Journal of Sound and Vibration (1998) **215**(3), 587–591
Article No. sv981627

DAMAGE INDEX ALGORITHM FOR A CIRCULAR CYLINDRICAL SHELL*

M. G. Srinivasan and C. A. Kot
Reactor Engineering Division, Argonne National Laboratory, Argonne, IL 60439, *U.S.A.*

(*Received* 13 *February* 1998, *and in final form* 9 *March* 1998)

1. INTRODUCTION

Among the different techniques proposed for damage detection using changes in measured modal parameters, the damage index method was found to be the most effective in an experimental investigation on comparative evaluation of these techniques [1]. The damage index algorithm is based on the observation that the change in modal strain energy of one or more modes was a sensitive indicator of damage. An expression for the index was first developed for linear elastic beam structures [2] and subsequently for plate-type structures [3]. The only modal parameter that is required for using this technique is the mode shape. The method requires only that the mode shapes be normalized consistently, but does not require mass normalized modes. This makes it possible to use this method if ambient excitation tests were used for modal parameter identification.

For beams and plates, only one degree of freedom is required for expressing the damage index. This is the displacement component normal to the beam's neutral plane or the plate's surface. This fact is quite significant because the test measurements are dependent on which degrees of freedom need to be instrumented and recorded. Having to measure only one component greatly simplifies the modal testing aspect. In the case of cylindrical shells, the general expression for the strain energy is a function of all three components of the displacement of the mid-surface. These are the longitudinal, circumferential and radial displacement components. For the most general situation, these three components would have to be measured in tests making the testing effort rather expensive.

However, for thin shells with a length much greater than the radius, certain approximations could be made. Previous work by others on shell theories has justified assuming the hoop strain and shear strain at mid-surface to be zero. This assumption results in relating some of the derivatives of one displacement component to that of another. Furthermore, the assumed form for the mode shapes, previously developed by other investigators, enables expressing all the three displacement components in terms of a single function of the axial co-ordinate. Thus, it is possible to derive an expression for the damage index that requires the measurement of only the radial component of shell vibrations.

2. DERIVATION OF DAMAGE INDEX FOR CYLINDRICAL SHELL

Consider the free vibrations of a cylindrical shell of mean radius R, uniform thickness h, and length l. It is assumed the shell is thin, i.e., h/R is small. The elastic modulus and the Poisson ratio are denoted E and ν, respectively. In the cylindrical co-ordinate system, the displacement components are given by (u, v, w), in the (x, θ, r) directions, respectively.

*This work has been created by the University of Chicago as Operator of Argonne National Laboratory ("Argonne") and is managed by the U.S. Department of Energy under Contract No. W-31-109-ENG-38.

0022–460X/98/330587 + 05 $30.00/0

An expression for strain energy of the cylindrical shell is given by [4]

$$U = \int_0^l \int_0^{2\pi} \frac{Eh}{2(1-\nu^2)} \times \left[\begin{array}{l} u_{,x}^2 + \frac{1}{R^2}(\upsilon_{,\theta} + w)^2 + \frac{2\nu}{R} u_{,x}(\upsilon_{,\theta} + w) + \frac{1}{2}(1-\nu)\left(\upsilon_{,x} + \frac{1}{R} u_{,\theta}\right)^2 \\ + \frac{h^2}{12} \left\{ \begin{array}{l} w_{,xx}^2 + \frac{1}{R^4}(w_{,\theta\theta} - \upsilon_{,\theta})^2 + \frac{2\nu}{R^2} w_{,xx}(w_{,\theta\theta} - \upsilon_{,\theta}) \\ + \frac{2(1-\nu)}{R^2}(w_{,x\theta} - \upsilon_{,x})^2 \end{array} \right\} \end{array} \right] R \, \mathrm{d}\theta \, dx,$$

where the subscript $,x$ denotes partial differentiation with respect to x and so on. This expression for strain energy is derived assuming σ_z, ε_{xr} and $\varepsilon_{\theta r}$ are zero. To further simplify the problem, it is assumed, following Sharma and Johns [5], that the hoop strain, ε_θ, and shear strain, $\varepsilon_{\theta x}$, at the mid-surface are also zero. Then the above expression for strain energy reduces to

