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FREE VIBRATION OF IMMERSED COLUMN CARRYING A TIP MASS

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An offshore structure having the form of a column partially immersed in a fluid is considered and its free vibration analysis is presented. The column is modelled as a uniform Bernoulli–Euler cantilever beam fixed at the bottom with a concentrated mass at the top. The effect of added mass on vibration in the fluid, the rotatory inertia of the concentrated mass and its eccentricity are all taken into account.

The analysis of the free vibration frequencies and eigenfunctions of the column, presented in the paper, enables one to obtain very accurate results. The problem of determining the natural frequencies leads to a sixth order determinant. The roots of the transcendental frequency equations are obtained numerically by an improved Regula–Falsi procedure.

The non-dimensional frequency coefficients are given in tabular form. The influence of different non-dimensional parameters on frequency coefficients is discussed. Some comparison with other results from the literature are presented.

1. INTRODUCTION

Offshore structures such as piles, dams or towers are subjected to various dynamic loads such as those due to wind, currents, waves, and if located in seismically active regions, to earthquakes. However, in most cases, wave loads are the predominant sources of excitation. The wave forces are usually evaluated by using the well known Morison equation modified to account for the movement of the structure. The evaluation of the dynamic response of these structures to wave excitation, or preliminary design process, requires calculation of natural frequencies and mode shapes in the first step.

In offshore engineering, structures such as towers or piles surrounded by water are usually modelled in analyses as cantilevers with a tip mass (see, for example, references [1–3]). Since a part of the cantilever is immersed in water it is in some sense a two-span cantilever beam. Part of beam immersed in water, due to added mass, has some effective density greater than the density of the material of the column.

The literature devoted to the calculation of the natural frequencies and mode shapes for free vibration of cantilever beams with attached inertia elements at the free end is relatively rich. In the majority of the relevant papers only uniform cantilevers [4–11] or tapered cantilever beams [12–15] were considered. Little attention has been paid to the analysis of piece wise uniform columns (having two parts of different density).

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The solutions presented in the literature can be roughly divided into exact and numerical ones. The exact solutions are those in which the governing equation and boundary conditions are fulfilled exactly. Then, the size of the matrix involved in the characteristic equation for natural frequencies is 4×4. In the case of a classical cantilever beam with a free end the characteristic matrix is sparse, and can be easily solved to yield a relatively simple frequency equation and mode shape functions. On the other hand, in the case of a cantilever beam carrying a mass with moment of inertia at the tip, the characteristic matrix becomes completely filled and the transcendental frequency equation is of relatively high order, with the end inertia elements appearing as parameters.

Two segment cantilever beams carrying a mass with moment of inertia at the tip, relating to the mechanical model adopted in this paper were considered in references [1, 16–21]. They were solved by exact methods [17, 18, 21] and by numerical solutions [1, 16, 19, 20]. In case of exact methods, the frequency equation and mode shapes are obtained by formulating the equations of motion for each of the beam segments and requiring satisfaction of the boundary and continuity conditions. This leads to an 8×8 matrix characteristic equation for the natural frequencies which results in a transcendental equation with rather complex algebraic expressions. This is probably the reason why numerical solutions for two-span beams are more popular. On the other hand an exact solution, in some situations, has an indubitable advantage in comparison with a numerical solution.

A study of the exact solutions for two-span beams presented in references [17, 18, 21], reveals that they cannot be directly applied to the analysis of the two-segment columns discussed in this paper. This fact motivated the authors to propose the exact method for determination of the natural frequencies and mode shapes of a partially immersed column with a tip mass. In the proposed method, the eccentricity and rotatory inertia of the tip mass are taken into account. The method seems to have a great potential, as demonstrated by the numerical examples included in the paper. The method paves the way for future application of the mode superposition method (see reference [22]) for determination of the dynamic response of a partially immersed column under wave loading.

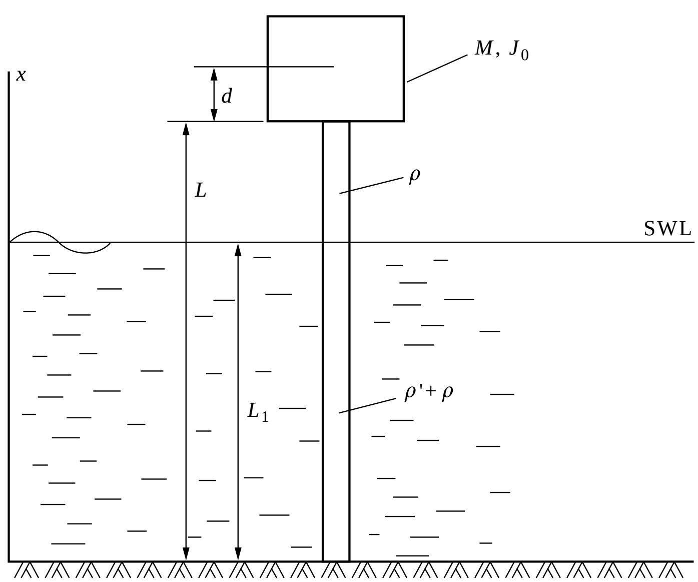


Figure 1. Sketch of the column system under study

2. NATURAL FREQUENCIES OF A COLUMN PARTIALLY IMMERSED IN FLUID

We consider a simple offshore structure consisting of a cylindrical column supporting the deck, as shown in Figure 1. The mass of the deck (M) is lumped at the top.

