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VIBROACOUSTICS OF CYLINDRICAL PIPES: INTERNAL RADIATION MODAL COUPLING

F. GAUTIER AND N. TAHANI
Institut d'Acoustique et de Mécanique (*Laboratoire d'Acoustique de l'Université du Maine*, *UMR CNRS* 6613), *Av. O. Messiaen*, 72085 *Le Mans Cedex* 9, *France*

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The study of the vibroacoustic characteristics of wave guides requires an appropriate analytical description of the acoustic fields inside pipes and cavities, taking into account wall vibrations. A new method for determining the internal acoustic field, called "Separate Modal Expansions" method, is presented and applied to a cylindrical shell. The method uses two modal bases, the one corresponding to the *in vacuo* shell modes and the one consisting of the acoustic modes of the two-dimensional Neumann transverse problem. The expressions for the internal radiation impedances, which completely describe the internal vibroacoustic problem, are obtained for various sets of acoustic boundary conditions imposed on the ends of the waveguide. A comparison with the internal method (using a modal formulation), frequently used for this type of problem, demonstrates the benefits of the proposed approach.

1. INTRODUCTION

The study of the vibroacoustic behaviour of cylindrical pipes has a wide variety of applications such as muffler acoustics, industrial pipes, wind musical instruments, or aeronautical shells. All these systems consist of an elastic structure containing an internal fluid surrounded by an external fluid. Depending on the application, the fluid can be either light or heavy, and at rest or not with respect to the structure. Mechanical forces and internal or external acoustical sources usually act on such a system, thus creating a coupling between the acoustic and vibration fields. This class of problem is usually solved using the classical integral method. First, the vibratory response is expressed as an expansion over the *in vacuo* structural modes. Then, the acoustic pressure in the internal and external media is written using the Helmholtz–Huygens surface integral. Finally, the momentum equation of the structure is projected on each of its modes to obtain the coupled governing equations, where the generalized excitations are written on the right hand side. Such a method was applied to internal [1–3] and external [4–9] vibroacoustic problems.

For the internal vibroacoustic problem, a Green's function satisfying Neumann boundary conditions is often chosen, leading to the expressions for the coupling coefficients or internal radiation impedances. In the case of a cylindrical cavity, these are written as sums of axial and radial acoustic indexes in which each elementary term describes the coupling between an acoustic mode and a structural mode [10]. The use of another Green's function, namely Green's function for the infinite tube is more appropriate and allows one to simplify these expressions by eliminating the sums over the axial indexes which are then implicitly contained in the equations. This Green's function is used by Ouelaa *et al*. [11]. They study the radiation of a finite cylindrical shell interacting with

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internal and external fluids which are separated by two semi-infinite rigid cylindrical baffles placed at both ends of the shell.

In order to further simplify the expressions for the internal radiation impedance, a new method, called "Separate Modal Expansions" (SME) method, is presented here [12, 13]. In this method, the internal acoustic field is described using two modal expansions, the *in vacuo* structural one and the one consisting of the eigenfunctions of the two-dimensional Neumann transverse problem. Thus, the expressions of the radiation impedance are simplified because the sum over the radial index does not appear anymore. This new method is used to study the interactions between a thin cylindrical elastic shell of finite length, and an internal fluid at rest, for various sets of acoustic boundary conditions applied at the ends of the waveguide.

Such a study has its origins in a musical acoustic problem whose aim is to quantify the effects of the vibrations of the body of a wind music instrument on the sound it emits. Several vibroacoustic interactions take place in such a system [14]: structure/internal fluid interaction, structure/external fluid interaction and inter-modal coupling due to the radiation of the open end of the waveguide. In the present study, the interaction between the waveguide and the internal fluid is solely considered, and results are integrated into a global model describing the vibroacoustic behaviour of a "simplified" wind music instrument [15]. In section 2, after presenting the internal vibroacoustic problem, the SME method is described and results in terms of internal radiation impedance are shown. The classical integral method is used in section 3 and the same results as with the SME method are obtained. Finally, both methods are compared in section 4, thus allowing the benefits of the SME method over the classical one to be demonstrated.

2. THE SEPARATE MODAL EXPANSIONS METHOD (SME)

2.1. FORMULATION OF THE PROBLEM

The problem under consideration is schematically depicted in Figure 1: a cylindrical shell of length ℓ, mean radius a, and thickness h, is characterized by its ring angular frequency ω_a and its density ρ_s. The cylinder is filled with a fluid characterized by its density ρ and the sound speed c. The surfaces S_0, S, S_ℓ correspond to the co-ordinate $z = 0$, the lateral surface of the cylinder ($r = a$), and the co-ordinate $z = \ell$, respectively. Let D_i be the fluid domain inside the cylinder, delimited by S_0, S, and S_ℓ, and let $\mathbf{n}$ be the unitary vector normal to the cylinder in the outward direction.

The shell motion is described by the displacement vector $\mathbf{X}$, whose components u, v, and w denote the longitudinal, circumferential, and radial displacements, respectively. The dynamic behaviour of the shell is described by the Donnell or Flügge operator denoted $\mathscr{L}$ [16]. The shell is assumed to be simply supported at both ends in order to simplify the *in vacuo* normal modes. The acoustic pressure and the particle velocities normal to the surfaces S_0, S, S_ℓ are noted p, v_{S0}, v_S and $v_{S\ell}$, respectively. The influence of the external

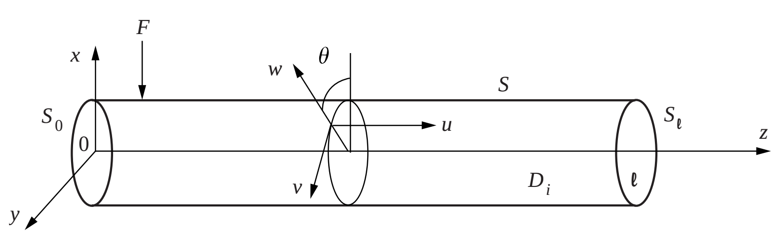

Figure 1. Notations for the vibrating cylinder.

fluid ($r > a$) is neglected since the present study aims at characterizing the influence of the internal fluid on the vibrations of the cylinder. The shell is excited by a point force $\mathbf{F}(\mathbf{r}) = \mathbf{F}_0\delta(\mathbf{r} - \mathbf{r}_0)\,\mathrm{e}^{-j\omega t}$. Several sets of acoustic boundary conditions on the surfaces S_0 and S_ℓ, denoted by $\mathscr{C}$, are considered: $\mathscr{C} = (NN)$ for Neumann boundary conditions on surfaces S_0 and S_ℓ; $\mathscr{C} = (DD)$ for Dirichlet boundary conditions on surfaces S_0 and S_ℓ; $\mathscr{C} = (ND)$ for Neumann boundary conditions on the surface S_0 and Dirichlet boundary conditions on the surface S_ℓ; $\mathscr{C} = (II)$ for conditions of continuity between the cylinder at S_0, or S_ℓ, and a rigid semi-infinite waveguide in $z < 0$, or $z > \ell$, respectively.