$$U = \int_0^1 \int_0^{2\pi} \frac{Eh}{2(1-\nu^2)} \times \left[u_{,x}^2 + \frac{h^2}{12} \left\{ \begin{array}{l} w_{,xx}^2 + \frac{1}{R^4}(w_{,\theta\theta} - \upsilon_{,\theta})^2 \\ + \frac{2\nu}{R^2} w_{,xx}(w_{,\theta\theta} - \upsilon_{,\theta}) + \frac{2(1-\nu)}{R^2}(w_{,x\theta} - v_{,x})^2 \end{array} \right\} \right] R \, \mathrm{d}\theta \, \mathrm{d}x. \quad (1)$$

The modes of free vibration of the shell are characterized by the number of circumferential waves, i, and the number of axial half-waves, j. The mode shape associated with any particular values of i ($i = 2, 3, 4, \ldots$) and j ($j = 1, 2, 3, \ldots$) is characterized by the following midsurface deformations [5, 6]:

$$u_{ij} = -\frac{1}{i^2} R\phi_j' \cos i\theta \cos \omega_{ij} t, \qquad v_{ij} = -\frac{1}{i} \phi_j \sin i\theta \cos \omega_{ij} \cos \omega_{ij} t,$$

$$w_{ij} = \phi_j \cos i\theta \cos \omega_{ij} t, \quad (2)$$

where ϕ_j is a function only of the axial co-ordinate, x, and is associated with the jth axial half-wave, and ω_{ij} is a natural frequency associated with i and j. The prime denotes the first derivative. We note that for the mode shapes defined by equation (2), it is necessary only to measure the radial component of displacement from which the other two components could be derived. From equations (1) and (2), we could obtain the modal strain energy associated with (i, j) as

$$U_{ij} = \int_0^l \int_0^{2\pi} \frac{D}{2} \left[\begin{array}{l} \left(1 + \frac{12}{i^4}\frac{R^2}{h^2}\right) w_{ij,xx}^2 + \left(\frac{1-i^2}{R^4}\right) w_{ij}^2 \\ + \frac{2\nu}{R^2}(1-i^2) w_{ij} w_{ij,xx} + \frac{2(1-\nu)}{R^2}\frac{(i^2-1)^2}{i^4}\omega_{ij,x\theta}^2 \end{array} \right] R \, \mathrm{d}\theta \, \mathrm{d}x, \quad (3)$$

where $i = 2, 3, 4, \ldots, j = 1, 2, 3, \ldots$, and $D = Eh^3/12(1-\nu^2)$.

We subdivide the shell into a number of small segments to facilitate localization of damage. Let N_α be the number of axial segments and N_θ the number of circumferential segments. We consider a segment defined by $x_n \leqslant x \leqslant x_{n+1}$, $\theta_m \leqslant \theta \leqslant \theta_{m+1}$, where $n = 1, 2, \ldots, N_\alpha$, $m = 1, 2, \ldots N_\theta$ and

$$\theta_m = (2\pi/N_\theta)(m-1), \qquad \theta_{m+1} = (2\pi/N_\theta)m, \qquad x_n = (l/N_\alpha)(n-1), \qquad x_{n+1} = (l/N_\alpha)n.$$

The modal strain energy in this segment is given by

$$U_{ijmn} = \frac{D_{mn}}{2}\int_{x_n}^{x_{n+1}}\int_{\theta_m}^{\theta_{m+1}} \left[\begin{array}{l} \left(1 + \dfrac{12}{i^4}\dfrac{R^2}{h^2}\right) w_{ij,xx}^2 + \left(\dfrac{1-i^2}{R^4}\right) w_{ij}^2 \\ + \dfrac{2v}{R^2}(1-i^2) w_{ij} w_{ij,xx} + \dfrac{2(1-v)}{R^2}\dfrac{(i^2-1)^2}{i^4}\, w_{ij,x\theta}^2 \end{array}\right] R \,\mathrm{d}\theta\,\mathrm{d}x, \tag{4}$$

where D_{mn} is the flexural rigidity of the shell segment. Quantities pertaining to this segment are indicated by the subscripts mn. The subscripts ij denote quantities pertaining to the mode characterized by i and j.

