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A STUDY ON MODE LOCALIZATION PHENOMENA: INFLUENCES OF THE STIFFNESS AND THE MASS OF THE COUPLER ON MODE LOCALIZATION

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The influences of the stiffness and the mass that constitute a coupler between two substructures of the periodic structures on mode localization are studied theoretically, and the results are confirmed by numerical examples. The results of this study show that the mass as well as the stiffness of the coupler has significant influences on mode localization and weak coupling conditions. The mass of the coupler makes a periodic structure sensitive to mode localization especially in higher modes while the stiffness does in all modes. By introducing the large mass and the large stiffness into the coupler, an interesting phenomenon of delocalization can be observed in some modes for which mode localization does not occur or is very weak although structural disturbances are severe. A simple structure consisting of two substructures, each with a lumped mass and two stiffnesses, and a coupler is analysed theoretically. For example, structures for numerical analysis, simply supported continuous two- and three-span beams with couplers having a rotational stiffness and a rotational mass are considered.

1. INTRODUCTION

The natural frequencies and mode shapes are the important dynamic characteristics of structural systems, which are functions of the geometric configuration and the material properties of the structures. Many engineering structures are made of identical substructures, and many useful results can be generated by a dynamic analysis that is based on the perfect periodicity assumption. However, in real structures, no substructures will be perfectly identical with each other. And the presence of small irregularities in nominally periodic structures may significantly affect their dynamic responses and lead to mode localization. In the area of structural dynamics, the term ‘mode localization’ refers to the vibration confinement, i.e. the magnitude of the specific part of the free vibrational mode is large relative to the rest of the mode. This means that the vibration energy of the mode is confined to that region.

Identifying the mode localization is very important to the design of the efficient motion controller. For periodic structures, some irregularities such as structural damages or manufacturing errors may produce undesirable or unpredicted mode localization. For

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flexible structural systems, natural frequencies and mode shapes must be calculated with high precision to design the efficient motion controller. The mode shapes, however, may be changed extremely by small disturbances in the system parameters when the mode localization occurs drastically. This may be the reason why the performance of the controller deteriorates. It is, therefore, very important not only to calculate the natural frequencies and the mode shapes but also to identify the occurrence of mode localization and its degree caused by small changes in system parameters.

In solid-state physics, the localization phenomenon of electron field in disordered solid was first observed by Anderson [1], Anderson and Mott [2] shared the 1977 Nobel Prize in physics for their work in this area. Early localization work in structural dynamics dealt with the field of turbomachinery. Many works were concerned with cyclically symmetric structures with weak coupling in order to explain the unpredicted fatigue failure of the mistuned blades of turbomachinery [3–5]. Hodges [6] was the first to recognize that wave localization may occur in the disordered periodic structures and leads to mode localization. Wave localization is a phenomenon in which the vibrational energy imparted to the structure by an external source cannot propagate to arbitrary long distances but is instead substantially confined to a region close to the source. In his work, a series of coupled pendulums and a vibrating string with point masses and springs, were used to show that the degree of localization was strongly influenced by the ratio of disorder strength to coupling strength. After that, there have been many studies on localization in periodic engineering structures.

Bendiksen [7], and Cornwell and Bendiksen [8–10] investigated the localization behaviour in a cyclically symmetric large space structure using analytical and numerical methods. In these works, the sensitivities of eigenvalues and eigenvectors to small amounts of disorder and the precise nature of coupling effects that cause localization are studied. And a measure for the extent of localization was introduced.

Pierre and Dowell [11] employed a regular perturbation method to obtain the localized modes of a vibrating disordered system of coupled pendula. Pierre *et al*. [12] studied the mode localization of a weakly coupled disordered two-span beam using the modified perturbation method and the experimental method. In their work, influences of the structural disorder and coupling strength on mode localization were studied. The structural disorder was introduced by perturbing the position of the centre support from the midpoint of the beam. And the variation of the coupling strength between the spans was realized by modifying the stiffness of a centre support rotational spring connecting the beam to ground. They concluded that the degree of localization depended on the ratio of disorder strength to coupling strength. The results are consistent with those of Hodges and Bendiksen. Bouzit and Pierre [13] demonstrated weak and strong localization behaviours and calculated the localization factor for a multi-span beam on randomly spaced simple supports. The localization factor is defined by the average exponential rate at which a structural wave decays with respect to the wave propagation distance in a disordered periodic structure.

Lust *et al*. [14] studied the influences of various effects on mode localization in multispan beams such as Timoshenko beam effects, boundary condition effects, viscous damping effects, axial force effects, and transverse and rotational stiffness at the centre support. Delocalization phenomena were observed by them for the first time. They concluded that the Timoshenko beam formulation has a significant influence on mode localization for higher modes and span length imperfection is more important than any other imperfection parameters. The delocalization phenomenon is influenced by interaction of bending and shear modes, beam length imperfection and transversely supported stiffness. Vakakis [15], Vakakis *et al*. [16] and Zevin [17] studied mode localization for non-linear systems. Many other works have been conducted for mode localization and wave localization of periodic structures [18–28].

Various structures have been considered and many methods have been proposed to discuss the occurrence of mode localization. It is well known that under conditions of weak internal coupling, the mode shapes undergo dramatic changes to become strongly localized when a small disorder is introduced. In previous works, it was confirmed that the coupling strength is closely related to the strength of coupling stiffness between substructures. To date, however, little attention has been paid to the influences of the mass of coupler between the spans or substructures on coupling strength and mode localization. Many engineering structures have the form that periodically supported and stiffened. If the mass of the stiffner is large relative to that of subparts between the stiffeners, the influences of the mass on mode localization must be considered.

The present study is an attempt to prove that the mass, as well as the stiffness, of couplers exerts an important influence upon the occurrence of mode localization and the weak coupling conditions in periodic structures. To accomplish this objective, a dynamic analysis of a simple structure with two substructures, each with a lumped mass and two stiffnesses, and a coupler, is performed and the influences of the mass and the stiffness of the coupler on mode localization are qualitatively discussed in the theoretical background section. In the numerical examples, mode localization of simply supported continuous multispan beams are analysed under various coupling conditions, and the results of the theoretical approach are confirmed. A measure for degree of mode localization is newly defined and used in the numerical examples, since the measure used in previous works [12, 14, 15, 21, 27] is not suitable for the cases of multispan beams.

2. THEORETICAL BACKGROUND

In this section the characteristics of mode localization of a simple structure consisting of two substructures and a coupler is discussed qualitatively. Figure 1 shows the structure considered. The equation of motion of the structure may be written as

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} + \begin{bmatrix} k_1 + k_3 & 0 & -k_3 \\ 0 & k_2 + k_4 & -k_4 \\ -k_3 & -k_4 & k_3 + k_4 + k_5 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{1}$$

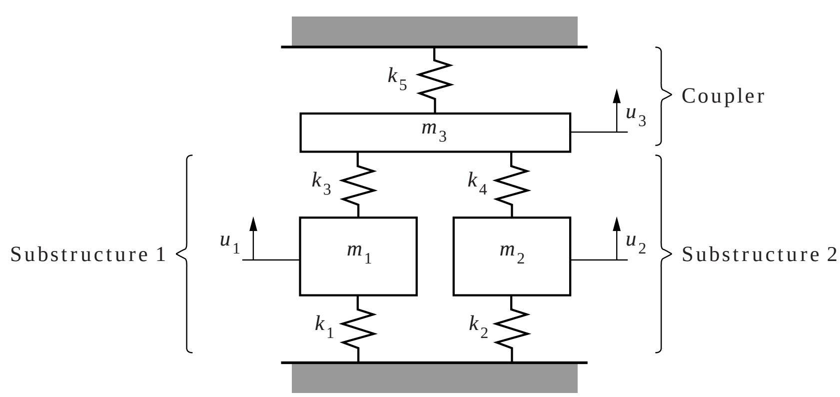

Figure 1. Simple structure constituted with two substructures and a coupler.

