as is made clear in reference [13].

3.2. *Case* 1: $\omega^2 < (\alpha_2 + \alpha_7)/\rho_0 y$

This case corresponds to $c < 0$ and hence to $(-b + \sqrt{b^2 - 4c}) > 0$. The four roots of equation (14) are $\pm\gamma$ and $\pm i\varepsilon$, where

$$\gamma = \left\{\frac{-b + \sqrt{b^2 - 4c}}{2}\right\}^{1/2}, \qquad \varepsilon = \left\{\frac{b + \sqrt{b^2 - 4c}}{2}\right\}^{1/2}. \tag{19}$$

For a fixed value of ω, the general solution is

$$U(z) = A_1 \cosh \gamma z + B_1 \sinh \gamma z + C_1 \cos \varepsilon z + D_1 \sin \varepsilon z, \tag{20}$$

$$\Delta(z) = \bar{\alpha} A_1 \sinh \gamma z + \bar{\alpha} B_1 \cosh \gamma z - \hat{\alpha} C_1 \sin \varepsilon z + \hat{\alpha} D_1 \cos \varepsilon z, \tag{21}$$

where

$$\bar{\alpha} = \frac{\rho_0 \omega^2}{2\alpha_8 \gamma} - \frac{\alpha_3 \gamma}{2\alpha_8}, \qquad \hat{\alpha} = \frac{\rho_0 \omega^2}{2\alpha_8 \varepsilon} - \frac{\alpha_3 \varepsilon}{2\alpha_8}. \tag{22}$$

For simply-supported ends, in light of equation (12b), the boundary conditions become

$$U(0) = 0, \qquad U(L) = 0, \qquad \frac{d\Delta}{dz}(0) = 0, \qquad \frac{d\Delta}{dz}(L) = 0. \tag{23}$$

The frequency equation in the form of a vanishing determinant is

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ \cosh \gamma L & \sinh \gamma L & \cos \varepsilon L & \sin \varepsilon L \\ \bar{\alpha}\gamma & 0 & -\hat{\alpha}\varepsilon & 0 \\ \bar{\alpha}\gamma \cosh \gamma L & \bar{\alpha}\gamma \sinh \gamma L & -\hat{\alpha}\varepsilon \cos \varepsilon L & \hat{\alpha}\varepsilon \sin \varepsilon L \end{vmatrix} = 0. \tag{24}$$

Since $\hat{\alpha}\varepsilon - \bar{\alpha}\gamma \neq 0$, equation (24) reduces to

$$\sinh \gamma L \sin \varepsilon L = 0. \tag{25}$$

For clamped ends, the boundary conditions are

$$U(0) = 0, \qquad U(L) = 0, \qquad \Delta(0) = 0, \qquad \Delta(L) = 0, \tag{26}$$

so that the frequency equation is

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ \cosh \gamma L & \sinh \gamma L & \cos \varepsilon L & \sin \varepsilon L \\ 0 & \bar{\alpha} & 0 & \hat{\alpha} \\ \bar{\alpha} \sinh \gamma L & \bar{\alpha} \cosh \gamma L & -\bar{\alpha} \sin \varepsilon L & \hat{\alpha} \cos \varepsilon L \end{vmatrix} = 0. \tag{27}$$

For free ends, one has

$$\frac{\mathrm{d}\Delta}{\mathrm{d}z}(0) = 0, \qquad \frac{\mathrm{d}\Delta}{\mathrm{d}z}(L) = 0, \qquad 2\alpha_8 \Delta(0) + \alpha_3 \frac{\mathrm{d}U}{\mathrm{d}z}(0) = 0, \qquad 2\alpha_8 \Delta(L) + \alpha_3 \frac{\mathrm{d}U}{\mathrm{d}z}(L) = 0, \tag{28}$$