Since the column is partially immersed in a fluid, in the equation of vibration of column, the density of immersed part must be modified to account for the added fluid mass. Therefore, instead of one, we have two equations of free vibration of column: one for the immersed part of column

$$EI\frac{\partial^4 w_i}{\partial x^4} + A(\rho + \rho')\frac{\partial^2 w_1}{\partial t^2} = 0, \tag{1}$$

and the other for the part above the fluid

$$EI\frac{\partial^4 w_2}{\partial x^4} = A\rho\frac{\partial^2 w_2}{\partial t^2} = 0. \tag{2}$$

In the above equations, EI is the bending stiffness, ρ is the mass density of the column, ρ' is the mass density of fluid (see reference [1]), A is the column cross-section area ($A = \pi D^2/4$, D is the diameter of beam) and $w_1 = w_1(x, t)$ is the displacement function for the immersed part of the column ($0 \leqslant x \ll L_1$), while $w_2 = w_2(x, t)$ is the pertinent function for $L_1 \leqslant x \leqslant L$ (see Figure 1).

The boundary and continuities conditions for equations (1) and (2) are as follows: zero displacement and slope at the fixed end

$$w_1(0, t) = 0, \qquad \frac{\partial w_1}{\partial x}(0, t) = 0, \tag{3, 4}$$

the bending moment and shear force at the tip of the colum ($x = L$) are equal

$$EI\frac{\partial^2 w_2}{\partial x^2}(L, t) = -(J_0 + Md^2)\frac{\partial^3 w_2}{\partial x\,\partial t^2}(L, t) - Md\frac{\partial^2 w_2}{\partial t^2}(L, t), \tag{5}$$

$$EI\frac{\partial^3 w_2}{\partial x^3}(L, t) = M\frac{\partial^2 w_2}{\partial t^2}(L, t) + Md\frac{\partial^3 w_2}{\partial x\,\partial t^2}(L, t), \tag{6}$$

equal deflections, slopes, moments and shear forces at $x = L_1$

$$w_1(l_1, t) = w_2(L_1, t), \qquad \frac{\partial w_1}{\partial x}(L_1, t) = \frac{\partial w_2}{\partial x}(L_1, t), \tag{7, 8}$$

$$\frac{\partial^2 w_1}{\partial x^2}(L_1, t) = \frac{\partial^2 w_2}{\partial x^2}(L_1, t), \qquad \frac{\partial^3 w_1}{\partial x^3}(L_1, t) = \frac{\partial^3 w_2}{\partial x^3}(L_1, t). \tag{9, 10}$$

In the above equations, M is the tip mass, J_0 is its moment of inertia, L is the length of column, L_1 is length of part of column immersed in fluid and d is distance between centre of gravity of the tip mass and the end of column.

Separating the variables, the transverse displacement functions can be written as:

$$w_1(x, t) = W_1(x)\cos\omega t \quad \text{for} \quad 0 \leqslant x \leqslant L_1, \tag{11}$$

$$w_2(x, t) = W_2(x)\cos\omega t \quad \text{for} \quad L_1 \leqslant x \leqslant L, \tag{12}$$

where $W_1(x)$ and $W_2(x)$ are the mode shapes for the immersed and above fluid parts of the column.

Substituting equations (11) and (12) into equations (1) and (2) and using conditions (3)–(10), the homogenous system of linear equations with respect to c_1, c_2,, c_5, c_6, c_7, c_8 is obtained. The system is given in dimensionless form by formula (13)

$$\begin{cases} c_5 l_{13} + c_6 l_{14} + c_7 l_{15} + c_8 l_{16} = 0 \\ c_5 l_{23} + c_6 l_{24} + c_7 l_{25} + c_8 l_{26} = 0 \\ c_1 l_{31} + c_2 l_{32} + c_5 l_{33} + c_6 l_{34} + c_7 l_{35} + c_8 l_{36} = 0 \\ c_1 l_{41} + c_2 l_{42} + c_5 l_{43} + c_6 l_{44} + c_7 l_{45} + c_8 l_{46} = 0 \\ c_1 l_{51} + c_2 l_{52} + c_5 l_{53} + c_6 l_{54} + c_7 l_{55} + c_8 l_{56} = 0 \\ c_1 l_{61} + c_2 l_{62} + c_5 l_{63} + c_6 l_{64} + c_7 l_{65} + c_8 l_{66} = 0 \end{cases}, \tag{13}$$