In the frequency domain ($\mathrm{e}^{-j\omega t}$), the governing equations of the problem can be written ($k = \omega/c$) as

$$(\Delta + k^2)p(\mathbf{r}) = 0 \quad \text{for} \quad \mathbf{r}\in D_i, \tag{1}$$

$$\text{acoustic boundary conditions } \mathscr{C} \quad \text{for} \quad \mathbf{r}\in S_0 \quad \text{and} \quad \mathbf{r}\in S_\ell, \tag{2}$$

$$v_S(\mathbf{r}) = \dot{w}(\mathbf{r}) \quad \text{for} \quad \mathbf{r}\in S, \tag{3}$$

$$\rho_s h(\omega_a^2\mathscr{L} + \omega^2)\mathbf{X}(\mathbf{r}) = -p(\mathbf{r})\mathbf{n} - \mathbf{F}(\mathbf{r}) \quad \text{for} \quad \mathbf{r}\in S, \tag{4}$$

$$\text{simply supported boundary conditions for the shell at } z = 0, \ell. \tag{5}$$

The solution of the acoustic problem, equations (1) and (2), coupled to the mechanical problem, equations (4) and (5), through the continuity equation, (3), can be obtained using the classical integral method [17, 18]. A new approach is presented here, based on the procedure described by Bruneau and Bruneau [12]. The main feature of the method is that it leads to an expression for the internal acoustic field which contains two separate modal expansions, one involving the *in vacuo* shell modes and the other one involving the acoustic modes of the two-dimensional internal transverse Neumann problem. It is shown that such a description of the internal acoustic field is particularly well suited for the problem that we are interested in here.

2.2. EQUATION OF SHELL MOTION

The shell displacement vector $\mathbf{X}$ can be expanded making use of the *in vacuo* normal modes of the simply supported shell:

$$\mathbf{X} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \sum_\mu A_\mu \mathbf{\Phi}_\mu. \tag{6}$$

The eigenvectors, $\mathbf{\Phi}_\mu$, are the solutions of the homogeneous problem

$$\rho_s h(\omega_a^2\mathscr{L} + \omega_a^2)\mathbf{\Phi}_\mu = 0, \tag{7}$$

where ω_μ is the resonance angular frequency of the mechanical mode μ. The parameter μ states for a set of four integers

$$\mu = (m, q, s, j), \tag{8}$$

where m is the circumferential index, q the axial index, s the symmetry index, and j the type of mode index (bending for $j = 1$, extension–compression for $j = 2$, and twisting for $j = 3$). In the classical form, the modes $\mathbf{\Phi}_\mu$ are written

$$\mathbf{\Phi}_\mu = \begin{bmatrix} U_\mu \cos(q\pi z/\ell)\sin(m\theta + s\pi/2) \\ V_\mu \sin(q\pi z/\ell)\cos(m\theta + s\pi/2) \\ \sin(q\pi z/\ell)\sin(m\theta + s\pi/2) \end{bmatrix}. \tag{9}$$

They satisfy the orthogonality relationship

$$\rho_s h\langle \mathbf{\Phi}_\mu, \mathbf{\Phi}_{\mu'}\rangle_S = \rho_s h \sum_{i=1}^{3} \langle \Phi_{i\mu}|\Phi_{i\mu'}\rangle_S = \frac{\rho_s h\pi a\ell}{\varepsilon_m}(U_\mu^2 + V_\mu^2 + 1)\delta_{\mu\mu'} = m_\mu \delta_{\mu\mu'}, \tag{10}$$

where m_μ denotes the modal mass associated to the mode μ, ε_m the Neumann factor ($\varepsilon_m = 1$ if $m = 0$, $\varepsilon = 2$ if $m > 0$), δ the Kronecker notation, $\Phi_{i\mu}$ ($i = 1, 2, 3$) the components of $\mathbf{\Phi}_\mu$, and where the inner product $\langle \cdot | \cdot \rangle_S$ is defined by $\langle g|h\rangle_S = \int_S gh^* \,\mathrm{d}s$. The relation obtained by substituting equation (6) into equation (4) describes the shell vibrations under the acoustic and mechanical loads; its inner product with the eigenfunction $\mathbf{\Phi}_\mu$, leads to

$$m_\mu A_\mu(-\omega^2 + \omega_\mu^2) = F_\mu + P_\mu, \tag{11}$$

where the generalized force

$$F_\mu = \langle \mathbf{F}, \mathbf{\Phi}_\mu\rangle_S = \sum_{i=1}^{3} \langle F_i|\Phi_{i\mu}\rangle_S \tag{12}$$

and generalized pressure

$$P_\mu = \langle p \cdot \mathbf{n}, \mathbf{\Phi}_\mu\rangle_S = \langle p|\Phi_{3\mu}\rangle_S \tag{13}$$

are the inner product of the given point force $\mathbf{F}$ and of the internal wall pressure $p(r = a)$ with the shell mode $\mathbf{\Phi}_\mu$, respectively.

2.3. THE INTERNAL ACOUSTIC PRESSURE FIELD

Solving equation (11) requires calculating the internal acoustic pressure field. The method employed here consists of constructing this field component by component. Using the shell displacement field equation, equation (6), and the continuity equation, equation (3), the radial particle velocity on the surface S can be written as a function of the radial component $\Phi_{3\mu}$ of $\mathbf{\Phi}_\mu$:

$$v_r(r = a, \theta, z) = v_S(\theta, z) = -\mathrm{j}\omega \sum_\mu A_\mu \Phi_{3\mu}(\theta, z). \tag{14}$$

This expression for the radial particle velocity can be extended to the fluid domain $r < a$ as follows:

$$v_r(r, \theta, z) = \sum_\mu v_{3\mu}(r)\Phi_{3\mu}(\theta, z) + \mathscr{V}(r, \theta, z), \tag{15}$$

where, using the continuity relation, equation (3), one gets

$$v_{3\mu}(r = a) = -\mathrm{j}\omega A_\mu \quad \text{and} \quad \mathscr{V}(r = a) = 0. \tag{16}$$

An acoustic velocity potential is defined from the relations

$$\mathbf{v} = -\nabla Q \quad \text{and} \quad p = -\mathrm{j}\omega\rho Q. \tag{17}$$