and the frequency equation may be written as

$$\begin{vmatrix} \bar{\alpha}\gamma & 0 & -\hat{\alpha}\varepsilon & 0 \\ \bar{\alpha}\gamma \cosh \gamma L & \bar{\alpha}\gamma \sinh \gamma L & -\hat{\alpha}\varepsilon \cos \varepsilon L & -\hat{\alpha}\varepsilon \sin \varepsilon L \\ 0 & 2\bar{\alpha}\alpha_8 + \gamma\alpha_3 & 0 & 2\hat{\alpha}\alpha_8 + \varepsilon\alpha_3 \\ (2\bar{\alpha}\alpha_8 + \gamma\alpha_3) \sinh L & (2\bar{\alpha}\alpha_8 + \gamma\alpha_3) \cosh \gamma L & -(2\hat{\alpha}\alpha_8 + \varepsilon\alpha_3) \sin \varepsilon L & (2\hat{\alpha}\alpha_8 + \varepsilon\alpha_3) \cos \varepsilon L \end{vmatrix} = 0. \tag{29}$$

These frequency equations provide conditions for the existence of the mode in question in terms of geometric and material properties of the rod. Furthermore, the matrices corresponding to the determinants in equations (24), (27) and (29) have a one-dimensional kernel implying that only three of the four boundary conditions in equation (23), (26) and (28) are functionally independent. These independent boundary conditions may be used to obtain the eigenfunction pair associated with the mode of vibration. This procedure is illustrated for Case 3 below.

3.3. *Case* 2: $\omega^2 > (\alpha_2 + \alpha_7)/\rho_0 y$

The four roots of equation (14) are given by $\mp i\gamma$ and $\pm i\varepsilon$, where γ and ε are given in equation (19). The solutions to this case can be gotten by replacing the hyperbolic functions in equations (20) and (21) by corresponding trigonometric ones. That is, the linearly independent solutions are $\{\cos \gamma x, \sin \gamma x, \cos \varepsilon x, \sin \varepsilon x\}$. The frequency equations follow in the same way as before and furthermore, the comments following equation (29) apply to this case as well. In light of the observations just made and for brevity's sake, the solutions and frequency equations are not reproduced.

3.4. *Case* 3: $\omega^2 = (\alpha_2 + \alpha_7)/\rho_0 y$

This value of ω^2 is equal to $\bar{\omega}^2$ in equation (5). By setting $u(z, t) = 0$, one can see from equation (7) that δ is forced to be constant in z so that for time-harmonic motions, equation (8) reduces to a simple harmonic oscillator with frequency $\bar{\omega}$. Thus, from a physical viewpoint, the rod undergoes pure radial oscillations with particles on its centreline remaining stationary. This situation is analogous to one in Timoshenko's theory in which the beam undergoes pure thickness-shear oscillations.* This is discussed under Case II in O'Reilly and Turcotte [13], and the analysis which follows presently is the extensional counterpart of that in reference [13].

* Here, the beam vibrates in the absence of flexural displacement of its centreline particles.

The roots of the characteristic equation (14) reduce to 0 occurring as a double root and $\pm i\bar{\gamma}$, where $\bar{\gamma}^2 = [2y\alpha_8^2 + \alpha_{10}(\alpha_2 + \alpha_7)]/y\alpha_3\alpha_{10}$. The general forms of $U(z)$ and $\Delta(z)$ are

$$U(z) = F_1 \cos \bar{\gamma}z + F_2 \sin \bar{\gamma}z + F_3, \tag{30}$$

$$\Delta(z) = \frac{\alpha_8}{\bar{\gamma}\alpha_{10}} F_1 \sin \bar{\gamma}z - \frac{\alpha_8 F_2}{\bar{\gamma}\alpha_{10}} \cos \bar{\gamma}z + F_3\left(\frac{\alpha_8}{\alpha_{10}} - \frac{\alpha_3\bar{\gamma}^2}{2\alpha_8}\right)z + F_4, \tag{31}$$

where F_1, F_2, F_3, F_4 are constants to be determined from boundary conditions. As before, this yields four linear equations which may be written as $[D]\{F\} = \{0\}$, where $[D]$ is a 4×4 matrix of coefficients, $\{F\}$ is the column of constants in equation (31) and $\{0\}$ is the column of zeroes. For a non-trivial solution, it is required that $|D| = 0$, where $|D|$ is the determinant of $[D]$. The comments just following equation (29) are recalled and the procedure to obtain eigenfunctions for two types of boundary conditions is illustrated.