In view of equation (4), the modal strain energy for the entire shell could be written as

$$U_{ij} = \sum_{m=1}^{N_\theta}\sum_{n=1}^{N_\alpha} U_{ijmn}. \tag{5}$$

We define F_{ijmn} as the ratio of U_{ijmn} to U_{ij}. It is assumed that the shell is subdivided into sufficiently large number of segments so that $F_{ijmn} \ll 1$. By definition,

$$\sum_{m=1}^{N_\theta}\sum_{n=1}^{N_\alpha} F_{ijmn} = 1 \tag{6}$$

All the above expressions pertain to the undamaged shell. Similar expressions for the damaged shell could be written in terms of the corresponding mode shape defined by w_{ij}^*. The expressions for the damaged shell are indicated by the superscript asterisk. Following the method proposed in reference [1] for deriving the damage index for beams, we define the damage index for the segment mn as

$$\beta_{ijmn} = \frac{f_{ijmn}^*}{f_{ijmn}}, \tag{7}$$

where

$$f_{ijmn} = \left\{ \frac{\left(\displaystyle\int_{x_n}^{x_{n+1}}\int_{\theta_m}^{\theta_{m+1}} \Omega_{ij}\,\mathrm{d}\theta\,\mathrm{d}x + \int_0^l\int_0^{2\pi} \Omega_{ij}\,\mathrm{d}\theta\,\mathrm{d}x\right)}{\displaystyle\int_0^l\int_0^{2\pi} \Omega_{ij}\,\mathrm{d}\theta\,\mathrm{d}x} \right\}, \tag{8}$$

and

$$f^*_{ijmn} = \left\{ \frac{\left(\int_{x_n}^{x_{n+1}} \int_{\theta_m}^{\theta_{m+1}} \Omega^*_{ij} \, \mathrm{d}\theta \, \mathrm{d}x + \int_0^l \int_0^{2\pi} \Omega^*_{ij} \, \mathrm{d}\theta \, \mathrm{d}x \right)}{\int_0^l \int_0^{2\pi} \Omega^*_{ij} \, \mathrm{d}\theta \, \mathrm{d}x} \right\}, \tag{9}$$

with

$$\Omega_{ij} = \left(1 + \frac{12}{i^4}\frac{R^2}{h^2}\right) w^2_{ij,xx} + \left(\frac{1 - i^2}{R^4}\right) w^2_{ij} + \frac{2v}{R^2}(1 - i^2) w_{ij} w_{ij,xx} + \frac{2(1 - v)}{R^2}\frac{(i^2 - 1)^2}{i^4} w^2_{ij,x\theta}, \tag{10}$$

and

$$\Omega^*_{ij} = \left(1 + \frac{12}{i^4}\frac{R^2}{h^2}\right) w^{*2}_{ij,xx} + \left(\frac{1 - i^2}{R^4}\right) w^{*2}_{ij} + \frac{2v}{R^2}(1 - i^2) w^*_{ij} w^*_{ij,xx} + \frac{2(1 - v)}{R^2}\frac{(i^2 - 1)^2}{i^4} w^{*2}_{ij,x\theta}, \tag{11}$$

the asterisk denoting quantity pertaining to damaged state.

If a single mode is used for localizing damage, (i.e., for given i and j), then the damage index, β_{ijmn}, is computed for each of the $(N_\alpha \times N_\theta)$ number of segments. If more than one mode is used, then we define the index, β_{mn}, as

$$\beta_{mn} = \frac{\sum_i \sum_j f^*_{ijmn}}{\sum_i \sum_j f_{ijmn}}, \tag{12}$$

and β_{mn} is computed for each of the segments. In either case, it is assumed that the distribution of the index over the segments of the shell is represented by a normal distribution. Then the normalized damage index is defined, following reference [1], as

$$z = \frac{(\beta - \overline{\beta})}{\sigma}, \tag{13}$$

where β stands for either β_{ijmn} or β_{mn}, the overbar denotes the mean and the σ denotes the standard deviation. A statistical decision making procedure similar to that described in reference [2] could be employed to determine the threshold value of z that indicates damage.