Making use of the relationship $v_r = -\partial_r Q$, the potential Q can be written as

$$Q(r, \theta, z) = \sum_\mu Q_{3\mu}(r)\Phi_{3\mu}(\theta, z) + \sum_\alpha Q_{r\alpha}(r) Q_{\theta z\alpha}(\theta, z), \tag{18}$$

with

$$\partial_r Q_{3\mu}(r=a) = +\mathrm{j}\omega A_\mu \quad \text{and} \quad \partial_r Q_{r\alpha}(r=a) = 0, \tag{19}$$

and where the last sum being related to the factor $\mathscr{V}$ in equation (15). This velocity potential must satisfy Helmholtz homogeneous equation

$$(\Delta + k^2)Q(\mathbf{r}) = 0, \quad \mathbf{r}\in D_i. \tag{20}$$

Considering the expression for Q, equation 18, the following two conditions for Q are sufficient to satisfy equation (20):

$$\forall\mu, \quad \frac{\Delta_r Q_{3\mu}(r)}{Q_{3\mu}(r)} + \frac{1}{r^2}\frac{\partial_\theta^2 \Phi_{3\mu}(\theta, z)}{\Phi_{3\mu}(\theta, z)} + \frac{\partial_z^2 \Phi_{3\mu}(\theta, z)}{\Phi_{3\mu}(\theta, z)} + k^2 = 0, \tag{21}$$

$$\forall\alpha, \quad \frac{\Delta_r Q_{r\alpha}(r)}{Q_{r\alpha}(r)} + \frac{1}{r^2}\frac{\partial_\theta^2 Q_{\theta z\alpha}(\theta, z)}{Q_{\theta z\alpha}(\theta, z)} + \frac{\partial_z^2 Q_{\theta z\alpha}(\theta, z)}{Q_{\theta z\alpha}(\theta, z)} + k^2 = 0, \tag{22}$$

where

$$\Delta_r \cdot = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\cdot\right).$$

Conclusions can be drawn as follows:

(i) equation (22) enables one to determine the general expression for $Q_{r\alpha}(r)Q_{\theta z\alpha}(\theta, z)$:

$$Q_{r\alpha}(r)Q_{\theta z\alpha}(\theta, z) = \frac{1}{-\mathrm{j}\rho\omega}\,[B_\alpha^+\,\mathrm{e}^{\mathrm{j}k_{mn}z} + B_\alpha^-\,e^{\mathrm{j}k_{mn}(\ell - z)}]\Psi_\alpha(r, \theta), \tag{23}$$

where B_α^+ and B_α^- are to be determined from equation (2). The functions Ψ_α denote the orthonormal eigenfunctions of the transverse two-dimensional Neumann problem, the subscript α being the triplet of integers

$$\alpha = (m, n, s), \tag{24}$$

where m is the circonferential index ($m > 0$), n the radial index ($n > 0$), and s the index of symmetry ($s = 0, 1$):

$$\Psi_\alpha = \mathrm{J}_m(k_{Wmn}r)\sin(m\theta + s\pi/2)/\Lambda_\alpha, \tag{25}$$

where

$$\Lambda_\alpha^2 = \frac{\pi a^2}{\varepsilon_m}(1 - \gamma_{mn}^2)\mathrm{J}_m^2(k_{Wmn}a), \quad \text{with} \quad \gamma_{mn}^2 = \begin{cases} 0 & \text{if } m = 0 \\ m^2/(k_{Wmn}a)^2 & \text{if } m > 0 \end{cases}, \tag{26}$$

the eigenvalues k_{mn} being given by $k_{mn}^2 = k^2 - k_{Wmn}^2$ $(0 \leqslant \arg(k_{mn}) \leqslant \pi/2)$, where $\mathrm{J}'_m(k_{Wmn}a) = 0$ $(k_{Wmn} \geqslant 0,\ n \geqslant 0)$.

(ii) From equation (21), the modes $\Phi_{3\mu}$ must satisfy the relationships (see Appendix):

$$\partial_\theta^2 \Phi_{3\mu}(\theta, z)/\Phi_{3\mu}(\theta, z) = C_1 \quad \text{and} \quad \partial_z^2 \Phi_{3\mu}(\theta, z)/\Phi_{3\mu}(\theta, z) = C_2, \tag{27}$$

where C_1 and C_2 are constants. These conditions show the limits of the SME method. The solution for the particle velocity in the fluid domain, equation (15), is valid only if the radial component $\Phi_{3\mu}(\theta, z)$ satisfy equation (27), which can therefore be called equations of admissibility for shell modes. For simply supported boundary conditions, these admissibility conditions can be written as

$$\partial_\theta^2 \Phi_{3\mu}(\theta, z)/\Phi_{3\mu}(\theta, z) = -m^2 \quad \text{and} \quad \partial_z^2 \Phi_{3\mu}(\theta, z)/\Phi_{3\mu}(\theta, z) = -(q\pi/\ell)^2. \tag{28}$$

(iii) Considering (ii), equation (21) which characterizes $Q_{3\mu}(r)$ is a Bessel equation and its solution which satisfies the boundary condition, equation (19) can be expressed as

$$Q_{3\mu}(r) = \frac{\mathrm{j}\omega A_\mu \mathrm{J}_m(k_q r)}{k_q \mathrm{J}'_m(k_q a)}, \quad \text{where} \quad k_q^2 = k^2 - (q\pi/\ell)^2. \tag{29}$$

Finally, from equations (18), (23) and (29), the acoustic pressure can be expressed as a sum of two terms

$$p(\mathbf{r}) = p_S^r(\mathbf{r}) + p_{S0\ell}^r(\mathbf{r}), \tag{30}$$

where

$$p_S^r(\mathbf{r}) = \rho c \sum_{\mu=(m,q,s,j)} -\mathrm{j}\omega A_\mu \left[\mathrm{j}\frac{k}{k_q}\frac{\mathrm{J}_m(k_q r)}{\mathrm{J}'_m(k_q a)} \right] \Phi_{3\mu}(\theta, z), \tag{31}$$

and

$$p_{S0\ell}^r(\mathbf{r}) = \sum_{\alpha=(m,n,s)} [B_\alpha^+ \, \mathrm{e}^{\mathrm{j}k_{mn}z} + B_\alpha^- \, \mathrm{e}^{\mathrm{j}k_{mn}(\ell - z)}] \Psi_\alpha(r, \theta). \tag{32}$$

This solution is called a Separate Modal Expansions (SME) solution. The first term, $p_S^r(\mathbf{r})$, describes the internal acoustic field generated by the shell (surface S), taking into account the boundary conditions on the surfaces S_0 and S_ℓ to be specified (see subsection 2.4). This term is expressed here in an adaptive manner using the *in vacuo* shell modes expansion and not using the acoustic eigenfunctions, as would be the case for the classical integral approach. The second term, $p_{s0\ell}^r(\mathbf{r})$ corresponds to the acoustic field generated by the surfaces S_0 adn S_ℓ and is written as a modal expansion function of the acoustic eigenfunctions of the transverse two-dimensional Neumann problem.