Assuming $\{u(t)\} = \exp(i\omega_n t)\{y\}$ where ω_n is circular natural frequency and $\{y\}$ is normal mode of the system, and substituting it into equation (1), yields an eigenvalue problem for free vibration of the structure as in equation (2).

$$\begin{bmatrix} k_1 + k_3 - \lambda m_1 & 0 & -k_3 \\ 0 & k_2 + k_4 - \lambda m_2 & -k_4 \\ -k_3 & -k_4 & k_3 + k_4 + k_5 - \lambda m_3 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{2}$$

where λ denotes an eigenvalue or the square of circular natural frequency of the structure. To have non-trivial solutions, the determinant of the coefficient matrix of equation (2) should be zero, that is,

$$(\lambda^{(1)} - \lambda)(\lambda^{(2)} - \lambda)(k_3 + k_4 + k_5 - \lambda m_3) - \frac{k_4^2}{m_2}(\lambda^{(1)} - \lambda) - \frac{k_3^2}{m_1}(\lambda^{(2)} - \lambda) = 0 \tag{3}$$

where $\lambda^{(1)}$ and $\lambda^{(2)}$ are the eigenvalues of the substructure 1 and 2 respectively:

$$\lambda^{(1)} = \frac{k_1 + k_3}{m_1}, \qquad \lambda^{(2)} = \frac{k_2 + k_4}{m_2}. \tag{4, 5}$$

Equation (3) is called frequency equation or characteristic equation of the structure. By solving equation (8), the eigenvalues, λs, of the structure can be determined.

Degree of mode localization may be assessed using a ratio of free vibration amplitude of the two substructures. To compare the free vibrational amplitudes of the two substructures with each other and to get an equation for the ratio, the degree of freedom of the coupler y_3 in equation (2) is eliminated first and we get

$$\begin{bmatrix} \lambda^{(1)} - \dfrac{k_3^2/m_1}{k_3 + k_4 + k_5 - \lambda m_3} - \lambda & -\dfrac{k_3 k_4/m_1}{k_3 + k_4 + k_5 - \lambda m_3} \\ -\dfrac{k_3 k_4/m_2}{k_3 + k_4 + k_5 - \lambda m_3} & \lambda^{(2)} - \dfrac{k_4^2/m_2}{k_3 + k_4 + k_5 - \lambda m_3} - \lambda \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

Rewriting equation (6) yields

$$\lambda^{(1)} - \lambda - \frac{k_3^2/m_1}{k_3 + k_4 + k_5 - \lambda m_3} = \frac{k_3 k_4/m_1}{k_3 + k_4 + k_5 - \lambda m_3} \frac{y_2}{y_1}, \tag{7}$$

$$\lambda^{(2)} - \lambda - \frac{k_4^2/m_2}{k_3 + k_4 + k_5 - \lambda m_3} = \frac{k_3 k_4/m_2}{k_3 + k_4 + k_5 - \lambda m_3} \frac{y_1}{y_2}. \tag{8}$$

Subtracting equation (8) from equation (7) and rearranging it gives

$$(k_3 + k_4 + k_5 - \lambda m_3)(\lambda^{(1)} - \lambda^{(2)}) - \left(\frac{k_3^2}{m_1} - \frac{k_4^2}{m_2}\right) = k_3 k_4 \left(\frac{1}{m_1}\frac{y_2}{y_1} - \frac{1}{m_2}\frac{y_1}{y_2}\right). \tag{9}$$

Multiplying each side of equation (9) by y_2/y_1 and simplifying it results in

$$(r - s_1)(r - s_2) = \alpha r \tag{10}$$

where

$$r = \frac{y_2}{y_1}, \qquad s_1 = -\frac{k_3}{k_4}, \quad s_2 = \frac{k_4}{k_3}\frac{m_1}{m_2} \tag{11, 12, 13}$$

and

$$\alpha = \frac{m_1}{k_3k_4}(k_3 + k_4 + k_5 - \lambda m_3)(\lambda^{(1)} - \lambda^{(2)}). \tag{14}$$

The free vibration amplitude ratio r of the two substructures is to be a measure for degree of mode localization. The left-hand side of equation (10) is a parabolic function having roots at s_1 and s_2, and the right-hand side is a line which passes origin with slope α as shown in Figure 2.

Using equation (10) and Figure 2, it is simple to discuss the influences of the stiffness and the mass of the coupler on mode localization phenomenon. The solutions of equation (10), r_1 and r_2, are determined by the functions on each side of the equation. If m_1, k_1 and k_3 are equal to m_2, k_2 and k_4 respectively, then both $|r_1|$ and $|r_2|$ are unity since $s_1 = -1$, $s_2 = 1$ and $\alpha = 0$ in equation (10) and Figure 2. That is, if the two substructures are identical to each other, then there is no localization. However, small disturbances introduced into the structure, such as small variations in the masses and/or the stiffnesses of the substructures, make the substructures different to each other and a mode becomes a localized one. Especially if r_1 and r_2 become zero and infinite, respectively or reversely, by small disturbances introduced into the substructures, then the corresponding mode becomes the perfectly localized one, i.e. one of y_1 and y_2 is zero and the other finite.

As one can see in Figure 2, r_1 and r_2 are strongly affected by the slope of the line. If α becomes positive or negative infinite by the small structural disturbances, then the degree of mode localization increases drastically and it can be said that the structure is very sensitive to mode localization. The influences of the stiffness and the mass of the coupler on mode localization are discussed by using the variation of α caused by the structural disturbance introduced into the structure. And for simple discussion, the structural disturbancers are realized by the variation of difference, $\lambda_d = \lambda^{(1)} - \lambda^{(2)}$, of the eigenvalues of the two substructures, and it is assumed that λ_d is equal to zero when the structure is a perfect and undisturbed one.

To compare the results of this study with those of previous works in which the effects of the stiffness of coupler were studied, a case of $m_3 = 0$ is considered first and the effect of the stiffness of coupler on mode localization is discussed. Neglecting the mass of coupler, equation (14) becomes equation (15),

$$\alpha = m_1\left(\frac{k_3 + k_4}{k_3k_4} + \frac{k_5}{k_3k_4}\right)\lambda_d. \tag{15}$$

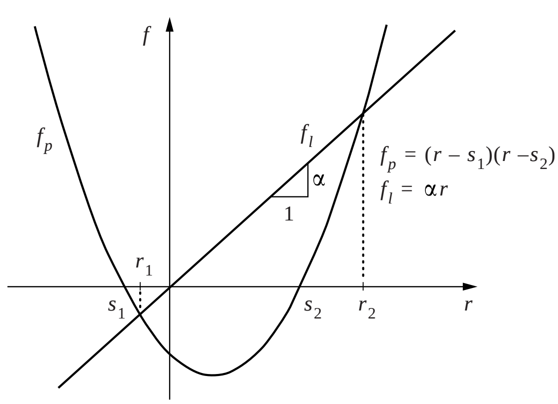

Figure 2. The two curves.

From equation (15), it is obvious that if $k_5/k_3k_4 \gg 1$, then the small variation in λ_d may lead to the significant change in α. That implies that when the ratio of k_5 to k_3k_4 is very large, the small disturbance is to be a cause of drastic occurrence of mode localization since it makes r_1 and r_2 become zero and infinite, respectively or reversely. These are the same conditions that discussed in previous research for weak couplings that are the pre-conditions for the drastic occurrence of mode localization by the structural disturbances. The high strength of stiffness of the coupler relative to the stiffnesses of the substructures $k_5 \gg k_3k_4$, makes the structure sensitive to mode localization and satisfies the coupling weak condition.

By considering equation (16) including the term of λm_3, an additional pre-condition for the drastic occurrence of mode localization may be derived.

$$\alpha = \frac{m_1}{k_3k_4}(k_3 + k_4 + k_5 - \lambda m_3)\lambda_d. \tag{16}$$

As one can see in equation (16), if the condition of $\lambda m_3 \gg k_3 + k_4 + k_5$ is satisfied, the small change in λ_d leads to the significant change in α and in the degree of mode localization. For a case of $\lambda m_3 \ll k_3 + k_4 + k_5$, the term of λm_3 can be neglected and reduced to the case considered in the previous paragraph. Therefore, $\lambda m_3 \gg k_3 + k_4 + k_5$ is an additional condition for the weak coupling and a pre-condition for the drastic occurrence of mode localization by the structural disturbances. The large mass of the coupler and/or the large eigenvalue of the structure make the structure sensitive to mode localization and the coupling weak.