3. DAMAGE DETECTION ALGORITHM

The inputs to the algorithm are as follows:

(1) *The shell geometry*. The radius, length and thickness of the shell and a table of co-ordinates of the reference and response measurement points, and their connectivity as defined for the tests. During the performance of the vibration tests, the measurement points are usually regularly spaced, with $(M_\alpha + 1)$ parallel to the axis and M_θ along the circumference. Relative to the grid for damage detection, $M_\alpha \ll N_\alpha$ and $M_\theta \ll N_\theta$.

(2) *The files containing the mass-normalized mode shapes*, obtained as a result of the experimental modal analysis. Since only the radial component of motion is measured, each mode shape is given as a table of values of the radial component at each of the

measurement points. Two sets of files are required, one from the baseline tests (before damage occurred) and the other from the post-damage tests.

(3) *The Poisson ratio for the shell material*, the only elastic property that explicitly occurs in the expression for the damage index, the elastic modulus having been cancelled out.

Most of the commercially available experimental modal analysis software allows for writing the modal results in ASCII tables. Therefore, the above input information is assumed to be available in text files, readable across platforms.

The first task is to perform an interpolation of any given mode shape to obtain mode shape vectors at each of the $(N_\alpha + 1)$ points along the longitudinal direction and N_θ points along the circumference, given the experimental mode shape at $(M_\alpha + 1)$ points along the longitudinal direction and M_θ points along the circumference.

The mode shapes are functions of x and θ. However, from equation (2), it may be seen the mode shape could be expressed in the form $f_j(x)\cos(i\theta + \alpha)$, where i, j and α are constants. The mathematical expression for $f_j(x)$ is same as that for the mode shapes of beams of the same boundary conditions [6]. Since it was found, in the investigations reported in reference [1], that a cubic polynomial fit gave a better means for interpolation than a cubic spline, the former is used for interpolation.

Thus, for a given mode, the mode shape is expressed locally as a product of a cubic polynomial and a cosine and the mode shape data is fitted to the above expression from which interpolated values of the mode shape and its derivatives, involved in equations (10) and (11), are obtained and tabulated at each of the damage-detection nodes.

This tabulated data is then used to compute the damage index for each segment of the shell using equations (7)–(13).

4. CONCLUDING REMARKS

The validity of the proposed method remains to be tested with appropriate experiments. Lab-scale circular cylindrical shells could be modal-tested both before and after a known flaw is introduced. The results of the experimental modal analysis will provide the input for the algorithm. Mathematical analysis packages such as MATLAB could be used to implement the algorithm. If the location of the flaw is correctly identified, additional tests with different flaw sizes could be performed to determine the sensitivity of the proposed technique.

ACKNOWLEDGMENT

This work was sponsored by the U.S. Department of Energy, under contract W-31-109-ENG-38.

REFERENCES

1. C. FARRAR and D. JAUREGUI 1996 *Report*, *Las Alamos National Laboratory*, *LA*-13074-*MS*. Damage detection algorithms applied to experimental and numerical modal data from the I-40 bridge.
2. N. STUBBS, J. KIM and C. R. FARRAR 1995 *Proceedings of the* 13*th International Modal Analysis Conference*, *Nashville*, *TN* **I**, 210–218. Field verification of a nondestructive damage localization and severity estimation algorithm.
3. P. CORNWELL, S. W. DOEBLING and C. R. FARRAR 1997 *Proceedings of the* 15*th International Modal Analysis Conference*, *Orlando*, *FL* **II**, 1312–1318. Application of the strain energy damage detection method to plate–like structures.
4. G. B. WARBURTON 1976 *The Dynamical Behaviour of Structures*. Oxford: Pergamon Press.
5. C. B. SHARMA and D. J. JOHNS 1971 *Journal of Sound and Vibration* **14**, 459–474. Vibration characteristics of a clamped-free and clamped-ring-stiffened circular cylindrical shell.
6. R. D. BLEVINS 1979 *Formulas for Natural Frequency and Mode Shape*. New York: Van Nostrand Reinhold.