For a rod clamped at both ends, the frequency equation may be determined to be

$$1 - \cos \bar{\gamma}L + \left(\frac{\alpha_3\alpha_{10}\bar{\gamma}^2}{4\alpha_8^2}\bar{\gamma}L - \frac{\bar{\gamma}L}{2}\right)\sin \bar{\gamma}L = 0. \tag{32}$$

If equation (32) is satisfied, $[D]$ has a one-dimensional kernel since only three of the four boundary conditions are functionally independent. These may be used to determine the eigenfunction pair $\{U, \Delta\}$:

$$U(z) = F_1\{\cos \bar{\gamma}z - 1 + (\operatorname{cosec} \bar{\gamma}L - \cot \bar{\gamma}L) \sin \bar{\gamma}z\}, \tag{33}$$

$$\Delta(z) = F_1\left\{\frac{\alpha_8}{\bar{\gamma}\alpha_{10}} \sin \bar{\gamma}z - \frac{\alpha_8}{\bar{\gamma}\alpha_{10}} (\operatorname{cosec} \bar{\gamma}L - \cot \bar{\gamma}L) \cos \bar{\gamma}z - \left(\frac{\alpha_8}{\alpha_{10}} - \frac{\bar{\gamma}^2\alpha_3}{2\alpha_8}\right)z + \frac{\alpha_8}{\bar{\gamma}\alpha_{10}} (\operatorname{cosec} \bar{\gamma}L - \cot \bar{\gamma}L)\right\}. \tag{34}$$

The constant F_1 may be determined from the normalization condition mentioned just following equation (13). It is clear that for specified values of y and v, only a rod of specific length L will possess eigenfunctions (33) and (34).

For the simply-supported rod, the matrix $[D]$ is

$$[D] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ \cos \bar{\gamma}L & \sin \bar{\gamma}L & 1 & 0 \\ \dfrac{\alpha_8}{\alpha_{10}} & 0 & \left(\dfrac{\alpha_8}{\alpha_{10}} - \dfrac{\bar{\gamma}^2\alpha_3}{2\alpha_8}\right) & 0 \\ \dfrac{\alpha_8}{\alpha_{10}} \cos \bar{\gamma}L & \dfrac{\alpha_8}{\alpha_{10}} \sin \bar{\gamma}L & \left(\dfrac{\alpha_8}{\alpha_{10}} - \dfrac{\bar{\gamma}^2\alpha_3}{2\alpha_8}\right) & 0 \end{bmatrix}. \tag{35}$$

Clearly, the rank of $[D]$ is 3 so that $[D]$ has a one-dimensional kernel. Hence, the eigenfunctions are given by

$$U(z) = 0, \qquad \Delta(z) = F_4, \tag{36}$$

and the normalization condition yields $F_4 = 1/\sqrt{\rho_0 y L}$. Furthermore, if $\sin \bar{\gamma} L = 0$, then $[D]$ has rank 2 and hence a two-dimensional kernel; only two of the four boundary conditions are independent. One sees that $\bar{\gamma}$ must satisfy

$$\bar{\gamma}^2 L^2 = n^2 \pi^2, \tag{37}$$

where n is a non-zero integer and since ω is fixed, only a rod of specific length satisfies this condition. It follows that there are two eigenfunction pairs associated with the eigenfrequency $\bar{\omega}$. One pair is given by equation (36) while the other orthogonal pair is

$$U(z) = F_2 \sin \bar{\gamma} z, \qquad \Delta(z) = \frac{-\alpha_8}{\bar{\gamma}\alpha_{10}} F_2 \cos \bar{\gamma} z, \tag{38}$$

in which the constant F_2 may be determined from the normalization condition.