2.4. ACOUSTIC BOUNDARY CONDITIONS ON THE SURFACES S_0 AND S_ℓ

The unknowns $B_\alpha^\pm$ of the equation for the acoustic field, equation (30), are determined using the boundary conditions on the surfaces S_0 and S_ℓ. These conditions can be written as an impedance relation:

$$\langle p | \Psi_\alpha \rangle_{S_i} = Z_{i\alpha} \langle v_{Si} | \Psi_\alpha \rangle_{S_i}, \tag{33}$$

where i corresponds to the surface considered ($i = 0$ or $i = \ell$). The impedances $Z_{i\alpha}$ are defined for each set of boundary conditions (see subsection 2.1)

$$\mathscr{C} = (NN), \qquad Z_{0\alpha} = Z_{\ell\alpha} = \infty,$$

$$\mathscr{C} = (DD), \qquad Z_{0\alpha} = Z_{\ell\alpha} = 0,$$

$$\mathscr{C} = (ND), \qquad Z_{0\alpha} = \infty, \quad Z_{\ell\alpha} = 0,$$

$$\mathscr{C} = (II), \qquad Z_{0\alpha} = -Z_{\ell\alpha} = -\rho c k / k_{mn}.$$

Substituting in equation (33) the expressions of the pressure $p(\mathbf{r})$, equation (30) and the axial particle velocity $v_{Si} = -\partial_z Q$ (obtained by combining equation (17) and (30)) gives the set of equations

$$\begin{bmatrix} 1 - \bar{Z}_{0\alpha} & \mathrm{e}^{\mathrm{j}k_{mn}\ell}(1 + \bar{Z}_{0\alpha}) \\ \mathrm{e}^{\mathrm{j}k_{mn}\ell}(1 - \bar{Z}_{\ell\alpha}) & 1 + \bar{Z}_{\ell\alpha} \end{bmatrix} \begin{bmatrix} B_\alpha^+ \\ B_\alpha^- \end{bmatrix} = \begin{bmatrix} Z_{0\alpha} G_{0\alpha} \\ z_{\ell\alpha} G_{\ell\alpha} \end{bmatrix}, \tag{34}$$

where the non-dimensional parameter $\bar{Z}_{i\alpha}$ is

$$\bar{Z}_{i\alpha} = \frac{Z_{i\alpha}}{\rho c}\frac{k_{mn}}{k}, \tag{35}$$

and where the generalized velocities $G_{i\alpha}$ are defined by

$$G_{i\alpha} = \sum_{\mu} -\mathrm{j}\omega A_\mu \frac{q\pi/\ell}{k_q \mathrm{J}'_m(k_q a)} \langle \mathrm{J}_m(k_q r)\sin(m\theta + s\pi/2)|\Psi_\alpha\rangle_{S_i} \begin{cases} 1 & \text{if } i = 0 \\ (-1)^q & \text{if } i = \ell \end{cases}, \tag{36}$$

$$= 2\left[\frac{\pi}{\varepsilon_m(1-\gamma_{mn}^2)}\right]^{1/2} \sum_q \frac{q\pi/\ell}{k_{mn}^2 - (q\pi/\ell)^2}\mathrm{j}\omega A_\mu \begin{cases} 1 & \text{if } i = 0 \\ (-1)^q & \text{if } i = \ell \end{cases}. \tag{37}$$

The solution of equation (34) leads to:

$$B_\alpha^+ = \frac{G_{0\alpha}Z_{0\alpha}/[1+\bar{Z}_{0\alpha}] - G_{\ell\alpha}\,\mathrm{e}^{\mathrm{j}k_{mn}\ell}Z_{\ell\alpha}/[1+\bar{Z}_{\ell\alpha}]}{[1-\bar{Z}_{0\alpha}]/[1+\bar{Z}_{0\alpha}] - \mathrm{e}^{2\mathrm{j}k_{mn}\ell}[1-\bar{Z}_{\ell\alpha}]/[1+\bar{Z}_{\ell\alpha}]}, \tag{38}$$

$$B_\alpha^- = \frac{G_{\ell\alpha}Z_{\ell\alpha}/[1-\bar{Z}_{\ell\alpha}] - G_{0\alpha}\,\mathrm{e}^{\mathrm{j}k_{mn}\ell}Z_{0\alpha}/[1-\bar{Z}_{0\alpha}]}{[1+\bar{Z}_{\ell\alpha}]/[1-\bar{Z}_{\ell\alpha}] - \mathrm{e}^{2\mathrm{j}k_{mn}\ell}[1+\bar{Z}_{0\alpha}]/[1-\bar{Z}_{0\alpha}]}. \tag{39}$$

Substituting this last result in equations (30)–(32) gives the solution of the problem equations (1)–(5). When $\mathscr{C} = (DD)$, the solution of the problem enables some interesting remarks to be made. The terms $B_\alpha^\pm$, equations (38) and (39) vanish and, therefore, the term $p_{S0\ell}^r$ is equal to zero, enabling conclusions on the pressure field p_S^r to be drawn: it corresponds to the internal field generated by the vibrating surface S when Dirichlet boundary conditions are applied on S_0 and S_ℓ. In that case, the continuity of the vibrational and acoustic velocity fields, given by equation (3), occuring at the co-ordinates ($r = a$, $z = 0$) and ($r = a$, $z = \ell$) is satisfied. Indeed, the condition of zero radial velocity required by the simply supported mechanical boundary conditions at $z = 0, \ell$ is compatible with the Dirichlet boundary conditions imposed on S_0 and S_ℓ: if the pressure is uniformly equal to zero on the surface S_i, so is the radial velocity gradient and, therefore, so is the radial acoustic velocity. The generalized velocities, $G_{i\alpha}$ ($i = 0$ and $i = \ell$), can then be interpreted as axial generalized acoustic velocities generated on the surface S_i by the vibrations of the surface S when the acoustic field obeys Dirichlet boundary conditions on S_0 and S_ℓ. For the other sets of acoustic boundary conditions $\mathscr{C} = (NN)$, (ND), (II), the extra term $p_{S0\ell}^r$ does not vanish. For example, if, in order to compare with the classical integral method, one considers the problem (NN), the expression for the internal acoustic field becomes

$$p(\mathbf{r}) = p_S^r + p_{S0\ell}^r \tag{40}$$

where

$$p_{S0\ell}^r = \rho c \sum_{\mu=(m,q,s,j)} (-\mathrm{j}\omega A_\mu)2\mathrm{j}ka\frac{q\pi a}{\ell}\sum_n \frac{\mathrm{J}_m(k_{Wmn}r)}{\mathrm{J}_m(k_{Wmn}a)[1-\gamma_{mn}^2]}F_{mnq}(z)\sin(m\theta + s\pi/2), \tag{41}$$

with

$$F_{mnq}(z) = \frac{1}{[k_{mn}a][(k_{mn}a)^2 - (q\pi a/\ell)^2]}\begin{cases} \sin(k_{mn}(z-\ell/2))/\cos(k_{mn}\ell/2) & q \text{ even} \\ \cos(k_{mn}(z-\ell/2))/\sin(k_{mn}\ell/2) & q \text{ odd} \end{cases}, \tag{42}$$

and where p_S^r is given by equation (31).