If one considers the mass of the coupler, another interesting characteristic of mode localization can be observed. If the eigenvalue of the system is close to $\lambda^{(3)}$ or equation (17) is satisfied, then the structure is not sensitive to mode localization although the weak coupling condition, such as $k_5 \gg k_3k_4$, is satisfied.

$$\lambda \cong \lambda^{(3)} \tag{17}$$

where $\lambda^{(3)}$ is the eigenvalue of the coupler and may be given by

$$\lambda^{(3)} = \frac{k_3 + k_4 + k_5}{m_3}. \tag{18}$$

This condition may lead to a delocalization phenomenon. That is, for a mode of which the natural frequency is close to that of the coupler, mode localization does not occur or is very weak although structural disturbances are severe. This delocalization phenomenon caused by the mass and stiffness of the coupler is observed in this study for the first time.

3. NUMERICAL EXAMPLES

Here, influences of the stiffness and the mass of the couplers on mode localization of multispan beams are verified using the results of dynamic analysis by the numerical method, and the results of the previous section are confirmed. As example structures, simply supported continuous two- and three-span beams with couplers on supports as shown in Figures 3 and 8 are considered. The two-span beam with a rotational stiffness at the midsupport is the most popular structure in the field of mode localization since it is very simple to analyse and it shows clearly many characteristics of mode localization. In this study the two-span beam with a rotational stiffness at the midsupport is considered also as an example and the results of the previous works were confirmed. However, if the mass of the coupler is considered, interesting results predicted in the previous section can be drawn. Mode localization characteristics of a general periodic multispan beam and the

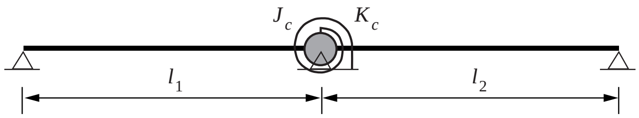

Figure 3. Simply supported continuous two-span beam with a coupler of the rotational stiffness and the mass at the midsupport.

influence of couplers can be shown in the second example of the three-span beam. The natural frequencies and the mode shapes of the perfect structures and the disturbed ones are computed by the finite element method. The structure having the same geometry and material proerties in all spans, is a perfect structure. The structural disorders or disturbances are realized by introducing the length variation into the last span for each example structure. The rotational stiffnesses and the rotational masses of the couplers are represented by the non-dimensional quantities;

$$\bar{K}_c = \frac{K_c l}{EI}, \tag{20}$$

$$\bar{J}_c = 6\,\frac{J_c}{Ml^2} \tag{21}$$

where K_c and J_c are the rotational stiffness and the rotational mass of the couplers respectively, EI is the flexural rigidity and l the span length of the perfect structures. Degrees of mode localization are computed and compared for various coupling conditions. A measure for degree of mode localization is newly defined and used in this section. A measure used in previous works [12, 14, 15, 21, 27] is not convenient for measuring and comparing the degree of mode localization for the cases of multispan beams with generality, since it was defined using only two of the maximum amplitudes associated with the spans.

3.1. MEASURE OF DEGREE OF MODE LOCALIZATION

In this study, to facilitate discussion for mode localization of multispan beams, a measure for the degree of mode localization (DML) is newly defined here as

$$DML \equiv \frac{m - m_c}{m - 1} \tag{22}$$

where m is the total number of spans and m_c is the number of spans in which vibrations are confined ($1 \leqslant m_c \leqslant m$). m_c can be computed by

$$m_c = \frac{\left(\sum_{i=1}^{m} \bar{y}_i\right)^2}{\sum_{i=1}^{m} \bar{y}_i^2} \tag{23}$$

where $\bar{y}_1$ is the absolute value of the maximum amplitude associated with the ith span. m_c is m when vibrations in all spans have the same amplitude, and unity when the vibration is confined within only one span. As one can see, the degree of mode localization is determined between zero and unity, $0 \leqslant DML \leqslant 1$. If DML is equal to unity, then the mode is extremely localized, and if DML is equal to zero, then the mode is not localized at all.

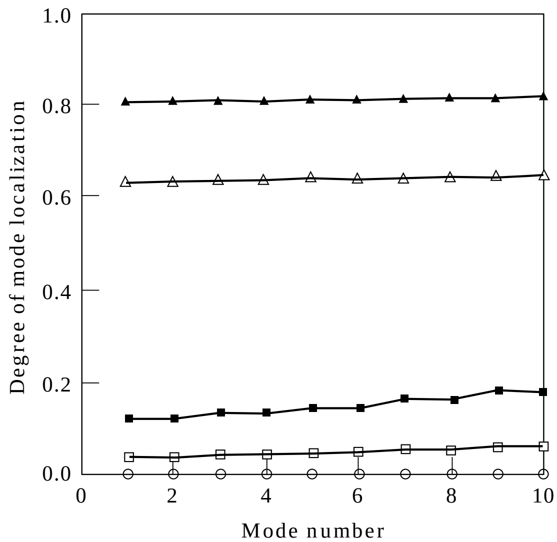

Figure 4. Localization curves showing the influence of the rotational stiffness of the coupler and the disturbance introduced into the second span on mode localization of the simply supported two-span beams. ○, Perfect structures; □, $\bar{K}_c = 100$, $\Delta l_3 = 0{\cdot}5\%$; ■, $\bar{K}_c = 100$, $\Delta l_3 = 1{\cdot}0\%$; △, $\bar{K}_c = 1000$, $\Delta l_3 = 0{\cdot}5\%$; ▲, $\bar{K}_c = 1000$, $\Delta l_3 = 1{\cdot}0\%$.

The span response ratio $\bar{A}$, a measure for mode localization used in previous works [12, 14, 21] may be written as

$$\bar{A} = \left|\frac{a}{b}\right| \tag{24}$$

where a is the maximum displacement associated with the span in which the response is smaller, and b the maximum displacement associated with the span in which the response is larger. A mode was considered as localized when $\bar{A} < 0{\cdot}1$ in previous works. To be consistent with that terminology, a mode is considered as localized when $DML > 0{\cdot}8$.

3.2. SIMPLY SUPPORTED CONTINUOUS TWO-SPAN BEAM

In this example, the simply supported continuous two-span beam with a coupler that consists of the rotational stiffness, K_c, and the rotational mass, J_c, is considered. Figure 3 shows the geometry of the example structure. Finite element method is used for the free vibration analysis of the structure. The influences of the rotational stiffness and the mass that constitute a coupler between two spans on mode localization occurred by the length disturbance in the second span are investigated. Excepting a rotational mass of the coupler, the structure has the same geometry studied in previous works [12, 14]. Young's modulus of the beam is $E = 30 \times 10^6$ lbf/in^2, the mass density $\rho = 0{\cdot}28$ lb/in^3, and the moment of inertia $I = 6{\cdot}51 \times 10^{-4}$ in^4. The span lengths are $l_1 = l_2 = 12$ in. in the undisturbed perfect structures.

To study the individual influences of K_c and J_c on mode localization, six cases are considered in each study. And to study the combined influences of K_c and J_c on mode localization, six cases are considered. In each case, the structural disturbance is realized by increasing the length of the second span. Degrees of mode localization are computed by using equation (22) and plotted as functions of mode number and disturbance for the

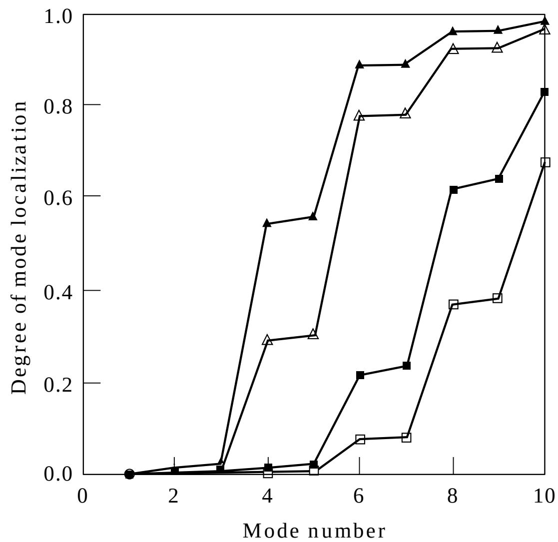

Figure 5. Localization curves showing the influence of the rotational mass of the coupler and the disturbances introduced into the second span on mode localization of the simply supported two-span beams. ○, Perfect structures; □, $\bar{J}_c = 0{\cdot}1$, $\Delta l_3 = 0{\cdot}5\%$; ■, $\bar{J}_c = 0{\cdot}1$, $\Delta l_3 = 1{\cdot}0\%$; △, $\bar{J}_c = 1{\cdot}0$, $\Delta l_3 = 0{\cdot}5\%$; ▲, $\bar{J}_c = 1{\cdot}0$, $\Delta l_3 = 1{\cdot}0\%$.

lowest ten modes of cases in Figures 4, 5 and 6. The first ten mode shapes of the perfect structures and the imperfect ones for selected cases are shown in Figure 7, and the first ten natural frequencies of the perfect ones are given in Table 1.