where

$$l_{13} = (1-\beta)\cosh(\lambda k) - \alpha \sinh(\lambda k), \qquad l_{23} = (1+\beta)\sinh(\lambda k) + \gamma\cosh(\lambda k),$$

$$l_{14} = (1-\beta)\sinh(\lambda k) - \alpha \cosh(\lambda k), \qquad l_{24} = (1+\beta)\cosh(\lambda k) + \gamma\sinh(\lambda k),$$

$$l_{15} = -(1+\beta)\cos(\lambda k) + \alpha \sin(\lambda k), \qquad l_{25} = (1-\beta)\sin(\lambda k) + \gamma\sin(\lambda k),$$

$$l_{16} = -(1+\beta)\sin(\lambda k) - \alpha \cos(\lambda k), \qquad l_{26} = -(1-\beta)\cos(\lambda k) + \gamma\cos(\lambda k),$$

$$l_{31} = \cosh(\lambda\eta) - \cos(\lambda\eta), \qquad l_{41} = \sinh(\lambda\eta) + \sin(\lambda\eta),$$

$$l_{32} = \sinh(\lambda\eta) - \sin(\lambda\eta), \qquad l_{42} = \cosh(\lambda\eta) - \cos(\lambda\eta),$$

$$l_{33} = -\cosh(k\lambda\eta), \qquad l_{43} = -k\sinh(k\lambda\eta),$$

$$l_{34} = -\sinh(k\lambda\eta), \qquad l_{44} = -k\cosh(k\lambda\eta),$$

$$l_{35} = -\cos(k\lambda\eta), \qquad l_{45} = k\sin(k\lambda\eta),$$

$$l_{36} = -\sin(k\lambda\eta), \qquad l_{46} = -k\cos(k\lambda\eta),$$

$$l_{51} = \cosh(\lambda\eta) + \cos(\lambda\eta), \qquad l_{61} = \sinh(\lambda\eta) - \sin(\lambda\eta),$$

$$l_{52} = \sinh(\lambda\eta) + \sin(\lambda\eta), \qquad l_{62} = \cosh(\lambda\eta) + \cos(\lambda\eta),$$

$$l_{53} = -k^2\cosh(k\lambda\eta), \qquad l_{63} = -k^3\sinh(k\lambda\eta),$$

$$l_{54} = -k^2\sinh(k\lambda\eta), \qquad l_{64} = -k^3\cosh(k\lambda\eta),$$

$$l_{55} = k^2\cos(k\lambda\eta), \qquad l_{65} = -k^3\sin(k\lambda\eta),$$

$$l_{56} = k^2\sin(k\lambda\eta), \qquad l_{66} = k^3\cos(k\lambda\eta),$$

and

$$\alpha = (\mu + \delta d^2)\left(\frac{k\lambda}{L}\right)^3, \qquad \beta = \delta d\left(\frac{k\lambda}{L}\right)^3, \qquad \gamma = \delta\frac{k\lambda}{L}, \qquad \delta = \frac{M}{\rho A}, \qquad \mu = \frac{J_0}{\rho A},$$

$$k = \frac{k_2}{k_1} = \sqrt[4]{\frac{\rho}{\rho + \rho'}}, \qquad k_1^4 = \frac{(\rho + \rho')A\omega^2}{EI_1}, \qquad k_2^4 = \frac{\rho A\omega^2}{EI_2}, \qquad \eta = \frac{L_1}{L}, \qquad \gamma = Lk_1.$$

TABLE 1

Frequency of free vibration of column with parameters: $d = 0{\cdot}5\ m,\ D = 0{\cdot}3$ m