2.5. INTERNAL RADIATION IMPEDANCES

The generalized pressure, P_μ, defined by equation (13) can be written as a function of the modal amplitudes A_μ using equations (30), (37), (38) and (39):

$$P_\mu = +\mathrm{j}\omega \sum_{\mu' = (m', q', s', j')} A_{\mu'} Z^i_{\mu\mu'}. \tag{43}$$

The impedance, $Z^i_{\mu\mu'}$, called internal radiation impedance, describes the interaction between the mechanical modes μ and μ'. Its normalized value $\bar{Z}^i_{\mu\mu'}$ can be split into real and imaginary parts as

$$\bar{Z}^i_{\mu\mu'} = \frac{Z^i_{\mu\mu'}}{\rho c 2\pi a \ell} = \bar{R}_{\mu\mu'} - \mathrm{j}\bar{I}_{\mu\mu'}, \tag{44}$$

where $\bar{R}_{\mu\mu'}$ and $\bar{I}_{\mu\mu'}$ are the normalized radiation resistance and reactance, respectively. This impedance $\bar{Z}^i_{\mu\mu'}$ can be written as follows

$$\bar{Z}^i_{\mu\mu'} = \delta_{mm'}\delta_{ss'}(\bar{Z}^c_{\mu\mu'} + \delta_{\mu\mu'}\bar{Z}^d_{\mu\mu}), \tag{45}$$

where $\mu = (m, q, s, j)$ and $\mu' = (m', q', s', j')$. The crossed impedance, $\bar{Z}^c_{\mu\mu'}$, and the complementary term, $\bar{Z}^d_{\mu\mu}$, (necessary to obtain the direct impedance $\bar{Z}^i_{\mu\mu} = \bar{Z}^c_{\mu\mu} + \bar{Z}^d_{\mu\mu}$) can then be expressed as (see equations (40) to (45))

$$\bar{Z}^c_{\mu\mu'} = -\mathrm{j}\,\frac{2}{\varepsilon_m}\frac{a}{\ell}\,ka\,\frac{q\pi a}{\ell}\frac{q'\pi a}{\ell}\sum_n \frac{\chi_{n\mu\mu'}}{k_{mn}a[1-\gamma^2_{mn}][(k_{mn}a)^2 - (q\pi a/\ell^2)][(k_{mn}a)^2 - (q'\pi a/\ell)^2]}, \tag{46}$$

$$\bar{Z}^d_{\mu\mu} = \frac{-\mathrm{j}}{2\varepsilon_m}\frac{k}{k_q}\frac{\mathrm{J}_m(k_q a)}{\mathrm{J}'_m(k_q a)}, \tag{47}$$

where the auxiliary parameter $\chi_{n\mu\mu'}$ is given by

$$\chi_{n\mu\mu'} = (-1)^q[1 + (-1)^{q+q'}]\tan(k_{mn}\ell/2)]^{(-1)^q}, \quad \text{if} \quad \mathscr{C} = (NN),$$

$$\chi_{n\mu\mu'} = 0, \quad \text{if} \quad \mathscr{C} = (DD),$$

$$\chi_{n\mu\mu'} = \tan(k_{mn}\ell), \quad \text{if} \quad \mathscr{C} = (ND),$$

$$\chi_{n\mu\mu'} = \mathrm{j}[1 - (-1)^q\,\mathrm{e}^{\mathrm{j}k_{mn}\ell}][1 + (-1)^{q+q'}]/2, \quad \text{if} \quad \mathscr{C} = (II).$$

Using equation (44) and isolating the direct impedance terms $Z^i_{\mu\mu}$, the governing equation of the shell motion, equation (11), can be written

$$[-m_\mu\omega^2 - \omega I_{\mu\mu} - \mathrm{j}\omega R_{\mu\mu} + m_\mu\omega^2_\mu]A_\mu = F_\mu + \mathrm{j}\omega\sum_{\mu'\neq\mu} Z^i_{\mu\mu'}A_{\mu'}. \tag{48}$$

The effect of the fluid on the structure is threefold: a reactive effect described by $(-\omega I_{\mu\mu})$, a damping due to internal radiation described by $(-\mathrm{j}\omega R_{\mu\mu})$, and an inter-modal coupling described by the term $(\mathrm{j}\omega\,\Sigma_{\mu'\neq\mu} Z^i_{\mu\mu'}A_\mu)$. Considering the definition of $I_{\mu\mu}$ given by equation (44) and the convention $(\mathrm{e}^{-\mathrm{j}\omega t})$, the reactive effect coming from the term $I_{\mu\mu}$ corresponds to an effect of added mass if $I_{\mu\mu} \geqslant 0$ or of added stiffness if $I_{\mu\mu} \leqslant 0$.

2.6. NUMERICAL RESULTS AND DISCUSSIONS

When the fluid is not dissipative (c is real), the radiation impedances, equations (46) and (47), are purely imaginary for the cases (NN), (DD) and (ND). The influence of the fluid is then reduced to a reactive effect. For these three cases, the acoustic power across the surfaces S_0 and S_ℓ is equal to zero. The domain D_i is then confined and the fluid does not introduce any loss by internal radiation. In the case (II), however, the internal domain is not confined any longer, and losses by internal radiation are not null, thus implying that $\overline{R}_{\mu\mu'} \neq 0$, even if c is real (see equation (46) in the case $\mathscr{C} = (II)$).

In order to take into account the damping in the internal fluid—which is an essential physical phenomenon—the sound speed c is a complex number: $c = c_0(1 - \mathrm{j}\eta)$, with $\eta > 0$. The damping factor in the internal fluid η is set to the constant value $\eta = 0{\cdot}01$. (A complete model which takes into account the visco-thermal losses effects can be found in reference [19]). When c is complex, all the cases, (NN), (DD), (ND), and (II), are such that $\overline{R}_{\mu\mu'} \neq 0$ and $\overline{I}_{\mu\mu'} \neq 0$.

Figure 2 shows the normalized direct radiation resistance and reactance for the circumferential index $m = 0$ and the longitudinal index $q = 1$ for the four considered cases.

Equations (46) and (47) show that the direct impedance reaches its maximum when the non-dimensional wavenumber ka is roughly equal to $k_c a = [(k_{Wmn}a)^2 + (q\pi a/\ell)^2]^{1/2}$. More precisely, for the four sets of boundary conditions, these maxima are reached for (see Figure 2):

$$(ka)^2 = (k_{Wmn}a)^2 + \begin{cases} [(2r+1)\pi a/\ell]^2, & q \text{ even,} \\ [(2r)\pi a/\ell]^2, & q \text{ odd,} \end{cases} \quad \text{if } \mathscr{C} = (NN),$$

$$(ka)^2 = (k_{Wmn}a)^2 + (q\pi a/\ell)^2, \quad \text{if } \mathscr{C} = (DD),$$

$$(ka)^2 = (k_{Wmn}a)^2 + [(2r+1)\pi a/2\ell]^2, \quad \text{if } \mathscr{C} = (ND),$$

$$(ka)^2 = (k_{Wmn}a)^2, \quad \text{if } \mathscr{C} = (II), \tag{49}$$

For the cases (NN) and (ND), an integer denoted r in equation (49), is used to identify the non-dimensional wavenumber near $k_c a$, for which local maximum radiation resistances are reached. For example, in the case $\mathscr{C} = (ND)$ (Figure 2), the values taken by r show these maximum, which are reached near $k_{W00}a = 0$ and $k_{W01} = 3{\cdot}83$. In this case, the particular choice of the parameter $\ell/a = \pi$ is such that the position of the maximum is