4. FURTHER DISCUSSION OF THE FREE–FREE ROD

Recall that the solution of equation (29) yields natural frequencies of the rod with free end boundary conditions. Although equation (29) was derived under the assumption $\omega^2 < (\alpha_2 + \alpha_7)/\rho_0 y$, it may be used for the case when $\omega^2 > (\alpha_2 + \alpha_7)/\rho_0 y$ if γ is replaced by $\gamma^* i$ and the relations $\sinh(\gamma L) = i \sin(\gamma^* L)$ and $\cosh(\gamma L) = \cos(\gamma^* L)$ are used. Equation (29) may be simplified and written as

$$2[1 - \cosh(\xi\zeta)\cos(\zeta)] + \left(\frac{\alpha_3 - \eta}{\eta\xi + \alpha_3\xi^3} - \frac{\eta\xi + \alpha_3\xi^3}{\alpha_3 - \eta}\right)\sinh(\xi\zeta)\sin(\zeta) = 0, \tag{39}$$

where

$$\xi = \frac{\gamma}{\epsilon}, \qquad \zeta = \epsilon L, \qquad \eta = \rho_0\omega^2/\epsilon^2. \tag{40a–c}$$

With the help of equation (40c), it is easily seen that the slenderness parameter L/a is given by

$$L/a = \frac{\bar{\omega}}{2\omega}\sqrt{\frac{\eta}{\alpha_2 + \alpha_7}}\,\zeta, \tag{41}$$

where $\bar{\omega} = (\alpha_2 + \alpha_7)/\rho_0 y$. Equations (39)–(41) are solved numerically to obtain the natural frequencies ω. One proceeds in the following manner.

First, note that for given values of the coefficients α_2, α_3, α_7 and α_{10} in equations (3) and (4) and for a given value of ω, the values of ξ and η in equation (40) are determined and hence the transcendental equation (39) involves only one unknown ζ. A solution to equation (39) yields ζ for a particular mode and therefore one may obtain a value for L/a for this mode from equation (41). Thus, by sweeping through various values of ω, one obtains corresponding values of L/a for each mode and hence, one may plot the variation of the natural frequencies of various modes with the slenderness parameter L/a. ω is non-dimensionalized by $\omega^* = \omega a(\mu A/\rho_0)^{1/2}$, μ being the shear modulus $j\omega^*$ which is the frequency associated with propagating shear waves in an infinite linear elastic body. This facilitates a comparison of our results with those of Rumerman and Raynor [7] which were obtained using a different approach.

Prior to presenting numerical results, a comment regarding the value of α_2 in equation (3a) is in order. In a previous paper [5], the authors chose the second non-zero root β of equation (6) to compute α_2 in order to better capture the three-dimensional high-frequency behaviour of higher modes of an infinite rod. However, it was seen in reference [5] that this compromised the predictions of the present theory in a small range of intermediate frequencies. Since it is precisely this range of frequencies that Rumerman and Raynor [7] report results for, α_2 is computed by choosing β to be the first non-zero root of equation (6). Of course, now the high-frequency three-dimensional behaviour of higher modes will be affected somewhat. In this connection, see reference [16].