k	L_1/L	P	Q	ω_1	ω_2	ω_3	ω_4	ω_5
0·9704572	2/3	0·00	0·00	5·9465	36·1711	101·6035	199·0316	327·7672
0·9704572	2/3	0·00	0·01	5·8234	35·2812	98·7969	192·6033	316·0996
0·9704572	2/3	0·00	0·10	4·9692	30·6631	87·3226	172·1122	287·0867
0·9704572	2/3	0·00	0·50	3·3272	26·1794	80·2084	163·4925	278·1971
0·9704572	2/3	0·01	0·00	5·7301	24·0482	56·14191	124·8375	229·9601
0·9704572	2/3	0·01	0·01	5·6179	23·8828	56·0355	124·0154	228·1210
0·9704572	2/3	0·01	0·10	4·8321	22·9047	55·4997	119·3847	218·6042
0·9704572	2/3	0·01	0·50	3·2819	21·6531	54·8201	114·0782	209·5111
0·9704572	2/3	0·10	0·00	4·2371	11·8302	50·1508	122·6376	228·8045
0·9704572	2/3	0·10	0·01	4·1919	11·7947	49·7178	121·5382	226·7339
0·9704572	2/3	0·10	0·10	3·8393	11·5454	46·8724	115·0830	215·7967
0·9704572	2/3	0·10	0·50	2·9193	11·0504	42·3808	107·2976	205·2087
0·9704572	2/3	0·50	0·00	2·2989	9·8684	49·6698	122·7042	228·7042
0·9704572	2/3	0·50	0·01	2·2926	9·7742	49·2007	121·3286	226·6131
0·9704572	2/3	0·50	0·10	2·2371	9·0648	46·0926	114·7103	215·5494
0·9704572	2/3	0·50	0·50	2·0244	7·4364	41·1149	106·7117	204·8328
0·9704572	1/3	0·00	0·00	6·0126	37·3900	103·1491	202·5036	335·7799
0·9704572	1/3	0·00	0·01	5·8852	36·4659	100·3509	195·9289	323·6606
0·9704572	1/3	0·00	0·10	5·0063	31·6899	88·6405	175·1862	293·5072
0·9704572	1/3	0·00	0·50	3·3377	27·0957	81·4999	166·6504	284·3100
0·9704572	1/3	0·01	0·00	5·7877	24·4144	57·4087	127·2044	234·0800
0·9704572	1/3	0·01	0·01	5·6718	24·2567	57·3055	126·3475	232·3778
0·9704572	1/3	0·01	0·10	4·8655	23·3263	56·7075	121·5297	222·2833
0·9704572	1/3	0·01	0·50	3·2919	22·1384	55·9717	116·0342	212·9657
0·9704572	1/3	0·10	0·00	4·2546	11·9870	51·6516	124·9958	232·9176
0·9704572	1/3	0·10	0·01	4·2091	11·9475	51·2170	123·8571	230·7613
0·9704572	1/3	0·10	0·10	3·8529	11·6712	48·3657	117·1882	219·4634
0·9704572	1/3	0·10	0·50	2·9257	11·1626	43·8851	109·1950	208·6745
0·9704572	1/3	0·50	0·00	2·3008	10·0253	51·1887	124·8097	232·8168
0·9704572	1/3	0·50	0·01	2·2945	9·9255	50·7191	123·6459	230·6396
0·9704572	1/3	0·50	0·10	2·2390	9·1778	47·6120	116·8113	219·2152
0·9704572	1/3	0·50	0·50	2·0260	7·4857	42·6607	108·6035	208·2999
0·9554427	2/3	0·00	0·00	6·8536	41·6540	115·4908	226·3303	371·9116
0·9554427	2/3	0·00	0·01	6·7137	40·0640	112·3695	219·0712	358·8295
0·9554427	2/3	0·00	0·10	5·7393	34·8449	99·4839	195·6866	326·1101
0·9554427	2/3	0·00	0·50	3·8526	29·7378	91·3822	185·7349	316·0004
0·9554427	2/3	0·01	0·00	6·6082	27·6326	64·0256	141·6740	261·5539
0·9554427	2/3	0·01	0·01	6·4803	27·4355	63·9234	140·7632	259·5004
0·9554427	2/3	0·01	0·10	5·5828	26·2703	63·3336	135·6241	248·8146
0·9554427	2/3	0·01	0·50	3·8004	24·7774	62·6903	129·7064	238·4962
0·9554427	2/3	0·10	0·00	4·9030	13·6165	56·9383	139·1088	260·2070
0·9554427	2/3	0·10	0·01	4·8511	13·5779	56·4451	137·8804	257·8866
0·9554427	2/3	0·10	0·10	4·4444	13·3605	53·1966	130·6489	245·5519
0·9554427	2/3	0·10	0·50	3·3821	12·7961	48·0398	121·8729	233·4732
0·9554427	2/3	0·50	0·00	2·6654	11·3414	56·3712	138·8937	260·0902
0·9554427	2/3	0·50	0·01	2·6581	11·2359	55·8359	137·6372	257·7461
0·9554427	2/3	0·50	0·10	2·5939	10·4385	52·2797	130·2187	245·2645
0·9554427	2/3	0·50	0·50	2·3472	8·5922	46·5509	121·1975	233·0341
0·9554427	1/3	0·00	0·00	6·9728	43·1709	118·3374	232·3747	385·7484
0·9554427	1/3	0·00	0·01	6·8251	42·1132	115·0541	224·8471	371·9612
0·9554427	1/3	0·00	0·10	5·8065	36·6335	101·7456	201·0008	337·4109
0·9554427	1/3	0·00	0·50	3·8716	31·3431	93·5827	191·0033	329·7907
0·9554427	1/3	0·01	0·00	6·7122	23·2844	66·1566	145·7899	268·8346

