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LETTERS TO THE EDITOR

COMMENTS ON "A RATIONAL HARMONIC BALANCE APPROXIMATION FOR THE DUFFING EQUATION OF MIXED PARITY"

R. E. Mickens
Department of Physics, Clark Atlanta University, Atlanta, GA 30314, *U.S.A.*

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The main purpose of this letter is to clarify certain concepts and their applications discussed in the paper of Sarma and Rao [1] which are based on the earlier work of Mickens and Semwogerere [2].

Consider a non-linear, conservative oscillator modelled by the differential equation

$$\ddot{x} + f(x) = 0, \qquad x(0) = x_0 \neq 0, \qquad \dot{x}(0) = 0. \tag{1}$$

This equation is of "odd parity" if

$$f(-x) = -f(x). \tag{2}$$

For this case, the periodic solutions have a Fourier representation which contains only odd multiples of the angular frequency ω [3, 4], i.e.,

$$x^{(1)}(t) = \sum_{k=1}^{\infty} a_k \cos(2k+1)\omega t. \tag{3}$$

Equation (1) is of "mixed parity" if

$$f(x) = f_+(x) + f_-(x), \tag{4a}$$

$$f_+(-x) = f_+(x), \qquad f_-(-x) = -f_-(x), \tag{4b}$$

$$f_+(x) \not\equiv 0 \qquad f_-(x) \neq 0. \tag{4c}$$

For this situation, the Fourier series has both even and odd multiples of ω [3, 4], i.e.,

$$x^{(2)}(t) = \frac{b_0}{2} + \sum_{k=1}^{\infty} b_k \cos(k\omega t). \tag{5}$$

The simplest rational harmonic balance approximations to the periodic solutions of equation (1) are given by the functions [5]

$$x_{HB}^{(1)}(t) = \frac{A_1 \cos\omega t}{1 + B_1 \cos(2\omega t)}, \tag{6}$$

$$x_{HB}^{(2)}(t) = \frac{C_2 + A_2 \cos\omega t}{1 + B_2 \cos(2\omega t)}, \tag{7}$$

where $x_{HB}^{(1)}$ and $x_{HB}^{(2)}$ are the representative forms for the odd and mixed parity cases.

The paper by Sarma and Rao [1] investigates the case for which

$$f(x) = \alpha x + \beta x^2 + \gamma x^3 + \delta. \tag{8}$$

Note that equation (1), with this particular function $f(x)$, is of odd parity only if $\beta = \delta = 0$; consequently, the harmonic balance approximation $x_{HB}^{(1)}(t)$ should be used. If this condition is not satisfied, then equation (1) is of mixed parity and $x_{HB}^{(2)}(t)$ should be used for the approximation to the periodic solutions.

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The three cases of equation (8) considered by Sarma and Rao can now be discussed in light of the above discussion.

Case I: $\alpha = 0$, $\beta = 0$, $\gamma = 1$, $\delta = 0$

This is an odd parity situation since $f(x) = x^3$. Therefore, $x_{HB}^{(1)}(t)$ from equation (6) must be used. Sarma and Rao started with $x_{HB}^{(2)}(t)$, but found that the constant $C_2 = 0$. Under this condition, their assumed harmonic balance expression reduces to the form given by $x_{HB}^{(1)}(t)$. Consequently, they obtained the previous excellent result of Mickens [5] for the angular frequency ω.

Case II: $\alpha = 1$, $\beta = -0{\cdot}2$, $\gamma = 0$, $\delta = -1$ and *Case III*: $\alpha = 1$, $\beta = -2{\cdot}2518$, $\gamma = 2{\cdot}54328$, $\delta = 0$

These two cases are of mixed parity and $x_{HB}^{(2)}(t)$ must be used. In fact, Sarma and Rao obtained very accurate values for ω by using $x_{HB}^{(2)}(t)$, and inaccurate values applying the function $x_{HB}^{(1)}(t)$. However, based on our earlier analysis, this is exactly what should be expected.

In summary, the inappropriate application of rational harmonic balance approximations can lead to large errors in the determination of the angular frequency ω for the periodic solutions to equation (1). For odd parity differential equations, $x_{HB}^{(1)}(t)$ must be used; likewise, $x_{HB}^{(2)}(t)$ is the proper form to use for the mixed parity case. These particular restrictions are based on mathematical reasoning which leads to the conclusion that odd parity equations have periodic solutions which contain only odd multiples of ω; there are no even angular frequencies in the Fourier series representation of the periodic solutions. Now the function $x_{HB}^{(1)}(t)$ has this property, but not $x_{HB}^{(2)}(t)$. For the mixed parity case, the periodic solutions to equation (1) have both even and odd multiples of ω, a feature shared by $x_{HB}^{(2)}(t)$, but not by $x_{HB}^{(1)}(t)$. Thus, the results obtained by Sarma and Rao [1] can be clearly and easily understood as consequences of this analysis.

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AUTHORS' REPLY

M. S. SARMA

Applied Mathematics Division

AND

B. NAGESWARA RAO

Structural Engineering Group, Vikram Sarabhai Space Centre, Trivandrum 695 022, *India*

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We are pleased to note a few comments by Professor Mickens [1] on our article [2], wherein a modification in the rational function of Mickens and Semwogerere [3] is proposed for the approximate periodic solution of the non-linear one-dimensional oscillator differential equation. He has appreciated the contents of our paper and its style of presentation, further elucidating it for the benefit of the readers. Although, Mickens and Semwogerere [3] write the Fourier expansion of the periodic solution to the equation of motion, the rational function in equation (3) of reference [4] does not accommodate the constant term of the above Fourier series. Hence, we proposed a modified rational function as given in equation (7) of reference [1], which is suitable for the restoring force function in the equation of the motion, either odd or mixed-parity. The constant, C_2 in equation (7) of reference [1] automatically vanishes for an odd restoring force function, as can be noted from equation (7) of reference [2]. It is not necessary to have different forms of the rational function when the restoring force function is odd or of mixed-parity. Equation (7) of reference [1] proposed by us, is reasonably accurate for obtaining periodic solutions of the Duffing equation.

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