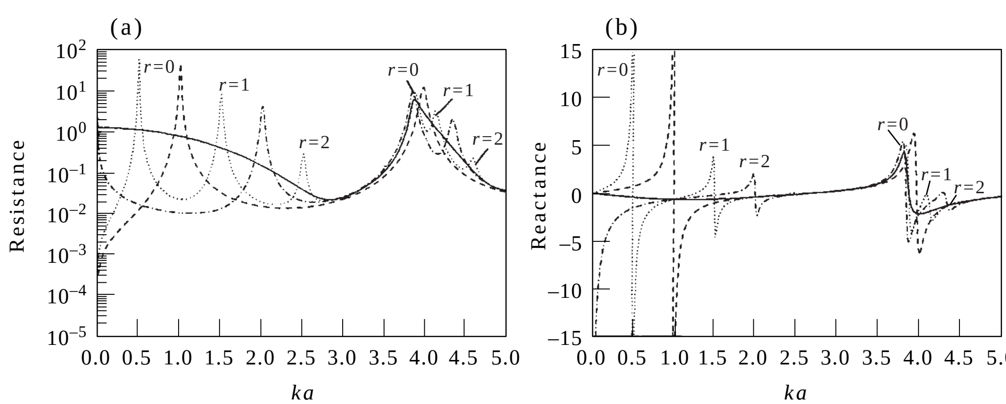

Figure 2. Normalized direct resistance and reactance for circumferential index $m = 0$ and longitudinal index $m = 1$ ($\eta = 0{\cdot}01$, $\ell/a = \pi$) for the different boundary conditions $\mathscr{C}$: ·—·–, (NN); –––, (DD); · · · , (ND); ——, (II).

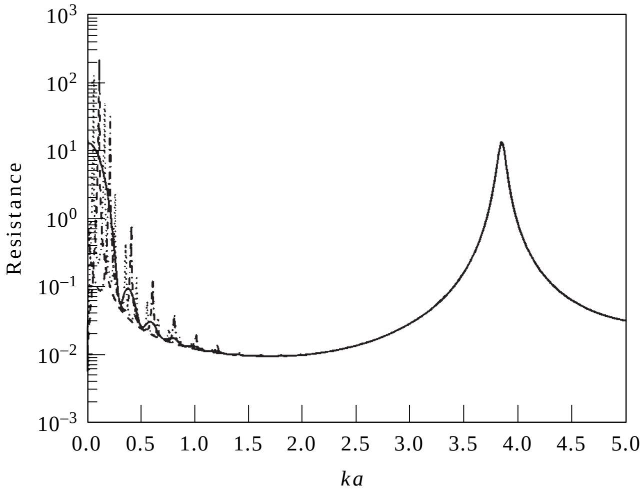

Figure 3. Direct normalized resistances for circumferential index $m = 0$ and longitudinal index $q = 1$. ($\eta = 0{\cdot}01$, $\ell/a = 10\pi$), $\mathscr{C}$: –·–·, (NN); ——, (DD); · · ·, (ND); ———, (II).

obtained for integer or half integer values for ka, depending on the problem considered. The maxima associated to the wavenumber $k_{W01} = 3{\cdot}83$, are all closer to $k_c a = [(k_{Wmn}a)^2 + (q\pi a/\ell)^2]^{1/2}$, when the parameter ℓ/a and the critical wavenumber $k_c a$ are increased. Thus, for high value of ℓ/a (case of a slender shell), and internal loss factor $\eta = 0{\cdot}01$, the secondary maximum does not exist anymore, except near the particular critical wavenumber associated to $k_{W00}a = 0$. This phenomenon is shown in Figure 3 for the direct normalized resistance and for $m = 0$ and $q = 1$. This result shows that radiation impedances are nearly independant of the boundary conditions set $\mathscr{C}$ when $n \geqslant 1$ for $m = 0$ (as shown in Figure 3), and when $m > 1$ for all values of n (as shown in Figure 4). Typical values for the direct resistance and reactance, for circumferential indexes corresponding to higher order modes ($m = 1$ and $m = 2$), are shown in Figure 4.

Typical values for cross radiation resistance and reactance for $m = 0$, $q = 1$ and $q' = 3$ are depicted in Figure 5. As already seen, in the case (DD), these quantities are strictly equal to zero. In the other cases, the cross radiation resistances can have positive and

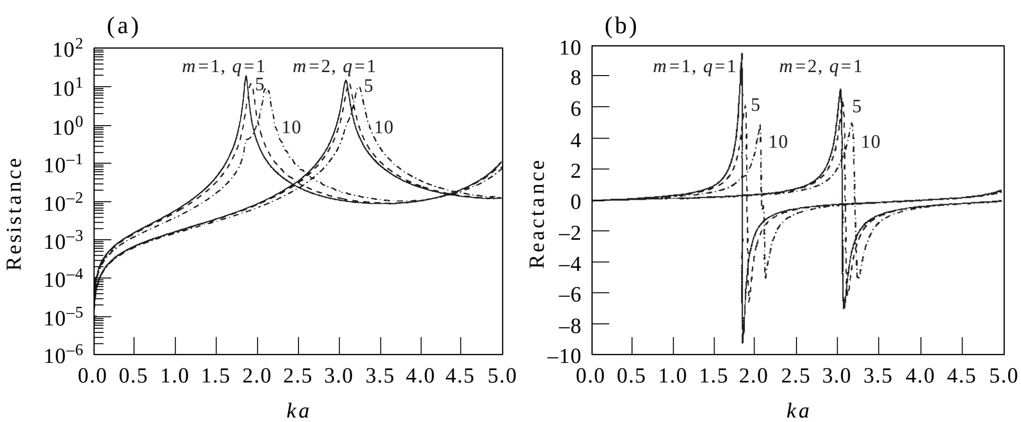

Figure 4. Direct normalized resistance and reactance for circumferential indexes $m = 1$, 2 and axial indexes $q = 1$, 5, 10 ($\eta = 0{\cdot}01$, $\ell/a = 10\pi$).