Table 1—(*continued on next page*)

TABLE 1—*continued*

k	L_1/L	P	Q	ω_1	ω_2	ω_3	ω_4	ω_5
0·9554427	1/3	0·01	0·00	6·5779	28·0114	66·0420	144·8203	266·6313
0·9554427	1/3	0·01	0·10	5·6433	27·0215	65·3793	139·3639	255·2934
0·9554427	1/3	0·01	0·50	3·8185	25·6422	64·5618	133·1207	244·5505
0·9554427	1/3	0·10	0·00	4·9350	13·8973	59·4989	143·297	267·4725
0·9554427	1/3	0·10	0·01	4·8822	13·8516	59·0041	141·9145	264·9968
0·9554427	1/3	0·10	0·10	4·4692	13·5327	55·7538	134·3176	251·9997
0·9554427	1/3	0·10	0·50	3·3937	12·9447	50·6285	125·1788	239·5336
0·9554427	1/3	0·50	0·00	2·6690	11·6228	58·9644	142·9926	267·3544
0·9554427	1/3	0·50	0·01	2·6616	11·5074	58·4292	141·6684	264·8545
0·9554427	1/3	0·50	0·10	2·5973	10·6419	54·8843	133·8799	251·7097
0·9554427	1/3	0·50	0·50	2·3502	8·6817	49·2148	124·4925	239·0958
0·9306048	2/3	0·00	0·00	9·4457	55·2202	155·4932	305·2030	499·6247
0·9306048	2/3	0·00	0·01	9·1736	53·6921	150·8362	294·2689	480·6208
0·9306048	2/3	0·00	0·10	7·9396	46·9119	134·4805	263·8696	438·6241
0·9306048	2/3	0·00	0·50	5·3538	39·9933	123·5620	250·0919	424·9553
0·9306048	2/3	0·01	0·00	9·1172	37·8913	86·7901	190·2750	352·5945
0·9306048	2/3	0·01	0·01	8·9447	37·6041	86·6675	189·0983	349·9237
0·9306048	2/3	0·01	0·10	7·7275	35·9047	85·9603	182·4384	335·8909
0·9306048	2/3	0·01	0·50	5·2819	33·7233	85·0923	174·7153	322·1001
0·9306048	2/3	0·10	0·00	6·8063	18·7253	76·3102	186·0817	349·2838
0·9306048	2/3	0·10	0·01	6·7346	18·6776	75·9311	185·0453	347·6719
0·9306048	2/3	0·10	0·10	6·1738	18·3405	71·5162	175·5207	331·3489
0·9306048	2/3	0·10	0·50	4·7050	17·6985	64·4376	163·8533	315·0509
0·9306048	2/3	0·50	0·00	3·7133	15·5530	75·7836	186·3544	350·5459
0·9306048	2/3	0·50	0·01	3·7030	15·4147	75·0586	184·7049	347·4759
0·9306048	2/3	0·50	0·10	3·6136	14·3642	70·2068	174·9251	330·9486
0·9306048	2/3	0·50	0·50	3·2702	11·8949	62·3088	162·9206	314·4336
0·9306048	1/3	0·00	0·00	9·7181	59·6858	161·5612	318·2602	528·8666
0·9306048	1/3	0·00	0·01	9·5125	58·2465	157·1637	307·9843	510·3282
0·9306048	1/3	0·00	0·10	8·0939	50·7563	139·2287	275·1923	463·2759
0·9306048	1/3	0·00	0·50	5·3979	43·4787	128·1347	261·3066	448·6198
0·9306048	1/3	0·01	0·00	9·3554	39·3462	91·1489	199·1244	368·6923
0·9306048	1/3	0·01	0·00	9·1685	39·0904	91·0019	197·8266	365·6837
0·9306048	1/3	0·01	0·10	7·8668	37·5817	90·1501	190·5092	350·1455
0·9306048	1/3	0·01	0·50	5·3139	35·6556	89·0979	182·0999	335·3236
0·9306048	1/3	0·10	0·00	6·8804	19·3587	81·9157	195·4741	366·7629
0·9306048	1/3	0·10	0·01	6·8067	19·2955	81·2487	193·7242	363·3727
0·9306048	1/3	0·10	0·10	6·2311	18·8538	76·8551	183·4328	346·5007
0·9306048	1/3	0·10	0·50	4·7319	18·0392	69·8887	170·9837	328·2378
0·9306048	1/3	0·50	0·00	3·7216	16·1901	81·1763	195·1674	366·5956
0·9306048	1/3	0·50	0·01	3·7114	16·0297	80·4538	193·3774	363·1715
0·9306048	1/3	0·50	0·10	3·6217	14·8272	75·6550	192·8208	345·0967
0·9306048	1/3	0·50	0·50	3·2771	12·1007	67·9356	170·0246	327·6196
0·6930980	2/3	0·00	0·00	24·1116	119·5139	316·1159	643·2245	1064·417
0·6930980	2/3	0·00	0·01	23·8808	116·7276	310·1343	631·0034	1032·127
0·6930980	2/3	0·00	0·10	21·4768	100·6635	282·5697	578·7918	925·0746
0·6930980	2/3	0·00	0·50	15·7417	81·8293	261·5765	546·4460	882·1998
0·6930980	2/3	0·01	0·00	23·6187	95·6352	208·1151	394·4489	719·4297
0·6930980	2/3	0·01	0·01	23·3285	94·3336	208·1149	392·5501	716·0535
0·6930980	2/3	0·01	0·10	21·0939	86·3774	208·1140	381·5896	697·4356
0·6930980	2/3	0·01	0·50	15·5682	75·4806	208·1128	368·2680	676·9798
0·6930980	2/3	0·10	0·00	19·4873	48·1759	168·3106	377·3331	713·4353
0·6930980	2/3	0·10	0·01	19·3149	48·1722	166·9309	373·9473	709·2536
0·6930980	2/3	0·10	0·10	17·9421	48·1439	157·0594	353·2065	685·4972