Influences of the stiffness of the coupler on mode localization

The influences of K_c on mode localization are studied by considering six cases. In the first three cases are $\bar{K}_c = 100$, and in the other cases $\bar{K}_c = 1000$. As structural disturbances,

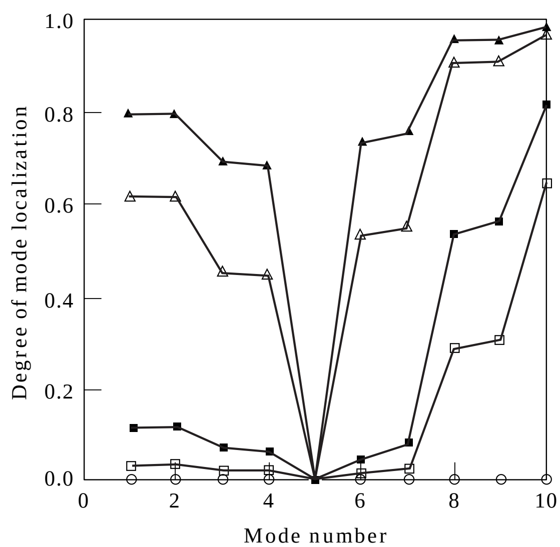

Figure 6. Localization curves showing the influence of the rotational stiffness and the rotational mass of the coupler disturbances introduced into the second span on mode localization of the simply supported two-span beams. ○, Perfect structure; □, $\bar{K}_c = 100$, $\bar{J}_c = 0{\cdot}1$, $\Delta l_3 = 0{\cdot}5\%$; ■, $\bar{K}_c = 100$, $\bar{J}_c = 0{\cdot}1$, $\Delta l_3 = 1{\cdot}0\%$; △, $\bar{K}_c = 1000$, $\bar{J}_c = 1{\cdot}0$, $\Delta l_3 = 0{\cdot}5\%$; ▲, $\bar{K}_c = 1000$, $\bar{J}_c = 1{\cdot}0$, $\Delta l_3 = 1{\cdot}0\%$.

the second span length increments of $\Delta l_2 = 0{\cdot}0\%$, $\Delta l_2 = 0{\cdot}5\%$ and $\Delta l_2 = 1{\cdot}0\%$ are considered in each $\bar{K}_c$. If $\Delta l_2 = 0{\cdot}0\%$, the structure is perfect.

Localization curves shown in Figure 4 indicate the typical localization behaviour and influences of $\bar{K}_c$ on mode localization. Degrees of mode localization increase with increasing disturbance, and with increasing $\bar{K}_c$.

Influences of the mass of the coupler on mode localization

Studies on the influences of J_c are conducted by considering cases for $\bar{J}_c = 0{\cdot}1$ and $\bar{J}_c = 1{\cdot}0$. In each case the span length increments of $\Delta l_2 = 0{\cdot}0\%$, $\Delta l_2 = 0{\cdot}5\%$ and $\Delta l_2 = 1{\cdot}0\%$ are considered where $\Delta l_2 = 0{\cdot}0\%$ is for the cases of perfect structures.

Figure 5 presents localization curves for lowest ten modes of cases, which show the influence of $\bar{J}_c$ on mode localization. Influence of $\bar{J}_c$ differs from that of $\bar{K}_c$. Degrees of mode localization increase with increasing length disturbance in the second span, with increasing mode number, and with increasing the mass of the coupler. It is interesting that degrees of mode localization are very small in lower modes, but they are significant in higher modes. As predicted in the section of theoretical background, this behaviour becomes more pronounced with increasing rotational mass of the coupler. The coupler of large mass makes the coupling weak and the structure sensitive to mode localization with increasing mode number.

Combined influences of the stiffness and the mass of the coupler on mode localization

The combined influences of the stiffness and the mass of the coupler on mode localization are also studied by considering cases of $\bar{K}_c = 100$ and $\bar{J}_c = 0{\cdot}1$, and $\bar{K}_c = 1000$ and $\bar{J}_c = 1{\cdot}0$. In each case the span length disturbances of $\Delta l_2 = 0{\cdot}0\%$, $\Delta l_2 = 0{\cdot}5\%$ and $\Delta l_2 = 1{\cdot}0\%$ are also considered.

Localization curves plotted in Figure 6 show the combined influences of $\bar{K}_c$ and $\bar{J}_c$ on mode localization. Degrees of mode localization decrease with increasing mode number until the fifth mode, but after that mode they increase abruptly with increasing mode number. The fifth mode is a delocalized one, and its frequency is close to the coupler's frequency 1780 Hz calculated by

$$\omega_c = \frac{1}{2\pi}\sqrt{\frac{K_c}{J_c}}, \tag{25}$$

where ω_c is the coupler's frequency, and neglecting k_3 and k_4 in equation (18), ω_c is equivalent to $\lambda^{(3)}$ in the theoretical background section.

The delocalization phenomenon is more drastic in cases of $\bar{K}_c = 1000$ and $\bar{J}_c = 1{\cdot}0$. The modes that are close to the frequency of the coupler or on it are delocalized, and for those modes mode localization does not occur or is very weak although the structural disturbance is severe. The localization behaviour is governed by $\bar{K}_c$ for lower modes but by $\bar{J}_c$ for higher modes on condition that the modes are far from the localized ones. These results are consistent with the results of the previous section on theoretical background.

3.3. SIMPLY SUPPORTED CONTINUOUS THREE-SPAN BEAM

In this example, the simply supported continuous three-span beam with couplers consisting of the rotational stiffness and the mass is considered, and finite element method is used for the free vibration analysis of the perfect structures and the imperfect ones. The influences of the rotational stiffness and the mass on mode localization are studied also. Figure 8 shows the geometry of the example structure. As one can see, each span has