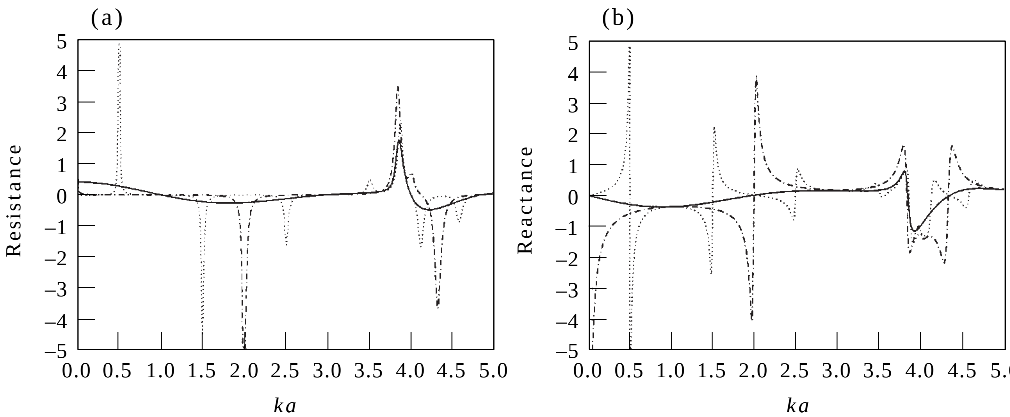

Figure 5. Cross normalized resistance and reactance for circumferential index $m = 0$ and longitudinal indexes $q = 1$ and $q' = 3$. ($\eta = 0{\cdot}01$, $\ell/a = 10\pi$). ——, (*II*); · · ·, (*ND*); ·–·–, (*NN*).

negative values, which is not the case of the direct resistances, as they are always positive quantities.

3. THE INTEGRAL METHOD

In order to show the benefits of the SME method, the expressions for the internal radiation impedance are derived in this section for the cases (*NN*), (*DD*), (*ND*) and (*II*), making use of the integral method. This classical method is applied with the Green's function for the infinite tube, which satisfies Neumann boundary conditions on the lateral surface S and Sommerfeld's conditions for $z = \pm\infty$. This kind of Green's function is widely used in propagation problems, but not very often in vibroacoustic problems dealing with finite length tubes. As it will be shown, the coupling terms obtained with this Green's function are quite simple to derive, but appear on a more complex form than with the SME method.

3.1. THE INTERNAL ACOUSTIC FIELD

The Green's function corresponding to an infinite tube is the solution of [20]

$$\begin{cases}(\Delta + k^2)G_\infty(\mathbf{r}, \mathbf{r}_0) = -\delta(\mathbf{r} - \mathbf{r}_0),\\ \partial_n G_\infty(r = a) = 0,\\ \text{Sommerfeld's conditions for } z = \pm\infty.\end{cases} \tag{50}$$

Making use of an expansion over the solutions of the two-dimensional transverse Neumann problem

$$\left(\Delta_r + \frac{1}{r^2}\partial_\theta^2 + k_{Wmn}^2\right)\Psi_\alpha(r, \theta) = 0, \tag{51}$$

$$\partial_r \Psi_\alpha(r = a) = 0, \tag{52}$$

which are denoted Ψ_α, where $\alpha = (m, n, s)$, the Green's function can be written as [18]:

$$G_\infty(\mathbf{r}, \mathbf{r}_0) = \frac{\mathrm{j}}{2}\sum_\alpha \Psi_\alpha(r, \theta)\Psi_\alpha^*(r_0, \theta_0)\frac{\mathrm{e}^{\mathrm{j}k_{mn}|z - z_0|}}{k_{mn}}. \tag{53}$$

The Helmholtz–Huygens integral

$$P(\mathbf{r}) = \int_{S0,S\ell} [\partial_{n_0} p(\mathbf{r}_0) G_\infty(\mathbf{r}, \mathbf{r}_0) - p(\mathbf{r}_0)\partial_{n_0} G_\infty(\mathbf{r}, \mathbf{r}_0)]\, \mathrm{d}S_0 + \int_S \partial_{n_0} p(\mathbf{r}_0) G_\infty(\mathbf{r}, \mathbf{r}_0)\, \mathrm{d}S_0, \quad (54)$$

leads to a sum of two terms, $p^c(\mathbf{r})$ and $p^d(\mathbf{r})$, corresponding to the radiation from the ends surfaces (S_0 and S_ℓ) and from the lateral surface S, respectively. Considering the expression for $G_\infty(\mathbf{r}, \mathbf{r}_0)$ (equation 53), each term can be expanded as the sum of two travelling waves as follows:

$$p(\mathbf{r}) = p^c(\mathbf{r}) + p^d(\mathbf{r}), \quad (55)$$

where

$$p^c(\mathbf{r}) = \sum_\alpha [C_\alpha^+ \, \mathrm{e}^{\mathrm{j}k_{mn} z} + C_\alpha^- \, \mathrm{e}^{\mathrm{j}k_{mn}(\ell - z)}] \Psi_\alpha(r, \theta),$$

and

$$p^d(\mathbf{r}) = \sum_\alpha [D_\alpha^+(z) \, \mathrm{e}^{\mathrm{j}k_{mn} z} + D_\alpha^-(z) \, \mathrm{e}^{\mathrm{j}k_{mn}(\ell - z)}] \Psi_\alpha(r, \theta).$$

The amplitude of the ingoing and outgoing waves, $D_\alpha^+(z)$ and $D_\alpha^-(z)$, can be expressed as the inner product of the axial velocity v_S over the surfaces $S_{(0,z)}$ and $S_{(z,\ell)}$ which denote the part of S between the axial co-ordinate (0 and z) and (z and ℓ), respectively:

$$D_\alpha^+(z) = -\rho c \frac{k}{2k_{mn}} \langle v_S | \Psi_\alpha \, \mathrm{e}^{\mathrm{j}k_{mn} z_0} \rangle_{S(0,z)}, \qquad D_\alpha^-(z) = -\rho c \frac{k}{2k_{mn}} \langle v_S | \Psi_\alpha \, \mathrm{e}^{\mathrm{j}k_{mn}(\ell - z_0)} \rangle_{S(z,\ell)}. \quad (56)$$

Making use of the modal expansion of equation (14), the amplitudes D_α^+ can be expressed in terms of the modal amplitudes A_μ.

3.2. ACOUSTIC BOUNDARY CONDITIONS ON THE SURFACES S_0 AND S_ℓ

The unknown coefficients $C_\alpha^\pm$ are obtained from impedance relationships, equation (33), which describe the boundary conditions on S_0 and S_ℓ:

$$C_\alpha^+ = \frac{-D_\alpha^-(0)\, \mathrm{e}^{\mathrm{j}k_{mn}\ell} + D_\alpha^+(\ell)\, \mathrm{e}^{2\mathrm{j}k_{mn}\ell}[1 - \bar{Z}_{\ell\alpha}]/[1 + \bar{Z}_{\ell\alpha}]}{[1 - \bar{Z}_{0\alpha}]/[1 + \bar{Z}_{0\alpha}] - \mathrm{e}^{2\mathrm{j}k_{mn}\ell}[1 - \bar{Z}_{\ell\alpha}]/[1 + \bar{Z}_{\ell\alpha}]}, \quad (57)$$

$$C_\alpha^- = \frac{-D_\alpha^+(\ell)\, \mathrm{e}^{\mathrm{j}k_{mn}\ell} + D_\alpha^-(0)\, \mathrm{e}^{2\mathrm{j}k_{mn}\ell}[1 + \bar{Z}_{0\alpha}]/[1 - \bar{Z}_{0\alpha}]}{[1 + \bar{Z}_{\ell\alpha}]/[1 - \bar{Z}_{\ell\alpha}] - \mathrm{e}^{2\mathrm{j}k_{mn}\ell}[1 + \bar{Z}_{0\alpha}]/[1 - \bar{Z}_{0\alpha}]}. \quad (58)$$