The non-dimensional natural frequencies of the first three modes are plotted versus the slenderness ratio L/a in Figures 1(a)–(c), respectively. For each mode, results are included for the two different values of α_2 discussed above. As in reference [7], a Poisson's ratio of $\nu = 0{\cdot}286$ is used in all calculations. The first mode compares extremely well with the results of reference [7]. In the limit $L/a \to 0$, which corresponds to a very thin plate, the value of ω/ω^* approaches 3·3. The exact value of ω/ω^* for this in-plane mode of a thin plate is given by $\omega/\omega^* = \beta\{2/(1-\nu)\}^{1/2}$, where β is the first non-zero root of $\beta J_0(\beta) = (1-\nu)J_1(\beta)$. For $\nu = 0{\cdot}286$, ω/ω^* approximately equals 3·4. However, for values

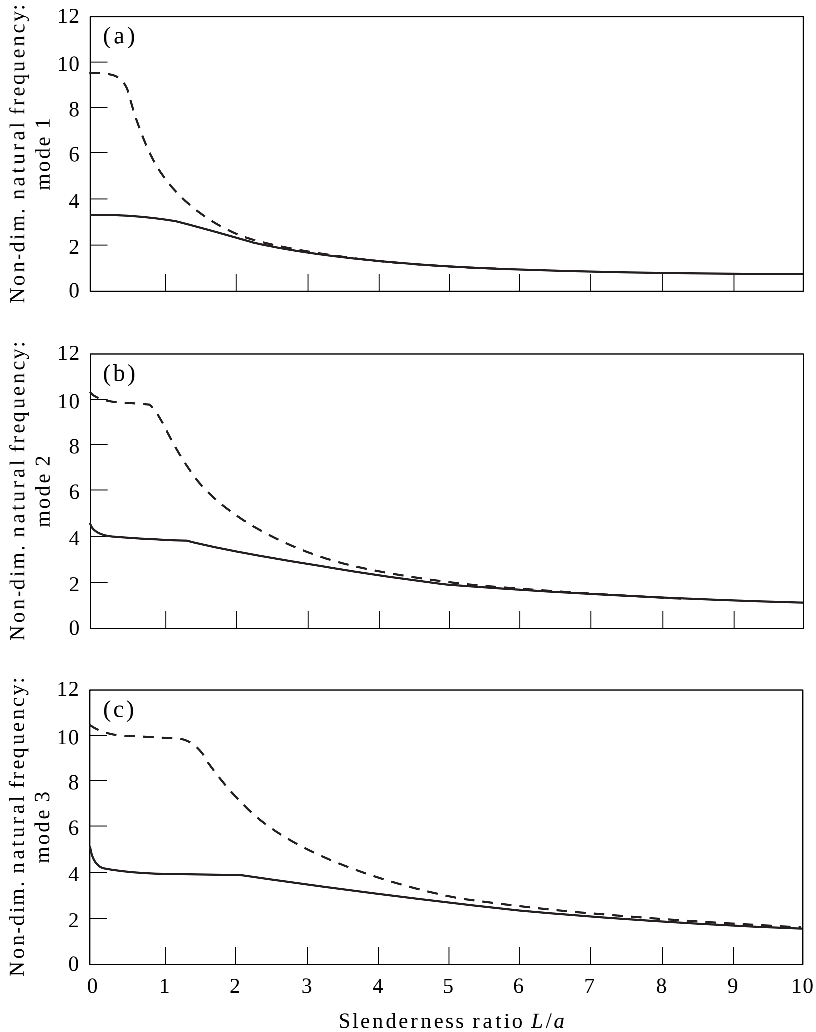

Figure 1. Variation of natural frequency with the slenderness ratio L/a for (a) mode 1, (b) mode 2, (c) mode 3: ——, α_2 computed using first root of equation (6); – –, second root of equation (6).

of $L/a < 2$ or so, the second mode here does not predict frequencies well. As $L/a \to 0$, this frequency should approach zero and not either value depicted in Figure 1(b). This discrepancy can be attributed to the fact that the theory used here is one-dimensional and cannot predict higher modes well for extremely short rods. The relatively flat portions of the higher two modes occur at $\omega/\omega^* \simeq 4$, whereas in reference [7] they occur near $\omega/\omega^* \simeq 3$. The plots indicate that for $L/a > 4$, the present results are insensitive to the manner in which α_2 is computed. However, for $L/a < 4$, reasonably good results may be obtained by using the first non-zero root β of equation (6) whereas if the second root is chosen, the results are somewhat poor.