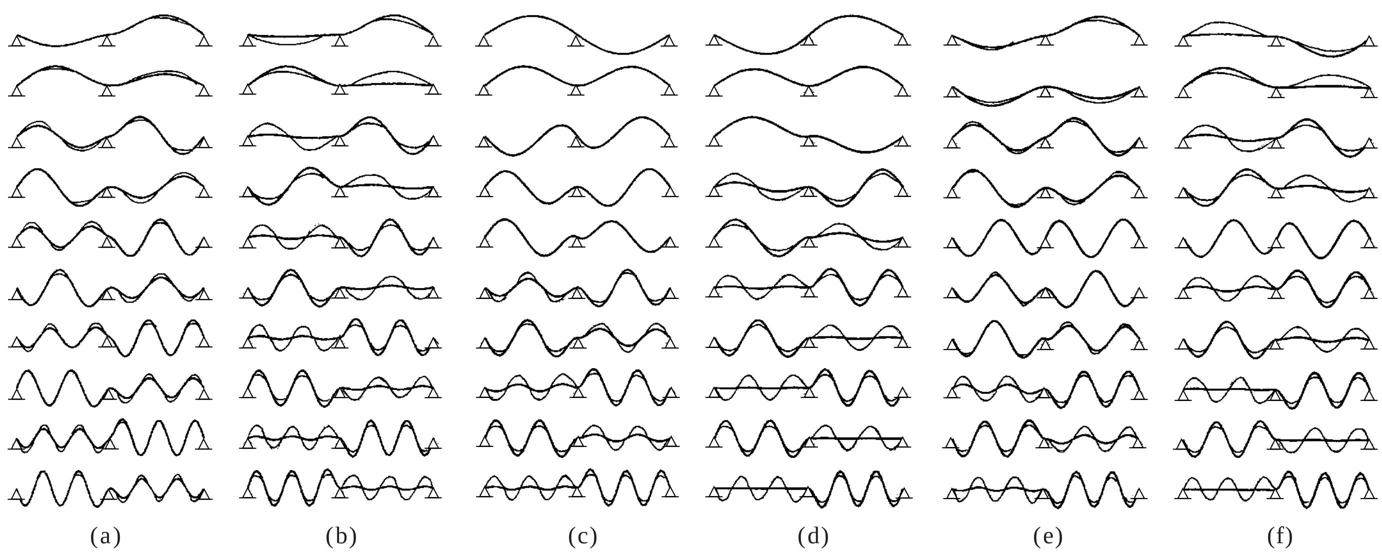

Figure 7. Localized or non-localized first ten mode shapes for selected cases for the simply supported continuous two-span beams with various coupling conditions. Mode shapes of perfect systems (——) and disturbed systems (——) are depicted in ascending order from the top to the bottom; (a) $\bar{K}_c = 100$, $\Delta l_2 = 1{\cdot}0\%$; (b) $\bar{K}_c = 1000$, $\Delta l_2 = 1{\cdot}0\%$; (c) $\bar{J}_c = 0{\cdot}1$, $\Delta l_2 = 1{\cdot}0\%$; (d) $\bar{J}_c = 1{\cdot}0$, $\Delta l_2 = 1{\cdot}0\%$; (e) $\bar{K}_c = 100$, $\bar{J}_c = 0{\cdot}1$, $\Delta l_2 = 1{\cdot}0\%$; and (f) $\bar{K}_c = 1000$, $\bar{J}_c = 1{\cdot}0$, $\Delta l_2 = 1{\cdot}0\%$.

TABLE 1

First ten natural frequencies (Hz) of the simply supported continuous two-span beams with various coupling conditions

	Stiffness effects		Mass effects		Mass and stiffness effects	
Mode	$\bar{K}_c = 10^2$ $\bar{J}_c = 0{\cdot}0$	$\bar{K}_c = 10^3$ $\bar{J}_c = 0{\cdot}0$	$\bar{K}_c = 0{\cdot}0$ $\bar{J}_c = 0{\cdot}1$	$\bar{K}_c = 0{\cdot}0$ $\bar{J}_c = 1{\cdot}0$	$\bar{K}_c = 10^2$ $\bar{J}_c = 0{\cdot}1$	$\bar{K}_c = 10^3$ $\bar{J}_c = 1{\cdot}0$
1	341·0	352·3	208·7	122·8	340·5	352·3
2	353·7	353·7	353·7	353·7	353·7	353·7
3	1106·1	1140·5	638·8	391·2	1088·0	1137·5
4	1145·0	1145·0	1145·0	1145·0	1145·0	1145·0
5	2308·0	2375·3	1274·0	1156·3	1868·7	1790·3
6	2384·6	2384·6	2384·6	2384·6	2384·6	2384·6
7	3944·0	4051·9	2441·9	2390·0	2508·1	2396·6
8	4067·6	4067·6	4067·6	4067·6	4067·6	4067·6
9	6009·2	6163·5	4099·8	4070·7	4107·5	4071·4
10	6187·1	6187·1	6187·1	6187·1	6187·1	6187·1

rotational stiffnesses, $0{\cdot}5K_c$, and masses, $0{\cdot}5J_c$, at both ends as coupler. Material properties of the beam are the same with those of the previous example structure. For perfect structures, span lengths are $l_1 = l_2 = l_3 = 12$ in.

The studies to determine the individual influences and the combined influences of K_c and J_c on the mode localization of the three-span beam are conducted by considering several cases. The disturbances are realized by introducing the span length variations into the third span and classified into decreasing case and increasing case since mode localization behaves with different manner in each case. The length variations of only the third span are considered since the influences show the same characteristics in all cases although a localization curve has a different shape according to the span number into which disturbances are introduced. Degrees of mode localization of the first twenty modes of each case are computed by equation (22) and plotted as functions of mode number and disturbance for the lowest ten modes of cases in Figures 9, 10 and 11. The first ten mode shapes are represented in Figure 12 for selected shortening cases, and in Figure 13 for selected lengthening ones. The natural frequencies of the first twenty modes of the perfect structures are presented in Table 2.

Influences of the stiffness of the couplers on mode localization

To study the influences of K_c on mode localization, the cases of $\bar{K}_c = 100$ and $\bar{K}_c = 1000$ are considered. The disturbances considered in the case of $\bar{K}_c = 100$ are $\Delta l_3 = -2{\cdot}5\%$ and $\Delta l_3 = -5{\cdot}0\%$ for shortening cases, and $\Delta l_3 = 2{\cdot}5\%$ and $\Delta l_3 = 5{\cdot}0\%$ for lengthening cases. The disturbances considered in the case of $\bar{K}_c = 1000$ are $\Delta l_3 = -2{\cdot}5\%$ and $\Delta l_3 = -0{\cdot}5\%$ for shortening cases, and $\Delta l_3 = 0{\cdot}25\%$ and $\Delta l_3 = 0{\cdot}5\%$ for lengthening cases.

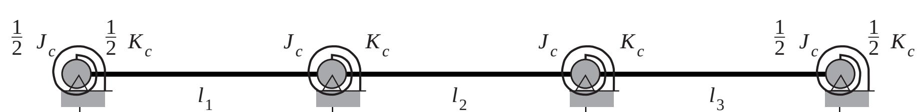

Figure 8. Simply supported continuous three-span beam with couplers of the rotational stiffness and the mass at each support.

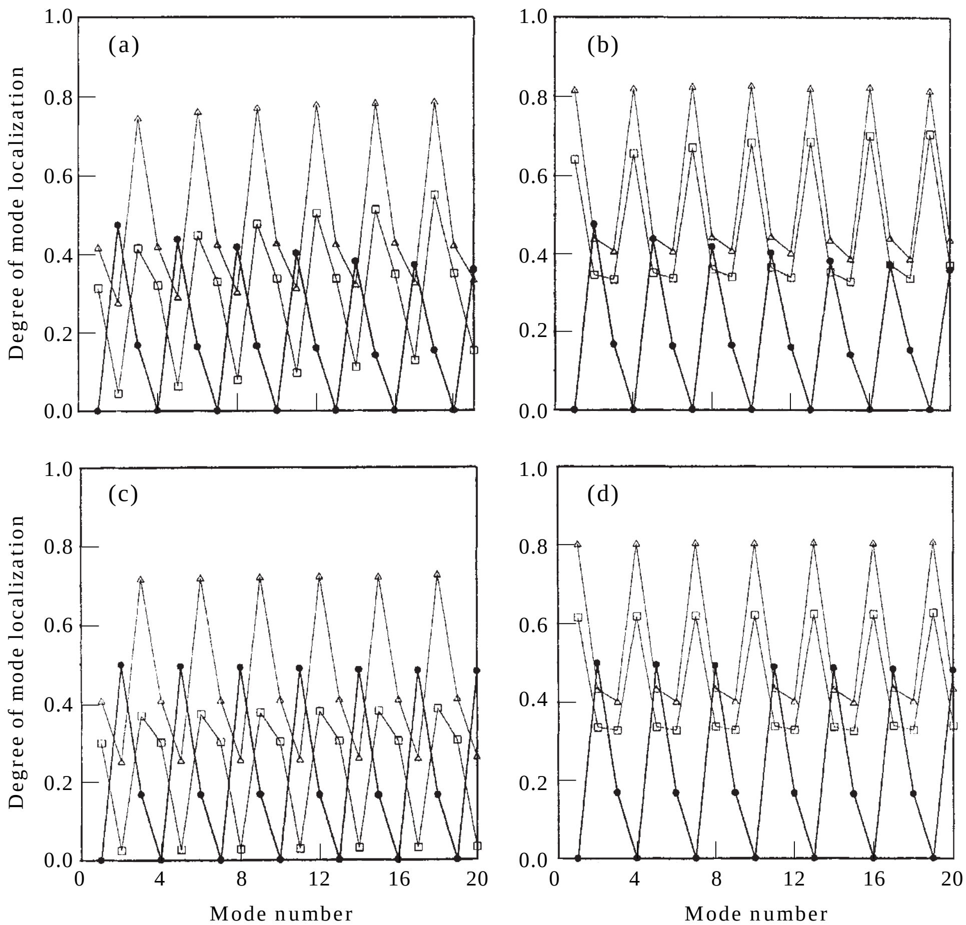

Figure 9. Localization curves showing the influence of the rotational stiffness of the couplers and the disturbances introduced into the third span on mode localization of the simply supported continuous three-span beams. The cases of $\bar{K}_c = 100$ and $\bar{K}_c = 1000$, and shortening and lengthening are considered. (a) $\bar{K}_c = 100$ and shortening cases. ●, Perfect structure; □, $\Delta l_3 = -2{\cdot}5\%$; △, $\Delta l_3 = -5{\cdot}0\%$; (b) $\bar{K}_c = 100$ and lengthening cases. ●, Perfect structure; □, $\Delta l_3 = 2{\cdot}5\%$; △, $\Delta l_3 = 5{\cdot}0\%$; (c) $\bar{K}_c = 1000$ and shortening cases●, Perfect structure; □, $\Delta l_3 = -0{\cdot}25\%$; △, $\Delta l_3 = -0{\cdot}5\%$; (d) $\bar{K}_c = 1000$ and lengthening cases. ●, Perfect structure; □, $\Delta l_3 = 0{\cdot}25\%$; △, $\Delta l_3 = 0{\cdot}5\%$.