These expressions show that C_α^+ are equal to zero in the case $\mathscr{C} = (II)$. Thus, the pressure p^c can be interpreted as the radiated contribution from the surface S, when the vibrating shell is extended with two semi-infinite cylindrical baffles. Splitting the pressure given by equation (55) into two terms, p^c and p^d, is a natural consequence of the integral expression equation (54). The case $\mathscr{C} = (DD)$, which is very particular for the reasons mentioned above, does not appear in this separate form. Furthermore, in the case $\mathscr{C} = (NN)$, the acoustic pressure equation (55), can be written as

$$p(\mathbf{r}) = p^c + p^d \quad (59)$$

$$p(\mathbf{r}) = p^r_{s0\ell} + \rho c \sum_{\mu=(m,q,s,j)} -\mathrm{j}\omega A_\mu \left[2\mathrm{j}ka \sum_n \frac{\mathrm{J}_m(k_{Wmn}r)}{[1-\gamma^2_{mn}][(k_{Wmn}a)^2 - (k_q a)^2]} \right] \Phi_{3\mu}, \tag{60}$$

where the term $p^r_{S0\ell}$ is given by equation (41). This solution for the (NN) problem is given in order to compare the integral method with the SME method (see section 4).

3.3. INTERNAL RADIATION IMPEDANCES

Inserting equations (55)–(58) into equation (4) leads to the coupled equation (11). The radiation impedance can then be obtained. The expressions for the cross terms $Z^c_{\mu\mu'}$ are the same as those obtained previously in equation (46). The complementary term $\bar{Z}^d_{\mu\mu}$ is expressed in the form

$$\bar{Z}^d_{\mu\mu} = \frac{-\mathrm{j}}{\varepsilon_m} ka \sum_n \frac{1}{[1-\gamma^2_{mn}][(k_{Wmn}a)^2 - (k_q a)^2]}, \tag{61}$$

where the sum involves the acoustic radial modal indexes n. The integral method leads to a more complex expression for the term $\bar{Z}^d_{\mu\mu}$ than the one obtained with the SME method (equation (47)).

4. BENEFITS OF THE SME METHOD

The expressions for the acoustic field, equations (40) and (60), give the same results. The aim of this section is to compare these expressions. Expressions (60) and (40) being equal, the following relationship is satisfied:

$$\sum_{n=0}^{+\infty} \frac{2\mathrm{J}_m(k_{Wmn}r)}{[1-\gamma^2_{mn}][(k_{Wmn}a)^2 - (k_q a)^2]\mathrm{J}_m(k_{Wmn}a)} = \frac{\mathrm{J}_m(k_q r)}{k_q a \mathrm{J}'_m(k_q a)}, \tag{62}$$

where $\mathrm{J}'_m(k_{Wmn}a) = 0$ and $k^2_q = k^2 - (q\pi/\ell)^2$. In order to demonstrate this result, the function

$$f\colon x \in [0, 1] \to f(x) = \frac{a}{k_q} \frac{\mathrm{J}_m(k_q a x)}{\mathrm{J}'_m(k_q a)}, \quad \text{where} \quad x = r/a, \tag{63}$$

is expressed as a Dini expansion ([21], p. 601). This kind of expansion corresponds to an inner product with over the functions $x \to \mathrm{J}_m(\lambda_n x)$, where λ_n denotes the nth strictly positive solution of $z\mathrm{J}'_m(z) + H\mathrm{J}_m(z) = 0$; H being a real constant. For this reason, expression (62) is not a Fourier–Bessel expansion. It can be shown that

$$f(x) = \mathscr{B}_0(x) + \sum_n b_n \mathrm{J}_m(\lambda_n x), \qquad \forall m \geqslant 0, \tag{64}$$

where

$$\mathscr{B}_0(x) = \begin{cases} 2(m+1)x^m \displaystyle\int_0^1 t^{m+1} f(t)\,\mathrm{d}t & \text{if } H + m = 0 \\ 0 & \text{if } H + m > 0, \end{cases} \tag{65}$$

$$b_n = \frac{2\lambda^2_n \displaystyle\int_0^1 t f(t) \mathrm{J}_m(\lambda_n t)\,\mathrm{d}t}{(\lambda^2_n - m^2)\mathrm{J}^2_m(\lambda_n) + \lambda^2_n \mathrm{J}'^2_m(\lambda_n)}. \tag{66}$$

Applying this result in the particular case $H = 0$, one obtains relation (62). $H \neq 0$ means that the impedance of the wall material would be taken into account. Such a relation shows the equality of expressions (40) and (60). The sum of the acoustic radial index n in equation (60) is implicitly done using the SME method. In a similar way, the same kind of summation can be found for the expression of the radiation impedance, equation (61), and is avoided in equation (47) making use of the Dini expansion (62) for $r = a$.

5. CONCLUSION

This paper presents the vibroacoustic behaviour of a cylindrical shell of finite length interacting with an internal fluid. Such a system is completely determined by the inter-modal radiation impedances of the cylinder which allow one to describe the effects of the internal fluid onto the structure: a damping effect by internal radiation, an effect of added mass or stiffness, and finally, an inter-modal coupling due to internal radiation. Inter-modal radiation impedances are obtained for four sets of boundary conditions applied at the end of surfaces of the tube. A new method, called Separate Modal Expansions, is presented in this paper and enables one to obtain simpler expressions for the internal radiation impedances than when using the classical integral method. Numerical results for the radiation impedances are presented for various sets of boundary conditions on the end sections of the tube. In order to demonstrate the benefits of the proposed approach, the expressions for the internal radiation impedance are compared with the ones obtained using the classical method. In the classical method, these expressions contain, for a cylindrical cavity, summations over the radial components of the acoustic modes. The use of the SME method allows one to simplify further the expressions for the internal acoustic field and for the internal radiation impedance, thus demonstrating the value of the present method.

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APPENDIX

In this appendix, the admissibility conditions for shell modes as defined by equation (27) is demonstrated. The condition (21) which has to be satisfied by the radial component of shell mode $\Phi_{3\mu}$ has the form

$$\forall \mathbf{r} \in D_i, \qquad f(r) = \frac{1}{r^2} g(\theta, z) + h(\theta, z) = 0. \tag{A1}$$

Such an equation, applied with two differents values of the variable r denoted r_1 and r_2, leads to:

$$g(\theta, z) = [f(r_1) - f(r_2)]/[1/r_1^2 - 1/r_2^2]. \tag{A2}$$

Because the variables, r, θ and z are independant, the function $g(\theta, z)$ is constant. Making use of this result and applying a similar reasoning for equation (67), it can be shown that $h(\theta, z)$ is constant too. Thus, the equation of admissibility can be written as equation (27).