5. CONCLUSIONS

A detailed analysis of extensional vibrations of an elastic rod has been presented using a theory which depends only on one independent spatial variable z. This theory has the simplicity of being one-dimensional and yet it can capture important three-dimensional effects through the kinematical variable δ and the associated equation of motion (1b). It is capable of generating analytical results for both infinite and finite rods of arbitrary section. Results for a circular section have been presented in order to compare our results with those of Rumerman and Raynor [7]. The reader is reminded that the three-dimensional theory of elasticity can analytically only handle an infinite rod of simple (e.g., circular) cross-section. For other rod cross-sections, the present theory is an attractive alternative and is a powerful tool for analyzing extensional vibrations. Furthermore, for finite rods, exact three-dimensional solutions are intractable for any cross-section and once again the present theory may be used to predict frequencies and mode shapes.

ACKNOWLEDGMENT

This work was partially supported by the NSF grant CMS9713453 to the Virginia Polytechnic Institute and State University.

REFERENCES

1. A. E. Green, N. Laws and P. M. Naghdi 1967 *Archive for Rational Mechanics and Analysis* **25**, 285–298. A linear theory of straight elastic rods.
2. A. E. Green, P. M. Naghdi and M. L. Wenner 1974 *Proceedings of the Royal Society of London* **A337**, 451–483. On the theory of rods. I. Derivations from the three-dimensional equations.
3. A. E. Green, P. M. Naghdi and M. L. Wenner 1974 *Proceedings of the Royal Society of London* **A337**, 485–507. On the theory of rods. II. Developments by direct approach.
4. R. D. Mindlin and G. Herrmann 1952 *Proceedings of the First U.S. National Congress of Applied Mechanics*, 187–191. A one-dimensional theory of compressional waves in an elastic rod.
5. S. Krishnaswamy and R. C. Batra 1998 *Mathematics and Mechanics of Solids*, to appear. On extensional oscillations and waves in elastic rods.
6. K. F. Graff 1975 *Wave Motion in Elastic Solids*. Ohio State University Press.
7. M. Rumerman and S. Raynor 1971 *Journal of Sound and Vibration* **15**, 529–543. Natural frequencies of finite circular cylinders in axially symmetric longitudinal vibration.
8. J. R. Hutchinson 1972 *Journal of the Acoustical Society of America* **51**, 233–240. Axisymmetric vibrations of a free finite length rod.
9. J. R. Hutchinson 1980 *Journal of Applied Mechanics* **47**, 901–907. Vibrations of solid cylinders.
10. S. N. Rasband 1975 *Journal of the Acoustical Society of America* **57**, 899–905. Resonant vibrations of free cylinders and disks.
11. H. D. McNiven and D. C. Perry 1962 *Journal of the Acoustical Society of America* **34**, 433–437. Axially symmetric waves in finite, elastic rods.
12. G. W. McMahon 1964 *Journal of the Acoustical Society of America* **36**, 85–92. Experimental study of the vibration of solid, isotropic elastic cylinders.
13. M. Levinson and D. W. Cooke 1982 *Journal of Sound and Vibration* **84**, 319–326. On the two frequency spectra of Timoshenko beams.

14. O. M. O'Reilly and J. S. Turcotte 1996 *Journal of Sound and Vibration* **198**, 517–521. Another mode of vibration in a Timoshenko beam.
15. C. Dolph 1954 *Quarterly of Applied Mathematics* **12**, 175–187. On the Timoshenko theory of transverse beam vibrations.
16. S. Krishnaswamy and R. C. Batra 1998 *Mathematics and Mechanics of Solids* (in press). Addendum to "On extensional oscillations and waves in elastic rods".