The localization curves depicted in Figure 9 show the influences of the stiffness of the couplers and the length disturbances in the third span on mode localization. All the modes can be classified into three localization groups by their degrees of mode localization, and every third mode belongs to a localization group, which is expected since the example structure has three repeated segments. The first localization group is not localized at all, but the rest of the groups are localized to some degree although the structure is perfect. Introducing length disturbances into the third span, each localization group show its own characteristics of mode localization. In addition, the shortening cases and the lengthening cases show the different behavior of mode localization in each localization group. The grouping characteristic is similar to an example of reference [21], where mode localization of a two-span beam was studied using the classical measure for mode localization and the wave approach, and it was shown that the localization curve has two branches and every second mode belongs to a branch.

Considering the case of $\bar{K}_c = 100$ and shortening shown in Figure 9(a), degrees of mode localization of the first and third localization group increase with increasing disturbance, and the shifts in degrees of mode localization of the third localization group is more severe. However, for the second localization group, degrees of mode localization decrease and increase according to the strengths of the disturbance. This is another delocalization phenomenon observed in reference [14] where it is shown that the delocalization can occur for specific combinations of the rotational stiffness and the length disturbance. In the lengthening cases shown in Figure 9(b), degrees of mode localization of the first localization group increase most significantly with increasing strength of disturbance. The shifts of degrees of mode localization of the second and third localization group are much smaller than that of the first localization group. Figure 9(c, d) shows behaviours of degrees of mode localization for the cases of $\bar{K}_c = 1000$. The span length variations of $\Delta l_3 = \pm 0{\cdot}25\%$ and $\Delta l_3 = \pm 0{\cdot}5\%$ are considered as the disturbances. The behaviour of

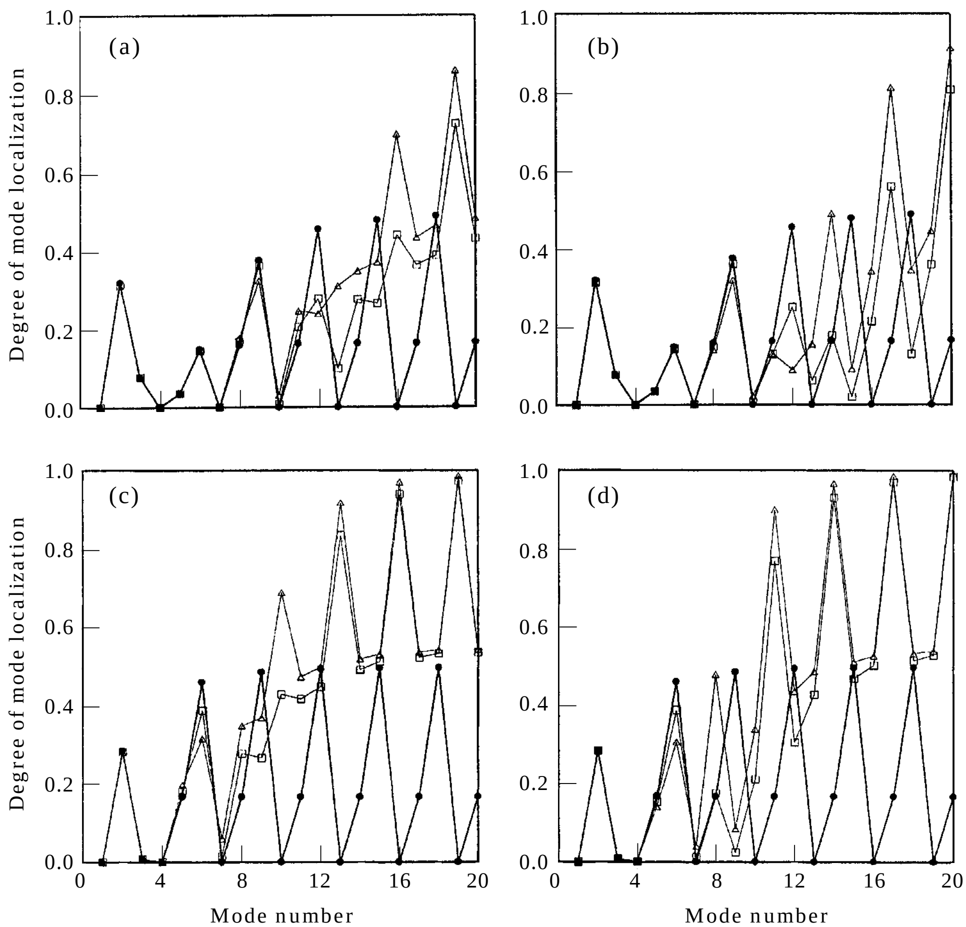

Figure 10. Localization curves showing the influences of the rotational mass of the couplers and the disturbances introduced into the third span on mode localization of the simply supported continuous three-span beams. The cases of $\bar{J}_c = 0{\cdot}1$ and $\bar{J}_c = 1{\cdot}0$, and shortening and lengthening are considered. (a) $\bar{J}_c = 0{\cdot}1$ and shortening cases. ●, Perfect structure; □, $\Delta l_3 = -0{\cdot}25\%$; △, $\Delta l_3 = -0{\cdot}5\%$; (b) $\bar{J}_c = 0{\cdot}1$ and lengthening cases. ●, Perfect structure; □, $\Delta l_3 = 0{\cdot}25\%$; △, $\Delta l_3 = 0{\cdot}5\%$; (c) $\bar{J}_c = 1{\cdot}0$ and shortening cases●, Perfect structure; □, $\Delta l_3 = -0{\cdot}25\%$; △, $\Delta l_3 = -0{\cdot}5\%$; (d) $\bar{J}_c = 1{\cdot}0$ and lengthening cases. ●, Perfect structure; □, $\Delta l_3 = 0{\cdot}25\%$; △, $\Delta l_3 = 0{\cdot}5\%$.