Table 1—(*continued on next page*)

TABLE 1—(*continued*)

k	L_1/L	P	Q	ω_1	ω_2	ω_3	ω_4	ω_5
0·6930980	2/3	0·10	0·50	14·1247	48·0792	138·1040	325·9694	658·5628
0·6930980	2/3	0·50	0·00	11·4920	37·4439	165·0343	376·0122	712·9342
0·6930980	2/3	0·50	0·01	11·4606	37·2983	163·4283	372·5027	708·6825
0·6930980	2/3	0·50	0·10	11·1869	36·1151	151·7500	350·9614	684·4765
0·6930980	2/3	0·50	0·50	5·36180	19·6628	128·5488	322·7505	656·9943
0·6930980	1/3	0·00	0·00	30·3751	158·4103	393·8983	829·0959	1295·882
0·6930980	1/3	0·00	0·01	29·7497	155·5939	383·7159	806·7252	1259·519
0·6930980	1/3	0·00	0·10	25·4019	139·8200	340·0668	725·3778	1158·978
0·6930980	1/3	0·00	0·50	17·0174	122·6180	310·6918	685·0807	1123·965
0·6930980	1/3	0·01	0·00	29·2799	117·4240	233·2127	506·6944	948·2404
0·6930980	1/3	0·01	0·00	28·7084	116·5947	233·1506	503·8091	942·4562
0·6930980	1/3	0·01	0·10	24·7049	111·7186	232·7929	487·2722	910·8155
0·6930980	1/3	0·01	0·50	16·7862	105·5352	232·3528	467·5243	877·2979
0·6930980	1/3	0·10	0·00	21·6756	59·7869	204·2673	491·4337	942·3392
0·6930980	1/3	0·10	0·01	21·4448	59·6198	203·0448	486·9711	935·5264
0·6930980	1/3	0·10	0·10	19·6404	58·4439	194·7462	459·9205	897·2267
0·6930980	1/3	0·10	0·50	14·9331	56·2298	180·6056	424·9244	855·7250
0·6930980	1/3	0·50	0·00	11·7672	49·9878	202·1084	490·1705	941·8316
0·6930980	1/3	0·50	0·01	11·7282	49·5251	200·7473	485·5667	934·9273
0·6930980	1/3	0·50	0·10	11·4448	46·0216	191·4206	457·5620	896·0325
0·6930980	1/3	0·50	0·50	10·3560	37·8835	175·2698	421·2364	853·8407

Equation (13) constitutes an eigenvalue problem and can be solved accordingly: the non-trivial solution of equation (13) exists if

$$\det\begin{bmatrix} 0 & 0 & l_{13} & l_{14} & l_{15} & l_{16} \\ 0 & 0 & l_{23} & l_{24} & l_{25} & l_{26} \\ l_{31} & l_{32} & l_{33} & l_{34} & l_{35} & l_{36} \\ l_{41} & l_{42} & l_{43} & l_{44} & l_{45} & l_{46} \\ l_{51} & l_{52} & l_{53} & l_{54} & l_{55} & l_{56} \\ l_{61} & l_{62} & l_{63} & l_{64} & l_{65} & l_{66} \end{bmatrix} = 0. \tag{14}$$

Expansion of this determinant leads to a transcendental equation which has an infinite number of roots. Equation (14) can be solved by "Regula falsi" with respect to λ, which are the eigenvalues of the problem: $\lambda_1, \lambda_2, \ldots, \lambda_i, \ldots$. Knowing λ_i the corresponding natural frequencies are $\omega_1, \omega_2, \ldots \omega_i, \ldots$.