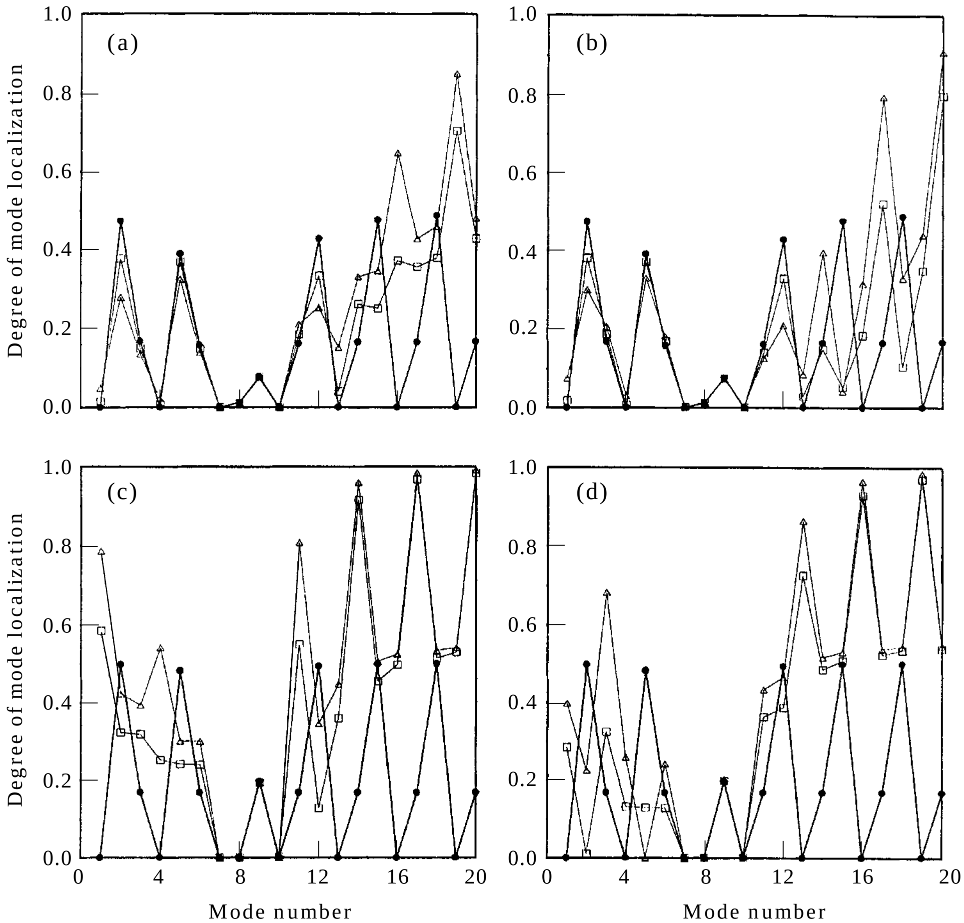

Figure 11. Localization curves showing the combined influence of the rotational mass and the rotational mass of the couplers and the disturbances introduced into the third span on mode localization of the simply supported continuous three-span beams. The cases of $\bar{K}_c = 100$ and $\bar{J}_c = 0{\cdot}1$, and $\bar{K}_c = 1000$ and $\bar{J}_c = 1{\cdot}0$, and shortening and lengthening are considered. (a) $\bar{K}_c = 100$, $\bar{J}_c = 0{\cdot}1$ and shortening cases. ●, Perfect structure; □, $\Delta l_3 = -0{\cdot}25\%$; △, $\Delta l_3 = -0{\cdot}5\%$; (b) $\bar{K}_c = 100$, $\bar{J}_c = 0{\cdot}1$ and lengthening cases. ●, Perfect structure; □, $\Delta l_3 = 0{\cdot}25\%$; △, $\Delta l_3 = 0{\cdot}5\%$; (c) $\bar{K}_c = 1000$, $\bar{J}_c = 1{\cdot}0$ and shortening cases. ●, Perfect structure; □, $\Delta l_3 = 0{\cdot}25\%$; △, $\Delta l_3 = 0{\cdot}5\%$; (d) $\bar{K}_c = 1000$, $\bar{J}_c = 1{\cdot}0$ and lengthening cases. ●, Perfect structure; □, $\Delta l_3 = -0{\cdot}25\%$; △, $\Delta l_3 = -0{\cdot}5\%$.

mode localization is similar to that of $\bar{K}_c = 100$, but more sensitive to the disturbances than that. Considering that, one can say that the high strength of K_c makes the coupling weak, and makes the system sensitive to mode localization.

Influences of the mass of the couplers on mode localization

The influences of the rotational mass of the couplers on mode localization is also studied by considering the cases of $\bar{J}_c = 0{\cdot}1$ and $\bar{J}_c = 1{\cdot}0$, and several span length disturbances. Degrees of mode localization of the first twenty modes for the perfect structures and the disturbed ones are measured. Disturbances considered are $\Delta l_3 = -0{\cdot}25\%$ and $\Delta l_3 = 0{\cdot}5\%$ for lengthening cases.

Localization curves presented in Figure 10 show the typical influences of the mass of the couplers. Although the shapes of localization curves are seen to be different from those

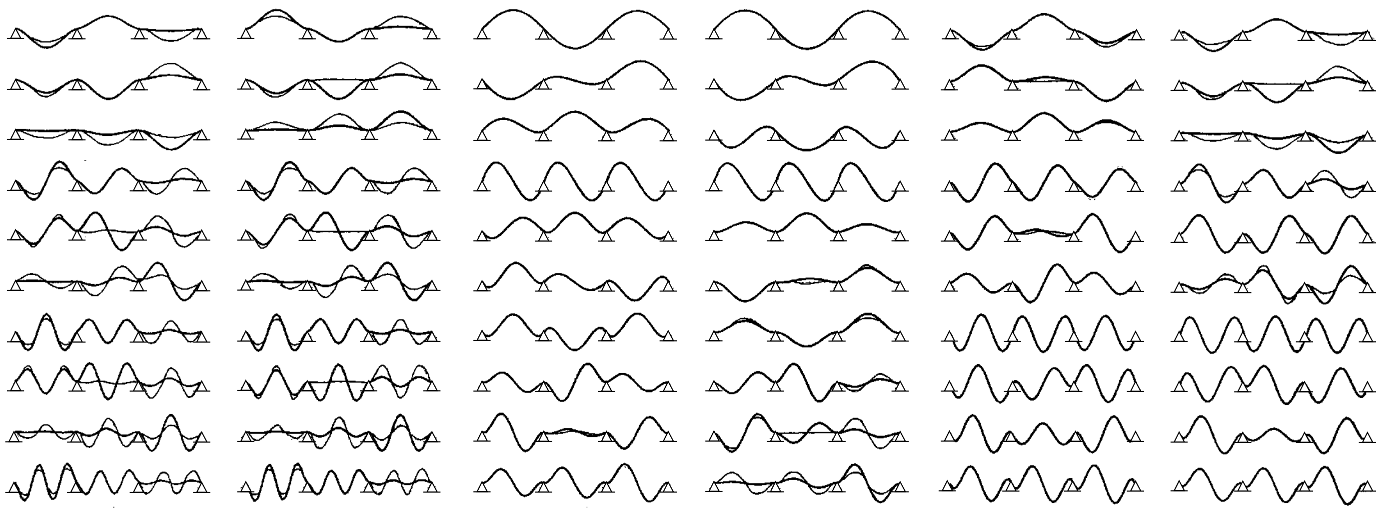

Figure 12. Localized or non-localized first ten mode shapes for selected cases for the simply supported continuous three-span beams with various coupling conditions. Mode shapes of perfect systems (——) and disturbed systems (——) are depicted in ascending order from the top to the bottom; (a) $\bar{K}_c = 100$, $\Delta l_2 = -5{\cdot}0\%$; (b) $\bar{K}_c = 1000$, $\Delta l_2 = -0{\cdot}5\%$; (c) $\bar{J}_c = 0{\cdot}1$, $\Delta l_2 = -0{\cdot}5\%$; (d) $\bar{J}_c = 1{\cdot}0$, $\Delta l_2 = -0{\cdot}5\%$; (e) $\bar{K}_c = 100$, $\bar{J}_c = 0{\cdot}1$, $\Delta l_2 = -0{\cdot}5\%$; and (f) $\bar{K}_c = 1000$, $\bar{J}_c = 1{\cdot}0$, $\Delta l_2 = -0{\cdot}5\%$.

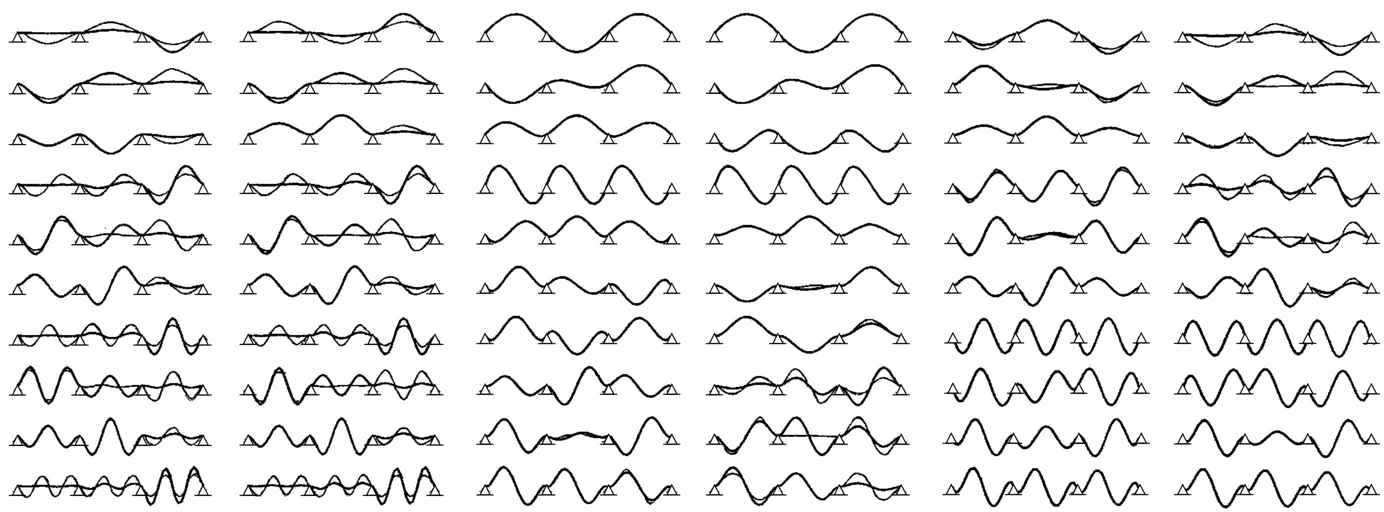

Figure 13. Localized or non-localized first ten mode shapes for selected lengthening cases for the simply supported continuous three-span beams with various coupling conditions. Mode shapes of perfect systems (——) and disturbed systems (——) are depicted in ascending order from the top to the bottom; (a) $\bar{K}_c = 100$, $\Delta l_2 = 5{\cdot}0\%$; (b) $\bar{K}_c = 1000$, $\Delta l_2 = 0{\cdot}5\%$; (c) $\bar{J}_c = 0{\cdot}1$, $\Delta l_2 = 0{\cdot}5\%$; (d) $\bar{J}_c = 1{\cdot}0$, $\Delta l_2 = 0{\cdot}5\%$; (e) $\bar{K}_c = 100$, $\bar{J}_c = 0{\cdot}1$, $\Delta l_2 = 0{\cdot}5\%$; and (f) $\bar{K}_c = 1000$, $\bar{J}_c = 1{\cdot}0$, $\Delta l_2 = 0{\cdot}5\%$.

TABLE 2

First twenty natural frequencies (Hz) of the simply supported continuous two-span beams with various coupling conditions

	Stiffness effects		Mass effects		Mass and stiffness effects	
Mode	$\bar{K}_c = 10^2$ $\bar{J}_c = 0{\cdot}0$	$\bar{K}_c = 10^3$ $\bar{J}_c = 0{\cdot}0$	$\bar{K}_c = 0{\cdot}0$ $\bar{J}_c = 0{\cdot}1$	$\bar{K}_c = 0{\cdot}0$ $\bar{J}_c = 1{\cdot}0$	$\bar{K}_c = 10^2$ $\bar{J}_c = 0{\cdot}1$	$\bar{K}_c = 10^3$ $\bar{J}_c = 1{\cdot}0$
1	477·7	509·2	195·0	102·3	475·3	508·9
2	486·2	510·2	250·4	127·5	484·4	510·0
3	504·0	512·3	373·3	171·2	503·3	512·2
4	1320·6	1402·1	534·2	192·0	1256·4	1385·9
5	1343·0	1404·9	683·8	526·6	1289·2	1392·4
6	1389·2	1410·4	828·6	550·7	1365·2	1406·0
7	2593·7	2743·8	897·2	561·8	1837·6	1785·2
8	2635·4	2749·2	1467·8	1417·7	1902·1	1794·3
9	2720·8	2759·9	1559·9	1426·8	2017·8	1811·7
10	4291·5	4524·7	1601·5	1431·2	2075·4	1820·2
11	4357·0	4533·4	2790·0	2767·6	2808·7	2769·2
12	4490·4	4551·0	2837·5	2772·2	2885·2	2777·0
13	6411·2	6738·8	2860·6	2774·4	2920·0	2780·8
14	6504·3	6751·7	4574·1	4561·1	4576·8	4561·4
15	6692·4	6777·5	4602·4	4563·9	4610·1	4564·7
16	8948·2	9378·7	4616·4	4565·3	4626·5	4566·3
17	9071·9	9396·5	6800·0	6791·5	6800·7	6791·5
18	9320·3	9432·2	6818·7	6793·3	6820·8	6793·5
19	11896·5	12436·6	6828·1	6794·2	6830·8	6794·5
20	12053·3	12460·0	9456·8	9450·8	9457·1	9450·8

of the two span beam example, the overall behaviours are similar to those. As predicted in the theoretical background section, the shifts in degrees of mode localization increase with increasing mode number, with increasing mass of couplers, and with increasnig strength of disturbance.

Additionally, two interesting aspects can be observed. The first is that the rule of classifying localization groups is broken in lowest modes. The other is that mode localization has a saturation point and mode localization is not sensitive to the structural disturbance in the vicinity of that point. Considering Figure 10(c, d), some modes are saturated at $DML = 1{\cdot}0$ and others at $DML \approx 0{\cdot}5$ as the strength of the disturbance increases. In the author's experience, the saturation of mode localization can be observed in all cases with an increase in the strength of disturbance, and it is a typical characteristic of mode localization of weakly coupled multispan beams.

Combined influences of the stiffness and the mass of the couplers on mode localization

Additional studies on the combined influences of K_c and J_c are conducted by considering $\bar{K}_c = 100$ and $\bar{J}_c = 0{\cdot}1$, and $\bar{K}_c = 1000$ and $\bar{J}_c = 1{\cdot}0$ under the span length disturbances of $\Delta l_3 = \pm 0{\cdot}25\%$ and $\Delta l_3 = \pm 0{\cdot}5\%$.

Localization curves depicted in Figure 11 show the influences of the stiffness and the mass at the same time. The overall behaviours are similar to those of the two-span beam example and agree with the theoretical background section. The stiffness of high strength makes the system sensitive to localization for lower modes and the large mass makes the system sensitive for higher modes. Delocalized modes by the mass of the couplers are also observed. As shown in the two-span beam example, the delocalized modes are also close to the coupler's frequency 1780 Hz. The rule of classifying localization groups can be adapted to only the modes far from the delocalized modes, since the delocalized modes do not experience the mode localization.

4. CONCLUDING REMARKS

In this work the influences of the stiffness and the mass of the coupler on mode localization have been studied by both the theoretical and numerical approaches and consistent results have been obtained.

In the theoretical study, a simple structure consisting of two substructures and a coupler was considered, and an equation for the ratio of free vibration amplitude of each substructure was used for qualitative discussion of the influences of the stiffness and the mass of a coupler on mode localization. In the numerical studies, the simply supported continuous two- and three-span beams with the couplers on supports were considered, and the influences of the rotational stiffness and the rotational mass of the couplers on mode localization were investigated. The two-span beam and three-span beam examples gave the results for the influences, although the localization curves of the examples were seen to be different from each other, and the results were consistent with those of the theoretical study.

Some important conclusions drawn in the course of this work can be summarized as follows:

(1) Degree of mode localization varies with the disturbances introduced into the structures.
(2) The sensitivity of the periodic structures to mode localization increases in all modes with increasing stiffness of couplers.
(3) Considering the mass of couplers, the sensitivity of the periodic structures to mode localization increases with increasing mass of couplers and with increasing mode number.
(4) The mass and stiffness of couplers causes a delocalization phenomenon for some modes for which mode localization does not occur or is very weak although structural changes are severe and the delocalization frequency is equal to that of a coupler.
(5) Introducing the large stiffness and the mass into the coupler, the behaviour of mode localization is governed by the stiffness for the lower modes but by the mass for the higher modes, and the delocalized ones can be observed between them.

The first two results agree with those of previous research [6–14] for occurrence of mode localization and for weak coupling conditions. The last three results are observed in this study for the first time. One of the most famous preconditions for dramatic occurrence of mode localization is the weak coupling induced by the stiffness of the coupler of the periodic structures. However, the mass as well as the stiffness of the couplers has important influences on mode localization. As shown in the study, for modes of which the natural frequencies are higher than delocalization frequencies or coupler frequencies, the behaviour of mode localization is governed by the mass of couplers although the mass of couplers is not so large. Two types of delocalization phenomena observed in this study are very interesting characteristics of mode localization. The first type of delocalization, observed by Lust *et al*. [14], is caused by the specific combination of the rotational stiffness and the span length variation. The other one, introduced by ourselves in this work, is caused by the stiffnesses and the masses of the couplers.

It is very interesting that the localization curves of a weakly coupled multispan beam consisting of three or more spans show different manner according to the span number into which disturbances are introduced. However, a detailed discussion for that is not contained in this work since it is beyond the scope of this study. It will be presented in a sequel to this paper by the authors.

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