3. EIGENFUNCTIONS OF A PARTIALLY IMMERSED COLUMN

In order to obtain an expression for the eigenfunctions, the eigenvectors $\mathbf{c}^{(1)}, \mathbf{c}^{(2)}, \ldots, \mathbf{c}^{(i)}, \ldots$ associated with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_i, \ldots$ are to be found, where

$$\mathbf{c}(i) = [c_1^{(i)} \quad c_2^{(i)} \quad c_5^{(i)} \quad c_6^{(i)} \quad c_7^{(i)} \quad c_8^{(i)}]^T,$$

where the T index denotes the transpose of the vector.

They can be found by consecutive substituting in equation (13) the values of λ_i, $i = 1, 2, \ldots$ and solving the resulting matrix equation

$$\begin{bmatrix} 0 & l_{13} & l_{14} & l_{15} & l_{16} \\ 0 & l_{23} & l_{24} & l_{25} & l_{26} \\ l_{32} & l_{33} & l_{34} & l_{35} & l_{36} \\ l_{42} & l_{43} & l_{44} & l_{45} & l_{46} \\ l_{52} & l_{53} & l_{54} & l_{55} & l_{56} \end{bmatrix} \begin{bmatrix} c_2^{(i)} \\ c_5^{(i)} \\ c_6^{(i)} \\ c_7^{(i)} \\ c_8^{(i)} \end{bmatrix} = \begin{bmatrix} m_1^{(i)} \\ m_2^{(i)} \\ m_3^{(i)} \\ m_4^{(i)} \\ m_5^{(i)} \end{bmatrix}, \tag{15}$$

where

$$c_1^{(i)} = 1, \qquad m_1^{(i)} = 0, \qquad m_2^{(i)} = -c_1^{(i)} \cdot l_{31}, \qquad m_3^{(i)} = -c_1^{(i)} \cdot l_{41},$$
$$m_4^{(i)} = -c_1^{(i)} \cdot l_{51}, \qquad m_5^{(i)} = -c_1^{(i)} \cdot l_{61}.$$

The mode shapes functions associated with the i-th natural frequency, in dimensionless form, are given by

$$Y_1^{(i)}(\lambda_i, X) = c_1^{(i)}(\cosh(\lambda_i X) - \cos(\lambda_i X)) + c_2^{(i)}(\sinh(\lambda_i X) - \sin(\lambda_i X)) \quad \text{for} \quad 0 \leqslant x \leqslant \eta, \tag{16}$$

$$Y_2^{(i)}(\lambda_i, X) = c_5^{(i)} \cosh(k\lambda_i X) + c_6^{(i)} \sinh(k\lambda_i X) + c_7^{(i)} \cos(k\lambda_i X) + c_8^{(i)} \sin(k\lambda_i X)$$
$$\text{for } \eta \leqslant x \leqslant 1, \qquad \text{for } i = 1, 2, \ldots. \tag{17}$$

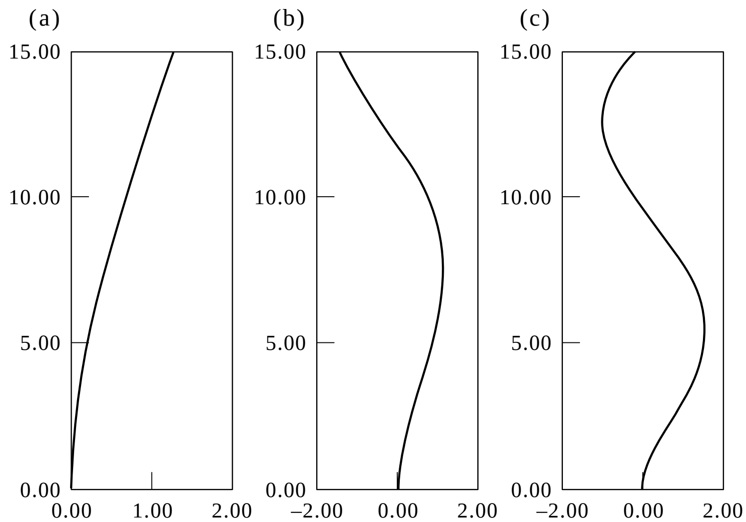
4. NUMERICAL RESULTS

As mentioned in the previous section, the functional relation (15) is highly transcendental in nature. In this equation the following dimensionless parameters are used: $k = \sqrt[4]{\rho/\rho + \rho'}$, $\eta = L_1/L$, $Q = M/\rho AL$, $P = J_0/\rho AL^3$. For each particular value of these parameters the transcendental equation has an infinite number of roots λ_i, which can

TABLE 2

Comparison of eigenfrequency parameter $(k_1L)_i (i = 1, 2, 3)$ for a solid column with a tip mass

						Reference [1]		
L_1/L	P	Q	$(k_1/L)_1$	$(k_1L)_2$	$(k_1L)_3$	$(k_1L)_1$	$(k_1L)_2$	$(k_1L)_3$
0·0	0·0	1·0	1·28589	4·15381	7·35123	1·24791	4·03105	7·13373
	0·0	2·0	1·10894	4·10377	7·31879	1·07619	3·98250	7·10227
0·5	0·0	1·0	1·28553	4·10890	7·23281	1·24755	3·98642	7·01864
	0·0	2·0	1·10878	4·05978	7·20006	1·07602	3·93877	6·98687
1·0	0·0	1·0	1·27812	4·04250	7·14178	1·24037	3·92303	6·93047
	0·0	2·0	1·10523	3·98914	7·10680	1·07259	3·87126	6·89655
0·0	1·0	1·0	0·95996	1·89739	5·05001	0·93161	1·84135	4·90076
	1·0	2·0	0·91265	1·74078	4·97632	0·78792	1·59862	4·82488
0·5	1·0	1·0	0·95991	1·98600	4·97223	0·93156	1·83996	4·82388
	1·0	2·0	0·91261	1·74004	4·89985	0·78789	1·59794	4·74919
1·0	1·0	1·0	0·95856	1·88350	4·91782	0·93025	1·82787	4·77247
	1·0	2·0	0·91147	1·73393	4·84047	0·78734	1·59165	4·69268



be determined by the "Regula falsi" method. In this paper the sequence of λ_i $(i = 1, 2, 3, 4, 5)$ has been obtained for selected values of the parameters k, η, Q and P. Next, using the formula in equation (13) for the natural frequencies, the $\omega_i (i = 1, 2, 3, 4, 5)$ have been calculated and presented in Table 1 as a function of the dimensionless parameters.

An examination of the results presented in Table 1 reveal that decreasing the parameters k and η leads to an increase in the natural frequencies. Since the parameter k is proportional to the density of fluid, it means that the values of the natural frequencies of an immersed column increase with the increase of the fluid density. On the other hand, when the parameters P and Q related to the tip mass and its moment of inertia, increase, the natural frequencies decrease.

In Table 2, the results obtained are compared with the data given in reference [1]. The comparison refers to the solid column with a tip mass, for which $L = 15$ m, $k = 0{\cdot}9704672$, $d = 0$ m, $D = 0{\cdot}3$ m, $\rho = 7850$ kg/m^3, $\rho_{ef} = 8850$ kg/m^3, $E = 2{\cdot}068 \times 10^{11}$ Pa.

For the cases $L_1/L = 0$, $Q = 1$, $P = 1$ and $Q = 2$, $P = 2$ and for cases $L_1/L = 0$, $Q = 1$, $P = 0$ and $Q = 2$, $P = 0$, the results obtained in this paper are identical to the data shown in refernce [1]. For other values of the parameters P and Q, small difference in the results can be observed. These can be attributed to the approximation inherent to the FEM solution used in reference [1].

As far as the eigenfunctions are concerned, they have been calculated for the column characterised with the following parameters: $Q = 0$, $P = 0{\cdot}005$, $L = 15$ m, $L_1 = 5$ m, $k = 0{\cdot}9704672$, $d = 0{\cdot}5$ m, $D = 0{\cdot}3$ m, $\rho = 7850$ kg/m^3, $\rho_{ef} = 8850$ kg/m^3.

The eigenfunctions have been determined for the first three natural frequencies: $\omega_1 = 3{\cdot}3045$ s^{-1}, $\omega_2 = 23{\cdot}7301$ s^{-1}, $\omega_3 = 62{\cdot}8525$ s^{-1}. They are shown in Figures 2(a)–(c), respectively.

5. CONCLUSIONS

Closed form, exact frequency and mode shape equations for a partially immersed column with excentrically located tip mass have been presented. The column is modelled as a distributed parameter cantilever with the lumped mass at the top. The rotatory inertia of the lumped mass is taken into account in the analysis. The lower part of the column, immersed in fluid, is considered as having larger density than the part above the fluid level, to account for the presence of fluid around the bar.

The equations derived in the paper were used next for collecting some numerical data that allows selected influences of the non-dimensional parameters on the natural frequencies of the column considered to be examined. The results were compared with those given in reference [1]. Small discrepancies between the result in reference [1] and those obtained in this paper are attributed to the difference between the methods used for calculation of the natural frequencies (the FEM in reference [1] and exact method in this paper).

The results presented in this work constitute the first stage of a wider research program devoted to the dynamic analysis of offshore structures.

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